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FUZZY COMPLETABLE OF FUZZY DISTANCE SPACE

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ABSTRACT

In this paper we recall the definition of fuzzy distance space on fuzzy set then we discuss several properties of this space after that we give an example to illustrate this notion. Then we prove the existence of a fuzzy distance space which is not fuzzy completable. Here we prove that every fuzzy completable fuzzy distance space admits a unique [up to fuzzy isodistance] fuzzy completation.

Key Words: *Fuzzy distance space on fuzzy set, fuzzy bounded set, fuzzy completable fuzzy distance space.*

INTRODUCTION

Theory of fuzzy sets was introduced by Zadeh in 1965 [21]. Many authors have introduced the concept of fuzzy metric in different ways [1, 2, 3, 7, 8, 9, 10, 13, 16, 17]. Kramosil and Michalek in 1975 [6] introduced the definition of fuzzy metric space which is called later KM-fuzzy metric space .George and Veeramani in 1994[3] introduced the definition of continuous $*$ t-norm to modify the concept of KM-fuzzy metric space which was introduced by Kramosil and Michalek which is called later GV-fuzzy metric space .

In section two of this paper we recall the definition of fuzzy distance space on fuzzy set [9] which is a modification of the definition GV-fuzzy metric space after that we introduce basic definitions ,basic concepts and properties of fuzzy distance space. Nevertheless, the theory of fuzzy distance fuzzy completion is very different from the classical theory of metric completion. Indeed there exists a fuzzy distance space which is not fuzzy completable. This fact suggest in a natural way the problem of obtaining conditions for a fuzzy distance space to be fuzzy completable. Here we present a solution to this problem.

1. FUZZY DISTANCE SPACE ON FUZZY SET

Definition 1.1: [21] Let X be a nonempty set of elements, a fuzzy set $\widetilde{A}$ in X is characterized by a membership function, $\mu_{\widetilde{A}}(x)$: X→ [0, 1]. Then we can write

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)): x\in X, 0 \le \mu_{\widetilde{A}}(x) \le 1\}.$$

We now recall an example of a continuous fuzzy set.

Example 1.2: [18] Let X = $\mathbb{R}$ and let $\widetilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function by:

$$\mu_{\widetilde{A}}(x) = \frac{1}{1+10x^2}.$$

Definition 1.3: [4] Let $\tilde{A}$ and $\widetilde{B}$ be two fuzzy sets in X. then

1. $\tilde{A} \subseteq \widetilde{B}$ if and only if $\mu_{\tilde{A}}(x) \le \mu_{\widetilde{B}}(x)$ for all $x\in X$.
2. $\tilde{A} = \widetilde{B}$ if and only if $\mu_{\tilde{A}}(x) = \mu_{\widetilde{B}}(x)$ for all $x\in X$.
3. $\tilde{C} = \tilde{A}\cup\widetilde{B}$ if and only if $\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) \vee \mu_{\widetilde{B}}(x)$ for all $x\in X$.
4. $\widetilde{D} = \tilde{A}\cap\widetilde{B}$ if and only if $\mu_{\widetilde{D}}(x) = \mu_{\tilde{A}}(x) \wedge \mu_{\widetilde{B}}(x)$ for all $x\in X$.
5. $\mu_{\tilde{A}^c}(x) = 1- \mu_{\tilde{A}}(x)$ for all $x\in X$

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Definition 1.4: [18] If $\tilde{A}$ and $\widetilde{B}$ are fuzzy sets in a nonempty sets X and Y respectively then the Cartesian product $\tilde{A} \times \widetilde{B}$ of $\tilde{A}$ and $\widetilde{B}$ is defined by:

$$\mu_{\tilde{A} \times \widetilde{B}}(x, y) = \mu_{\tilde{A}}(x) \wedge \mu_{\widetilde{B}}(y) \text{ for all } (x, y) \in X \times Y$$

Definition 1.5: [20] A fuzzy point p in X is a fuzzy set with member $p(x) = \alpha$ if $x = y$ and $p(x) = 0$ otherwise.

For all y in X where $0 < \alpha < 1$. We denote this fuzzy point by x_α. Two fuzzy points x_α and y_β are said to be distinct if and only if $x \neq y$.

Definition 1.6: [21] Let x_α be a fuzzy point and $\tilde{A}$ be a fuzzy set in X. then x_α is said to be in $\tilde{A}$ or belongs to $\tilde{A}$ which is denoted by $x_\alpha \in \tilde{A}$ if and only if $\mu_{\tilde{A}}(x) > \alpha$.

Definition 1.7: [11] Let f be a function from a nonempty set X into a nonempty set Y. If $\widetilde{B}$ is a fuzzy set in Y then $f^{-1}(\widetilde{B})$ is a fuzzy set in X defined by:
$\mu_{f^{-1}(\widetilde{B})}(x) = (\mu_{\widetilde{B}} \circ f)(x)$ for all x in X. Also if $\tilde{A}$ is a fuzzy set in X then $f(\tilde{A})$ is a fuzzy set in Y defined by:
$\mu_{f(\tilde{A})}(y) = \vee \{\mu_{\tilde{A}}(x): x \in f^{-1}(y)\}$, if $f^{-1}(y) \neq \emptyset$ and $\mu_{f(\tilde{A})}(y) = 0$, otherwise.

Proposition 1.8: [12] Let $f: X \rightarrow Y$ be a function. Then for a fuzzy point x_α in X, $f(x_\alpha)$ is a fuzzy point in Y and $f(x_\alpha) = (f(x))_\alpha$.

Definition 1.9: [3] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is associative and commutative.
2. $*$ is continuous.
3. $a * 1 = a$ for all $a \in [0,1]$.
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ where $a,b, c,d \in [0,1]$.

Remark 1.10: [3] For any $a > b$ we can find c such that $a * c \geq b$ and for any d we can find an e such that $e * e \geq d$ where $a, b, c, d, e \in (0, 1)$.

We introduce the following definition.

Definition 1.11: [14] A triple $(\widetilde{A},\widetilde{D},*)$ is said to be fuzzy distance space if $\widetilde{A}$ is a fuzzy set of the nonempty set X, $*$ is a continuous t- norm and $\widetilde{D}$ is a fuzzy set on $\widetilde{A}^2$ satisfying the following conditions:
(FD_1) $\widetilde{D}(x_\alpha,y_\beta) > 0$ for all $x_\alpha, y_\beta \in \widetilde{A}$
(FD_2) $\widetilde{D}(x_\alpha,y_\beta) = 1$ if and only if $x_\alpha = y_\beta$
(FD_3) $\widetilde{D}(x_\alpha,y_\beta) = \widetilde{D}(y_\beta, x_\alpha)$ for all $x_\alpha, y_\beta \in \widetilde{A}$
(FD_4) $\widetilde{D}(x_\alpha,z_\sigma) \geq \widetilde{D}(x_\alpha,y_\beta) * \widetilde{D}(y_\beta, z_\sigma)$ for all x_α, y_β and $z_\sigma \in \widetilde{A}$
(FD_5) $\widetilde{D}(x_\alpha,y_\beta)$ is a continuous fuzzy set

Example 1.12:[14] Let $X = \mathbb{N}$, and let $\widetilde{A}$ be a fuzzy set in X. Suppose that $a * b = a.b$ for all $a, b \in [0, 1]$.

Define $\widetilde{D}(x_\alpha,y_\beta) = \frac{x}{y}$ If $x \leq y$ and $\widetilde{D}(x_\alpha,y_\beta) = \frac{y}{x}$ If $y \leq x$, for all $x, y \in \mathbb{N}$.

Then $(\widetilde{A},\widetilde{D},*)$ is a fuzzy distance space.

Example 1.13: [14] Let $X=\mathbb{R}$ and let $\widetilde{A}$ be a fuzzy set in X. Suppose that $a * b = a.b$ for all $a, b \in [0, 1]$.

Define $\widetilde{D}(x_\alpha,y_\beta) = \frac{1}{e^{|x_\alpha - y_\beta|}}$ for all $x_\alpha, y_\beta \in \widetilde{A}$.

Then $(\widetilde{A},\widetilde{D},*)$ is a fuzzy distance space.

Definition 1.14: [14] Let $(\tilde{A},\widetilde{D},*)$ be a fuzzy distance space then $\widetilde{D}$ is continuous fuzzy set if whenever $(x_n,\alpha_n) \rightarrow x_\alpha$ and $(y_n,\beta_n) \rightarrow y_\beta$ in $\tilde{A}$ then $\widetilde{D}((x_n, \alpha_n),(y_n, \beta_n)) \rightarrow \widetilde{D}(x_\alpha, y_\beta)$ that is $\lim_{n\rightarrow\infty}\widetilde{D}((x_n, \alpha_n),(y_n, \beta_n)) = \widetilde{D}(x_\alpha, y_\beta)$.

Lemma 1.15: [14] Suppose that (X, d) is an ordinary metric space and assume that $\tilde{A}$ is a fuzzy set in X.
Define $d(x_\alpha, y_\beta) = d(x, y)$ for all $x_\alpha, y_\beta \in \tilde{A}$. Then $(\tilde{A}, d)$ is a metric space.

Example 1.16:[14] Let X= $\mathbb{R}$ and let $\tilde{A}$=[2,∞] be a fuzzy set in X. consider the mapping $\widetilde{D} : \tilde{A} \times\tilde{A} \rightarrow [0, 1]$ defined by:
$\widetilde{D}(a_\alpha,b_\beta) = 1$ if a = b and $\widetilde{D}(a_\alpha,b_\beta) = \left(\frac{1}{a}\right).\alpha + (\frac{1}{b}).\beta$ if a ≠ b, where α∗ β = α. β for all α, β ∈[0,1]

(FM_4) We show that $\widetilde{D}(a_\alpha,c_\sigma) \geq \widetilde{D}(a_\alpha,b_\beta) * \widetilde{D}(b_\beta, c_\sigma)$ is not satisfied for all a_α, b_β, $c_\sigma \in \tilde{A}$. Let a =10, b = 3 and c= 100 where $\alpha=\frac{1}{a}$, $\beta = \frac{1}{b}$, $\sigma = \frac{1}{c}$ Since a ≠ b ≠ c

Then $\widetilde{D}(a_\alpha, b_\beta)=(\frac{1}{a}).\alpha + \left(\frac{1}{b}\right).\beta = \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{100} + \frac{1}{9}$ = 0.01+0.111= 0.121

And $\widetilde{D}(b_\beta,c_\sigma)= \left(\frac{1}{b}\right).\beta + \left(\frac{1}{c}\right).\sigma = \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9} + \frac{1}{10000}$ = 0.111+0.0001= 0.1112

$\widetilde{D}(a_\alpha,c_\sigma)= (\frac{1}{a}).\alpha + \left(\frac{1}{c}\right).\sigma = \frac{1}{a^2} + \frac{1}{c^2} = \frac{1}{100} + \frac{1}{10000}$ = 0.01+0.0001= 0.0101

Therefore $\widetilde{D}(a_\alpha, b_\beta) * \widetilde{D}(b_\beta,c_\sigma) > \widetilde{D}(a_\alpha,c_\sigma)$=(0.121) + (0.1112) = 0.0134552 >0.0101

Thus $(\tilde{A},\widetilde{D},*)$ is not a fuzzy distance space.

Proposition 1.17: [14] Suppose that (X, d) is an ordinary metric space and assume that a∗b = a.b for all a, b∈[0,1]. Then by lemma 2.15, ($\tilde{A}$, d) is a metric space. Define $\widetilde{D}_d(x_\alpha, y_\beta) = \frac{t}{t+d(x_\alpha,y_\beta)}$, then $(\tilde{A},\widetilde{D}_d,*)$ is a fuzzy distance space and it is called the fuzzy distance on the fuzzy set $\tilde{A}$ induced by d.

Definition 1.18: [14] Let $(\tilde{A},\widetilde{D},*)$ be a fuzzy distance space on the fuzzy set $\tilde{A}$, we define
$\widetilde{B}(x_\alpha,r) = \{y_\beta \in \tilde{A}: \widetilde{D}(x_\alpha, y_\beta) > (1- r)\}$ then $\widetilde{B}(x_\alpha,r)$ is called an fuzzy open fuzzy ball with center the fuzzy point $x_\alpha \in \tilde{A}$ and radius $0 < r < 1$.

Proposition 1.19: [14] Suppose that $\widetilde{B}(x_\alpha,r_1)$ and $\widetilde{B}(x_\alpha,r_2)$ be two fuzzy open fuzzy balls with the same center $x_\alpha \in \tilde{A}$ and with radiuses $r_1,r_2 \in (0,1)$. Then we either have $\widetilde{B}(x_\alpha,r_1) \subseteq \widetilde{B}(x_\alpha,r_2)$ or $\widetilde{B}(x_\alpha,r_2) \subseteq \widetilde{B}(x_\alpha,r_1)$.

Definition 1.20: [14] A sequence $\{(x_m, \alpha_m)\}$ of fuzzy points in a fuzzy distance space $(\tilde{A},\widetilde{D},*)$ is called fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if whenever 0 <ε< 1, we can find a positive integer K with, $\widetilde{D}((x_m, \alpha_m), x_\alpha) > (1-\varepsilon)$ whenever m ≥ K.

Definition 1.21: [14] A sequence $\{(x_n, \alpha_n)\}$ of fuzzy points in a fuzzy distance space $(\tilde{A},\widetilde{D},*)$ is called fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if $\lim_{n\to\infty} \widetilde{D}((x_n, \alpha_n), x_\alpha) = 1$.

Theorem 1.22:[14] Definition 2.21 and definition 2.20 are equivalent.

Proposition 1.23: [14] Suppose that (X, d) is a metric space and assume that $(\tilde{A},\widetilde{D}_d,*)$ is the fuzzy distance space induced by d. Let $\{(x_n, \alpha_n)\}$ be a sequence of fuzzy points in $\tilde{A}$.

Then $\{(x_n, \alpha_n)\}$ converges to $x_\alpha \in \tilde{A}$ in ($\tilde{A}$, d) if and only if $\{(x_n, \alpha_n)\}$fuzzy converges to x_α in $(\tilde{A},\widetilde{D}_d,*)$.

Definition 1.24: [14] A fuzzy subset $\tilde{C}$ of a fuzzy distance space $(\tilde{A}, \widetilde{D},*)$ is called fuzzy open if for each $x_\alpha \in \tilde{C}$ there is $\widetilde{B}(x_\alpha,q) \subset \tilde{C}$ with 0 < q < 1. A fuzzy set $\widetilde{E} \subseteq \tilde{A}$ is said to be fuzzy closed if its complement is fuzzy open that is $\widetilde{E}^c = \tilde{A} \setminus \widetilde{E}$ is fuzzy open.

Theorem 1.25: [14] If $\widetilde{B}(x_\alpha, q)$ is fuzzy open fuzzy ball in a fuzzy distance space $(\tilde{A}, \widetilde{D},*)$ on a fuzzy set $\tilde{A}$ then $\widetilde{B}(x_\alpha,q)$ is a fuzzy open fuzzy set with 0 < q < 1.

Definition 1.26: [14] Suppose that $(\tilde{A}, \widetilde{D},*)$ is a fuzzy distance space on a fuzzy set $\tilde{A}$ and let $\tilde{C} \subset \tilde{A}$ then the fuzzy closure of $\tilde{C}$ is denoted by $\bar{\bar{\tilde{C}}}$ or FCL($\tilde{C}$) and is defined to be the smallest fuzzy closed fuzzy set contains $\tilde{C}$.

Definition 1.27: [14] A fuzzy subset $\tilde{C}$ of a fuzzy distance space $(\tilde{A}, \widetilde{D},*)$ on a fuzzy set $\tilde{A}$ is said to be fuzzy dense in $\tilde{A}$ if $\bar{\bar{\tilde{C}}} = \tilde{A}$.

Lemma 1.28: [14] Let $\tilde{C}$ be a fuzzy subset of $\tilde{A}$ and let $(\tilde{A}, \widetilde{D},*)$ be a fuzzy distance space on the fuzzy set $\tilde{A}$ then $a_\alpha \in \bar{\bar{\tilde{C}}}$ if and only if there is a sequence $\{(a_n,\alpha_n)\}$ in $\tilde{C}$ such that $(a_n,\alpha_n) \rightarrow a_\alpha$, where $\alpha,\alpha_n \in [0,1]$.

Theorem 1.29: [14] Suppose that $\tilde{C}$ is a fuzzy subset of a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ then $\tilde{C}$ is fuzzy dense in $\tilde{A}$ if and only if for every $x_\alpha \in \tilde{A}$ there is $a_\beta \in \tilde{C}$ such that $\tilde{D}(x_\alpha, a_\beta) > (1-\varepsilon)$ for some $0 < \varepsilon < 1$.

Definition 1.30: [14] A sequence $\{(x_n, \alpha_n)\}$ of fuzzy points in a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ is said to be fuzzy Cauchy if whenever $0 < \varepsilon < 1$ we can find K with $\tilde{D}((x_n, \alpha_n), (x_m, \alpha_m)) > (1-\varepsilon)$ for all n, m $\geq$K. A fuzzy distance space $(\tilde{A}, \tilde{D}_{\tilde{A}},*)$ is fuzzy complete if every fuzzy Cauchy sequence in $\tilde{A}$ fuzzy converges to a fuzzy point in $\tilde{A}$.

Theorem 1.31: [14] Let $(\tilde{A}, \tilde{D},*)$ be a fuzzy distance space on the fuzzy set $\tilde{A}$ if $\{(x_n, \alpha_n)\}$ is a sequence of fuzzy points in $\tilde{A}$ that is fuzzy converges to $x_\alpha \in \tilde{A}$ then $\{(x_n, \alpha_n)\}$ is fuzzy Cauchy.

Proposition 1.32: [14] Suppose that (X, d) is a metric space and let $\tilde{D}_d(x_\alpha, y_\beta) = \frac{t}{t+d(x_\alpha, y_\beta)}$ where t= min{α, β}.Then $\{(x_n, \alpha_n)\}$ is a Cauchy sequence in $(\tilde{A}, d)$ if and only if $\{(x_n, \alpha_n)\}$ is a fuzzy Cauchy sequence in $(\tilde{A}, \tilde{D}_d,*)$.

Definition 1.33:[14] Suppose that $(\tilde{A},\tilde{D},*)$ be a fuzzy distance space. A fuzzy subset $\tilde{C}$ of $\tilde{A}$ is called fuzzy bounded if we can find $0 < q < 1$ with, $\tilde{D}(x_\alpha, y_\beta) > (1-q)$, whenever $x_\alpha, y_\beta \in \tilde{C}$.

Proposition 1.34: [14] Let (X, d) be a metric space and let $\tilde{D}_d(x_\alpha, y_\beta) = \frac{t}{t+d(x_\alpha, y_\beta)}$ where $t=\alpha \wedge \beta$ then a fuzzy subset $\tilde{C}$ of $\tilde{A}$ is fuzzy bounded if and only if it is bounded.

Definition 1.35:[14] Let $(\tilde{A},\tilde{D},*)$ be a fuzzy distance space, then we define a fuzzy closed fuzzy ball with center $x_\alpha \in \tilde{A}$ and radius r, $0 < r < 1$ by $\tilde{B}[x_\alpha, r] = \{y_\beta \in X: \tilde{D}(x_\alpha, y_\beta) \geq (1-r)\}$.

Lemma 1.36: [14] If $\tilde{B}[x_\alpha, q]$ is fuzzy closed fuzzy ball in a fuzzy distance space $(\tilde{A}, \tilde{D},*)$ on a fuzzy set $\tilde{A}$ then $\tilde{B}[x_\alpha, q]$ is a fuzzy closed fuzzy set with $0 < q < 1$.

Theorem 1.37: [14] Suppose that $(\tilde{A},\tilde{D},*)$ is a fuzzy distance space . Put $\tau_{\tilde{D}} = \{\tilde{C} \subset \tilde{A} : x_\alpha \in \tilde{C}$ if and only if there is $0 < q < 1$ with $\tilde{B}(x_\alpha, q) \subset \tilde{C}\}$.Then $\tau_{\tilde{M}}$ is a fuzzy topology on $\tilde{A}$.

Proposition 1.38: [14] Suppose that (X, d) is an ordinary metric space. Let $\tilde{D}_d(x_\alpha, y_\beta) = \frac{t}{t+d(x_\alpha, y_\beta)}$ be the fuzzy distance induced by d. Then the topology τ_d induced by d and the fuzzy topology $\tau_{\tilde{D}_d}$ induced by $\tilde{D}_d$ are the same. That is $\tau_d = \tau_{\tilde{D}_d}$.

Theorem 1.39: [14] Every fuzzy distance space on a fuzzy set is a fuzzy Hausdorff space.

Definition 1.40:[14] Suppose that $(\tilde{A}, \tilde{D}_{\tilde{A}},*)$ and $(\tilde{E}, \tilde{D}_{\tilde{E}},*)$ are fuzzy distance spaces and $\tilde{C} \subseteq \tilde{A}$. The mapping h: $\tilde{C} \to \tilde{E}$ is said to be fuzzy continuous at $a_\beta \in \tilde{C}$, if whenever $0 < \varepsilon < 1$, we can find $0 < \delta < 1$, with $\tilde{D}_{\tilde{E}}(h(x_\alpha), h(a_\beta)) > (1-\varepsilon)$ whenever $x_\alpha \in \tilde{C}$ and $\tilde{D}_{\tilde{A}}(x_\alpha, a_\beta) > (1-\delta)$. When f is fuzzy continuous at every fuzzy point of $\tilde{C}$, then it is called to be fuzzy continuous on $\tilde{C}$.

Theorem 1.41: [14] Let $(\tilde{A}, \tilde{D}_{\tilde{A}},*)$ and $(\tilde{E}, \tilde{D}_{\tilde{E}},*)$ be fuzzy distance spaces and $\tilde{C} \subseteq \tilde{A}$. The mapping h: $\tilde{C} \to \tilde{E}$ is fuzzy continuous at $a_\beta \in \tilde{C}$ if and only if whenever a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{C}$ fuzzy converge to a_β, then sequence of fuzzy points $\{(h(x_n, \alpha_n))\}$ fuzzy converges to $h(a_\beta)$.

Proposition 1.42:[11] Let $\tilde{A}$ be a fuzzy set in X and let $\tilde{B}$ be a fuzzy set in Y. let f: $\tilde{A} \to \tilde{B}$ be a function and let $\tilde{C} \subseteq \tilde{A}$ and $\tilde{E} \subseteq \tilde{B}$ .Then $f(\tilde{C}) \subseteq \tilde{E}$ if and only if $\tilde{C} \subseteq f^{-1}(\tilde{E})$.

Theorem 1.43: [14] A mapping f: $\tilde{A} \to \tilde{E}$ is fuzzy continuous on $\tilde{A}$ if and only if the inverse image of $\tilde{C}$ is fuzzy open in $\tilde{A}$ for all fuzzy open fuzzy subset $\tilde{C}$ of $\tilde{E}$. Where $\tilde{A}$ and $\tilde{E}$ are fuzzy distance spaces.

Theorem 1.44: [14] A mapping f: $\tilde{A} \to \tilde{E}$ is fuzzy continuous on $\tilde{A}$ if and only if the inverse image of $\tilde{C}$ is fuzzy closed in $\tilde{A}$ for all fuzzy closed fuzzy subset $\tilde{C}$ of $\tilde{E}$.

Theorem 1.45: Let $\tilde{C}$ be a fuzzy dense fuzzy subset of a fuzzy distance space $(\tilde{A},\tilde{D},*)$. If every fuzzy Cauchy sequence of fuzzy point of $\tilde{C}$ fuzzy converges in $\tilde{A}$ then $(\tilde{A},\tilde{D},*)$ is fuzzy complete.

Theorem 1.46: Suppose that (X, d) is a metric space and let $\tilde{A}$ be fuzzy set in X then $(\tilde{A}, d)$ is a metric space. let $(\tilde{A},\tilde{D}_d,*)$ be the induced fuzzy distacnce space. Then $(\tilde{A}, d)$ is complete iff $(\tilde{A},\tilde{D}_d,*)$ is fuzzy complete.

Definition 1.47: A mapping f from a fuzzy distance space $(\tilde{A}, \widetilde{D}_{\tilde{A}},*)$ into a fuzzy distance space $(\widetilde{E}, \widetilde{D}_{\widetilde{E}},*)$ is called isodistance if $\widetilde{D}_{\widetilde{E}}(f(x_\alpha),f(y_\beta)) = \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta)$ for all $x_\alpha, y_\beta \in \tilde{A}$.

It is clear that isodistance is one-to-one and uniformly fuzzy continuous.

2. Fuzzy COMPLETABLE FUZZY DISTANCE SPACE

Definition 2.1: Let $(\tilde{A}, \widetilde{D}_{\tilde{A}},*)$ be a fuzzy distance space. A fuzzy completion of $(\tilde{A}, \widetilde{D}_{\tilde{A}},*)$ is a fuzzy complete fuzzy distance space $(\widetilde{E}, \widetilde{D}_{\widetilde{E}},\star)$ such that $(\tilde{A}, \widetilde{D}_{\tilde{A}},*)$ is fuzzy isodistance to a fuzzy dense fuzzy subset $\widetilde{W}$ of $\widetilde{E}$.

Next, we show that unfortunately there exists a fuzzy distance space that does not admit any fuzzy completion in the sense of Definition 2.1

Example 2.2: Let $a*b = \max\{0, a+b-1\}$ for all $a, b\in[0,1]$. Suppose that $\{((t_n,\alpha_n)): n \geq 1\}$ and $\{((z_n,\beta_n)): n \geq 1\}$ are two fuzzy sequences of fuzzy points with $t_i \neq t_j$ and $z_i \neq z_j$ for all $i, j\geq 1$ such that $\tilde{C} \cap \widetilde{E} = \emptyset$
where $\tilde{C}= \{((t_n,\alpha_n)): n \geq 3\}$ and $\widetilde{E} = \{((z_n,\beta_n)): n \geq 3\}$. Put $\tilde{A} = \tilde{C} \cup \widetilde{E}$, define $\widetilde{D} : \tilde{A} \times \tilde{A} \to [0,1]$ as follows:
$\widetilde{D}((t_n,\alpha_n),(t_m,\alpha_m)) = \widetilde{D}((z_n,\beta_n),(z_m,\beta_m)) = [1- \frac{1}{n\wedge m} + \frac{1}{n\vee m}]$
Where $n\wedge m = \min\{n,m\}$ and $n\vee m = \max\{n,m\}$.
$\widetilde{D}((t_n,\alpha_n),(z_m,\beta_m)) = \widetilde{D}((z_m,\beta_m),(t_n,\alpha_n)) = \frac{1}{n} + \frac{1}{m}$

First we show that $(\tilde{A},\widetilde{D},*)$ is a fuzzy distance space.

The following four properties are almost obvious:
(FM_1) $0 < \widetilde{D}(t_\alpha,z_\beta) \leq 1$, for all $t_\alpha, z_\beta \in \tilde{A}$
(FM_2) $\widetilde{D}(t_\alpha,z_\beta) = 1 \Leftrightarrow t_\alpha = z_\beta$
(FM_3) $\widetilde{D}(t_\alpha,z_\beta) = \widetilde{D}(z_\beta,t_\alpha)$ for all $t_\alpha, z_\beta \in \tilde{A}$
(FM_4) $\widetilde{D}(t_\alpha,z_\beta)$ is fuzzy continuous

On the other hand, an easy computation shows that for all n, m, k≥3
$\widetilde{D}((t_n,\alpha_n),(t_m,\alpha_m)) * \widetilde{D}((t_m,\alpha_m),(t_k,\alpha_k)) \leq \widetilde{D}((t_n,\alpha_n),(t_k,\alpha_k))$ &
$\widetilde{D}((z_n,\beta_n),(z_m,\beta_m)) * \widetilde{D}((z_m,\beta_m),(z_k,\beta_k)) \leq \widetilde{D}((z_n,\beta_n),(z_k,\beta_k))$

Finally, the following relation is straight forward:
$\widetilde{D}((t_n,\alpha_n),(t_m,\alpha_m))* \widetilde{D}((t_m,\alpha_m),(z_k,\beta_k))\leq \widetilde{D}((t_n,\alpha_n),(z_k,\beta_k))$
And similarly $\widetilde{D}((t_n,\alpha_n),(z_m,\beta_m)) * \widetilde{D}((z_m,\beta_m),(z_k,\beta_k))\leq \widetilde{D}((t_n,\alpha_n),(z_k,\beta_k))$
And $\widetilde{D}((t_n,\alpha_n),(z_m,\beta_m))* \widetilde{D}((z_m,\beta_m),(t_k,\alpha_k)) \leq \widetilde{D}((t_n,\alpha_n),(t_k,\alpha_k))$

Therefore, for all $t_\alpha, z_\beta, e_\sigma \in \tilde{A}$, we have $\widetilde{D}(t_\alpha,z_\beta)* \widetilde{D}(z_\beta,e_\sigma) \leq \widetilde{D}(t_\alpha,e_\sigma)$

Hence $(\tilde{A},\widetilde{D},*)$ is a fuzzy distance space.

Now we prove that $\{(t_n,\alpha_n): n \geq 3\}$ is a fuzzy Cauchy sequence in $(\tilde{A},\widetilde{D},*)$. Fix $0 < \varepsilon < 1$ therefore there is K, such that $|\frac{1}{n} - \frac{1}{m}|< \varepsilon$ for all $n, m \geq K$. Suppose without loss of generality $m \geq n$.

Thus $\widetilde{D}((t_n,\alpha_n),(t_m,\alpha_m)) = [1- (\frac{1}{n} - \frac{1}{m})] > (1- \varepsilon)$ for $n, m \geq K$. Hence $\{(t_n,\alpha_n): n \geq 3\}$ is a fuzzy Cauchy sequence in $(\tilde{A},\widetilde{D},*)$.Similarly $\{(z_n,\beta_n): n \geq 3\}$ is also a fuzzy Cauchy sequence in $(\tilde{A},\widetilde{D},*)$.However $\{(t_n,\alpha_n): n \geq 3\}$ and $\{(z_n,\beta_n): n \geq 3\}$ do not fuzzy converge in $\tilde{A}$ with respect to the fuzzy topology $\tilde{\tau}_{\widetilde{D}}$.In fact $\tilde{\tau}_{\widetilde{D}}$ is the discrete fuzzy topology on $\tilde{A}$ because for each $n \geq 3$, we have $\widetilde{B}((t_n,\alpha_n),\frac{1}{n(n+1)}) = \{(t_n,\alpha_n)\}$ and $\widetilde{B}((z_n,\beta_n),\frac{1}{n(n+1)}) = \{(z_n,\beta_n)\}$ .In order to prove the two preceding equalities it suffices to observe that for $n, m \geq 3$ with $n \neq m$, we have:
$\widetilde{D}((t_n,\alpha_n),(t_m,\alpha_m)) = [1- \frac{1}{n\wedge m} - \frac{1}{n\vee m}] \leq [1- (\frac{1}{n} - \frac{1}{n+1})] = [1- \frac{1}{n(n+1)}]$
and similarly, $\widetilde{D}((z_n,\beta_n),(z_m,\beta_m)) \leq [1- \frac{1}{n(n+1)}]$
and for $n, m \geq 3$, we have:
$\widetilde{D}((t_n,\alpha_n),(z_m,\beta_m)) = \frac{1}{n} + \frac{1}{m} \leq [1- (\frac{1}{n} - \frac{1}{n+1})]$
And similarly, $\widetilde{D}((z_n,\beta_n),(t_m,\alpha_m)) \leq [1- (\frac{1}{n} - \frac{1}{n+1})]$

Therefore $(\tilde{A},\tilde{D},*)$ is not fuzzy complete. Suppose that $(\tilde{A},\tilde{D},*)$ admits a fuzzy completion $(\tilde{C},\tilde{N},\star)$. Then there exists a fuzzy isodistance T from $\tilde{A}$ onto a fuzzy dense fuzzy subset of $\tilde{C}$.Since $\{T(t_n,\alpha_n): n \geq 3\}$ and $\{T(z_n,\beta_n): n \geq 3\}$ are fuzzy Cauchy sequences in $(\tilde{C},\tilde{N},\star)$ There exists $\hat{t}_\alpha,\hat{z}_\beta \in \tilde{C}$ such that $\lim_{n\to\infty}\tilde{N}(T(t_n,\alpha_n),\hat{t}_\alpha) = 1$ and $\lim_{n\to\infty} \tilde{N}(T(z_n,\beta_n),\hat{z}_\beta) = 1$.

Put $\tilde{N}(\hat{t}_\alpha,\hat{z}_\beta) = r$ with $0 < r < 1$, for $k \geq 3$ there exists $K_n \geq n$, such that $\tilde{N}(T(t_{K_n},\alpha_{K_n}),\hat{t}_\alpha) > (1-\frac{1}{n})$ and $\tilde{N}(T((z_{K_n},\beta_{K_n})),\hat{y}_\beta) > (1-\frac{1}{n})$

Since $\tilde{N}(T(t_{K_n},\alpha_{K_n}),T(z_{K_n},\beta_{K_n})) = \frac{2}{K_n}$ It follows from the relation:

$\tilde{N}(T(t_{K_n},\alpha_{K_n}),\hat{x}_\alpha) \star \tilde{N}(\hat{t}_\alpha,\hat{z}_\beta) \star \tilde{N}(T(z_{K_n},\beta_{K_n}),\hat{y}_\beta) \leq \tilde{N}(T(t_{K_n},\alpha_{K_n}), T(z_{K_n},\beta_{K_n}))$

$(1-\frac{1}{n}) \star r \star (1-\frac{1}{n}) \leq \frac{2}{K_n}$...... (1) For each $k \geq 3$

On the other hand since $\star$ is a continuous t-norm

$(1-\frac{1}{n}) \star r \star (1-\frac{1}{n}) \to r$ as $k \to \infty$

So by (1), we deduce that r = 0 a contradiction.

Hence $(\tilde{A}, \tilde{D},*)$ has no fuzzy completion.

Definition 2.3: A fuzzy distance space $(\tilde{A}, \tilde{D},*)$ is called fuzzy completable if it admits a fuzzy completion.

Theorem 2.4: Every fuzzy completable fuzzy distance space admits a fuzzy completion.

Proof: We subdivide the proof into three steps:

(a) Construction of $(\tilde{X},\tilde{M},*)$

(b) A fuzzy isodistance f from X onto $f(\tilde{A})$, $f(\tilde{A})$ is fuzzy dense fuzzy subset of $\tilde{X}$.

(c) fuzzy Completeness of $\tilde{X}$.

Proof of (a): Suppose that $(\tilde{A}, \tilde{D}_{\tilde{A}},*)$ is a fuzzy distance space and assume that $\tilde{E}$ be the fuzzy set of all fuzzy Cauchy sequence in $\tilde{A}$. A relation $\sim$ on $\tilde{E}$ is defined by:

$(x_n,\alpha_n) \sim (x'_n,\alpha'_n)$ if and only if $\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(x'_n,\alpha'_n)) = 1$

We prove that $\sim$ is an fuzzy equivalence relation on $\tilde{E}$

1. $\sim$ is reflexive that is $(x_n,\alpha_n)\sim(x_n,\alpha_n)$ because $\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(x_n,\alpha_n)) = 1$.
2. $\sim$ is symmetric that is if $(x_n,\alpha_n) \sim (y_n,\beta_n)$ it is immediately follows that $(y_n,\beta_n)\sim(x_n,\alpha_n)$ because $\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) = \tilde{D}_{\tilde{A}}((y_n,\beta_n),(x_n,\alpha_n))$ so that $\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((y_n,\beta_n),(x_n,\alpha_n)) = 1$.
3. Finally, $\sim$ is transitive , suppose that $(x_n,\alpha_n)\sim (y_n,\beta_n)$ and $(y_n,\beta_n) \sim (z_n,\sigma_n)$ we will prove that $(x_n,\alpha_n)\sim (z_n,\sigma_n)$.

Now, $\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(z_n,\sigma_n))\geq \tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n))* \tilde{D}_{\tilde{A}}((y_n,\beta_n),(z_n,\sigma_n))$

$\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(z_n,\sigma_n))\geq \lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n))* \lim_{n\to\infty}\tilde{D}_{\tilde{A}}((y_n,\beta_n),(z_n,\sigma_n)) = 1 * 1 = 1$

Hence $\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(z_n,\sigma_n)) = 1$. Therefore $(x_n,\alpha_n) \sim (z_n,\sigma_n)$.

Now denote $\tilde{X}$ the quotient $\tilde{E} / \sim$ and by:

$[(x_n,\alpha_n)] = \{(x'_n,\alpha'_n) \in \tilde{A}: (x'_n,\alpha'_n) \sim (x_n,\alpha_n)\}$ .The class of the element (x_n,α_n) of $\tilde{E}$.

Define $\tilde{D}_{\tilde{X}} : \tilde{X} \times \tilde{X} \to [0, 1]$ by

$$\tilde{D}_{\tilde{X}}([(x_n,\alpha_n)],[(y_n,\beta_n)]) = \lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) \quad (2)$$

We claim that the fuzzy limit in (2) is independent of the particular choice of the representative . In fact if $(x_n,\alpha_n)\sim (x'_n,\alpha'_n)$ and $(y_n,\beta_n)\sim (y'_n,\beta'_n)$.

Then $\tilde{D}_{\tilde{A}}((x'_n,\alpha'_n),(y'_n,\beta'_n))\geq \tilde{D}_{\tilde{A}}((x'_n,\alpha'_n),(x_n,\alpha_n)) * \tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) * \tilde{D}_{\tilde{A}}((y_n,\beta_n),(y'_n,\beta'_n))$

$\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x'_n,\alpha'_n),(y'_n,\beta'_n))\geq[\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x'_n,\alpha'_n),(x_n,\alpha_n))] *$

$[\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n))] * [\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((y_n,\beta_n),(y'_n,\beta'_n))]$

$\lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x'_n,\alpha'_n),(y'_n,\beta'_n))\geq \lim_{n\to\infty}\tilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n))$.

In a similar way we obtain that $\lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) \geq \lim_{n\to\infty}\widetilde{D}_{\tilde{A}}\left((x'_n,\alpha'_n),(y'_n,\beta'_n)\right)$.

Now we prove that $(\widetilde{X},\widetilde{D}_{\widetilde{X}},*)$ is a fuzzy distance space.

(FM_1) $\widetilde{D}_{\widetilde{X}}([(x_n,\alpha_n)],[(y_n,\beta_n)]) > 0$ for all $[(x_n,\alpha_n)],[(y_n,\beta_n)]\in\widetilde{X}$ because $\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n))>0$ implies that $\lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n))>0$.

(FM_2) Obviously, $\widetilde{D}_{\widetilde{X}}([(x_n,\alpha_n)],[(x_n,\alpha_n)]) = 1$ for all $[(x_n,\alpha_n)]\in\widetilde{X}$.

Moreover if $\widetilde{D}_{\widetilde{X}}([(x_n,\alpha_n)],[(y_n,\beta_n)]) = 1$ so $\lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) = 1$

Which implies that $(x_n,\alpha_n) \sim (y_n,\beta_n)$ hence $[(x_n,\alpha_n)] = [(y_n,\beta_n)]$.

(FM_3) clearly we have $\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) = \widetilde{D}_{\tilde{A}}((y_n,\beta_n),(x_n,\alpha_n))$ for all $[(x_n,\alpha_n)]$, $[(y_n,\beta_n)]\in\widetilde{X}$.

(FM_4) for each $[(x_n,\alpha_n)]$, $[(y_n,\beta_n)]$, $[(z_n,\sigma_n)]\in\widetilde{X}$.

we obtain
$$\begin{aligned}\widetilde{D}_{\widetilde{X}}([(x_n,\alpha_n)],[(z_n,\sigma_n)]) &= \lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(z_n,\sigma_n))\\ &\geq \lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) * \lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((y_n,\beta_n),(z_n,\sigma_n))\\ &= \widetilde{D}_{\widetilde{X}}([(x_n,\alpha_n)],[(y_n,\beta_n)]) * \widetilde{D}_{\widetilde{X}}([(y_n,\beta_n)],[(z_n,\sigma_n)])\end{aligned}$$

(FM_5) Finally $\widetilde{D}_{\widetilde{X}}$ is fuzzy continuous since $\widetilde{D}_{\tilde{A}}$ is fuzzy continuous.

Proof of (b): The function $f\colon \tilde{A} \to \widetilde{X}$ is defined by by $f(a_\gamma) = [(a_n,\gamma_n)]$,where $[(a_n,\gamma_n)] = \{(a_\gamma,a_\gamma,a_\gamma,\ldots,a_\gamma,..)\}$. Obviously f is one to one mapping , moreover for each $x_\alpha, y_\beta \in \tilde{A}$,we clearly have $\widetilde{D}_{\widetilde{X}}(f(x_\alpha),f(y_\beta)) = \widetilde{D}_{\tilde{A}}(x_\alpha,y_\beta)$ so f is an fuzzy isodistance from $\tilde{A}$ onto $f(\tilde{A})$. Next we show that $f(\tilde{A})$ is fuzzy dense in $\widetilde{X}$, indeed given $[(a_n,\gamma_n)]\in\widetilde{X}$ we can find K such that $\widetilde{D}_{\tilde{A}}((a_n,\gamma_n),(a_m,\gamma_m)) > (1-\varepsilon)$ for given $0<\varepsilon<1$ and for all $n, m \geq K$.

Then by Remark 1.10 there is $0 < r < 1$ such that:
$$(1-\varepsilon) * (1-\varepsilon) > (1-r).$$

Therefore $\widetilde{D}_{\widetilde{X}}([(a_n,\gamma_n)], f(a_K,\gamma_K)) = \lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((a_n,\gamma_n),(a_K,\gamma_K)) \geq (1-\varepsilon)$

We conclude that $f((a_K,\gamma_K))\in\widetilde{B}_{\widetilde{N}}(\tilde{a},\varepsilon)$ so $f(\tilde{A})$ is fuzzy dense in $\widetilde{X}$.

Proof of (c): By Theorem 1.45 it suffices to show that each fuzzy Cauchy sequence in $f(\tilde{A})$ fuzzy converges in $\widetilde{X}$. Let $\{(a_k,\gamma_k)\}$ be a sequence of fuzzy points in $\tilde{A}$ such that $\{(f(a_k,\gamma_k))\}$ is a fuzzy Cauchy sequence in $(\widetilde{X},\widetilde{D}_{\widetilde{X}},*)$.Since f is a fuzzy isodistance $\{(a_k,\gamma_k)\}$ is fuzzy Cauchy sequence in $\tilde{A}$. Put $\hat{a}_\gamma = [(a_k,\gamma_k)]$, then $\hat{a}_\gamma\in\widetilde{X}$ and we shall show that $f((a_k,\gamma_k))\to\hat{a}_\gamma$. Indeed we have: $\lim_{k\to\infty}\widetilde{D}_{\widetilde{X}}(\hat{a}_\gamma,f((a_k,\gamma_k))) = \lim_{k\to\infty}\lim_{n\to\infty}\widetilde{D}_{\tilde{A}}((a_n,\gamma_n),(a_k,\gamma_k)) = 1$

Since $\{(a_k,\gamma_k)\}$ is fuzzy Cauchy sequence in $\tilde{A}$. Consequently $f((a_k,\gamma_k))\to\hat{a}_\gamma$

Hence $(\widetilde{X},\widetilde{M}_{\widetilde{X}},*)$ is fuzzy complete.

Theorem 2.5: Suppose that $(\tilde{A},\widetilde{D}_{\tilde{A}},*)$ is a fuzzy distance space and let $(\widetilde{E},\widetilde{D}_{\widetilde{E}},\star)$ be a fuzzy complete fuzzy distance space. If there is a fuzzy isodistance mapping f from a fuzzy dense fuzzy subset $\tilde{C}$ of $\tilde{A}$ to $\widetilde{E}$ then f has a unique extension $f^*\colon \tilde{A}\to\widetilde{E}$.

Proof: We consider any $x_\alpha\in\tilde{A}$ but $\tilde{A} = \overline{\overline{\tilde{C}}}$ so $x_\alpha\in\overline{\overline{\tilde{C}}}$ then there is a sequence $\{(x_n,\alpha_n)\}$ of fuzzy points in $\tilde{C}$ such that $\{(x_n,\alpha_n)\}$ fuzzy converges to x_α by Lemma 1.28. Then $\{(x_n,\alpha_n)\}$ is fuzzy Cauchy.

Since f is fuzzy isodistance $\{f((x_n,\alpha_n))\}$ is fuzzy Cauchy in $\widetilde{E}$ but $\widetilde{E}$ is fuzzy complete hence there is $y_\alpha\in\widetilde{E}$ such that $\{f(x_n,\alpha_n)\}$ fuzzy converges to y_α. Now we define $f^*(x_\alpha) = y_\alpha$.

We now show that this definition is independent of the particular choice of the sequence in $\tilde{C}$ converging to x_α. Suppose that $\{(x_n,\alpha_n)\}$ in $\tilde{C}$ fuzzy converges to x_α and $\{(z_n,\sigma_n)\}$ in $\tilde{C}$ fuzzy converges to x_α. Then $\{(v_m,\gamma_m)\}$ fuzzy converges to x_α where $\{(v_m,\gamma_m)\} = ((x_1,\alpha_1),(z_1,\sigma_1),(x_2,\alpha_2),(z_2,\sigma_2),\ldots)$. Hence $\{(f(v_m,\gamma_m))\}$ fuzzy converges and the two subsequence $\{(f(x_n,\alpha_n))\}$ and $\{(f(z_n,\sigma_n))\}$ of $\{(f(v_m,\gamma_m))\}$ must have the same fuzzy limit. This prove f^* is uniquely defined at every $x_\alpha\in\tilde{A}$. Clearly $f^*(x_\alpha) = f(x_\alpha)$ for every $x_\alpha\in\tilde{C}$ so that f^* is an extension of f.

Theorem 2.6: Suppose that $(\tilde{A}, \widetilde{D}_{\tilde{A}},*)$ is a fuzzy distance space and let $(\widetilde{E},\widetilde{D}_{\widetilde{E}},\star)$ be a fuzzy complete fuzzy distance space .If f is an fuzzy isodistance mapping from a fuzzy dense fuzzy subset $\tilde{C}$ of $\tilde{A}$ to $\widetilde{E}$ then the unique extension $f^*:\tilde{A}\rightarrow\widetilde{E}$ is a fuzzy isodistance.

Proof: Let x_α, $y_\beta \in \tilde{A}$ then there exists two sequences $\{(x_n,\alpha_n)\}$ and $\{(y_n,\beta_n)\}$ of fuzzy points in $\tilde{C}$ such that $(x_n,\alpha_n)\rightarrow x_\alpha$ and $(y_n,\beta_n)\rightarrow y_\beta$. Choose an arbitrary $0 <\varepsilon< 1$.

Now: $\varepsilon + \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta) > \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta)$. Furthermore, it follows that $\{(x_n,\alpha_n)\}$ and $\{(y_n,\beta_n)\}$ are fuzzy Cauchy sequences in $\tilde{C}$ so $\{f^*((x_n,\alpha_n))\}$ and $\{f^*((y_n,\beta_n))\}$ are fuzzy Cauchy sequences in $\widetilde{E}$. But $\widetilde{E}$ is fuzzy complete hence $\{f^*((y_n,\beta_n))\}$ fuzzy converges to $f^*(y_\beta)$ and $\{f^*((x_n,\alpha_n))\}$ fuzzy converges to $f^*(x_\alpha)$. Then we can find K with $\widetilde{D}_{\tilde{A}}(x_\alpha,(x_n,\alpha_n)) > (1-\varepsilon)$, $\widetilde{D}_{\tilde{A}}((y_n,\beta_n), y_\beta) > (1-\varepsilon)$ $\widetilde{D}_{\widetilde{E}}(f^*((x_n,\alpha_n)),f^*(x_\alpha)) > (1-\varepsilon)$ and $\widetilde{D}_{\widetilde{E}}(f^*((y_n,\beta_n)),f^*(y_\beta)) > (1-\varepsilon)$ for all $n \geq K$. Thus we have $\varepsilon + \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta) > \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta) \geq \widetilde{D}_{\tilde{A}}(x_\alpha,(x_n,\alpha_n)) * \widetilde{D}_{\tilde{A}}((x_n,\alpha_n),(y_n,\beta_n)) * \widetilde{D}_{\tilde{A}}((y_n,\beta_n), y_\beta) \geq (1-\varepsilon) * \widetilde{D}_{\widetilde{E}}(f^*((x_n,\alpha_n)),f^*((y_n,\beta_n))) * (1-\varepsilon)$

But $\widetilde{D}_{\widetilde{E}}(f^*((x_n,\alpha_n)),f^*((y_n,\beta_n))) \geq \widetilde{D}_{\widetilde{E}}(f^*((x_n,\alpha_n)),f^*(x)) \star \widetilde{D}_{\widetilde{E}}(f^*(x_\alpha),f^*(y_\beta)) \star \widetilde{D}_{\widetilde{E}}(f^*((y_n,\beta_n)),f^*(y_\beta)) \geq (1-\varepsilon) \star \widetilde{D}_{\widetilde{E}}(f^*(x_\alpha),f^*(y_\beta)) \star (1-\varepsilon)$ for all $n \geq K$.

Therefore
$\varepsilon + \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta) > (1-\varepsilon) *[(1-\varepsilon) \star \widetilde{D}_{\widetilde{E}}(f^*(x_\alpha),f^*(y_\beta)) \star (1-\varepsilon)] *(1-\varepsilon)$

By fuzzy continuity of $*$ and $\star$ it follows that $\widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta) \geq \widetilde{D}_{\widetilde{E}}(f^*(x_\alpha),f^*(y_\beta))$.

A similar argument shows that $\widetilde{D}_{\widetilde{E}}(f^*(x_\alpha),f^*(y_\beta)) \geq \widetilde{D}_{\tilde{A}}(x_\alpha, y_\beta)$ For all $x_\alpha, y_\beta \in \tilde{A}$ We conclude that f^* is an fuzzy isodistance from $(\tilde{A}, \widetilde{D}_{\tilde{A}},*)$ to $(\widetilde{E},\widetilde{D}_{\widetilde{E}},\star)$.

Theorem 2.7: Every fuzzy completable fuzzy distance space admits a unique [up to fuzzy isodistance] fuzzy completion.

Proof: Let $(\widetilde{E},\widetilde{M}_1,\star)$ and $(\tilde{Z},\widetilde{M}_2,\circ)$ be two fuzzy completions of $(\tilde{A},\widetilde{D},*)$ then we will prove that $(\widetilde{E},\widetilde{M}_1,\star)$ and $(\tilde{Z},\widetilde{M}_2,\circ)$ are fuzzy isodistance. Since $(\widetilde{E},\widetilde{M}_1,\star)$ is a fuzzy completion of $(\tilde{A},\widetilde{D},*)$ then there is an fuzzy isodistance f from $(\tilde{A},\widetilde{D},*)$ to a fuzzy dense fuzzy subset of $(\widetilde{E},\widetilde{M}_1,\star)$. By Theorem 2.5 and Theorem 2.6 f admits a unique extension f^* onto $(\widetilde{E},\widetilde{M}_1,\star)$which is also a fuzzy isodistance. Similarly $\acute{f}$ is a fuzzy isodistance extension $(\tilde{A},\widetilde{D},*)$ onto $(\tilde{Z},\widetilde{M}_2,\circ)$.To prove that f^* and $\acute{f}$ are fuzzy isodistance it remains to see that $f^*$and $\acute{f}$ are onto we will show that f^* is onto. Indeed given $y_\alpha \in \widetilde{E}$ there is a sequence $\{(x_n,\alpha_n)\}$ of fuzzy points in $\tilde{A}$ such that $f^*(x_n,\alpha_n) \rightarrow y_\alpha$. Since f^* is an fuzzy isodistance $\{(x_n,\alpha_n)\}$ is a fuzzy Cauchy sequence, so it fuzzy converges to some fuzzy point $x_\alpha \in \tilde{A}$. Consequently $f^*(x_\alpha) = y_\alpha$. Similarly we can prove that $\acute{f}$ is onto. Hence f^* and $\acute{f}$ are fuzzy isodistance.

Now $(\widetilde{E},\widetilde{M}_1,\star)$ is fuzzy isodistance to $(\tilde{A},\widetilde{M},*)$ and $(\tilde{A},\widetilde{M},*)$ is fuzzy isodistance to $(\tilde{Z},\widetilde{M}_2,\circ)$. Hence $(\widetilde{E},\widetilde{M}_1,\star)$ is fuzzy isodistance to $(\tilde{Z},\widetilde{M}_2,\circ)$.

Theorem 2.8: [12] For a metric space (X, d) there exists a complete metric space $(\widehat{X},\hat{d})$ which has a subspace W that is isometric with $\tilde{A}$ and is dense in $\widehat{X}$.

This $(\widehat{X},\hat{d})$ is unique except for isometries. In the proof of this theorem $\widehat{X}$ will be the set of all equivalence classes $\hat{x}$,$\hat{y}$, … of Cauchy sequence of points that is if (x_n) and $(\acute{x}_n)$ are Cauchy sequences in (X,d) then $(x_n)\sim(\acute{x}_n)$ if and only if $\lim_{n\to\infty} d((x_n),(y_n)) = 0$. Also $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d((x_n),(y_n))$.

Proposition 2.9: Suppose that (X, d) is a metric space and let $\widetilde{A}$ be a fuzzy set in X. then $(\widetilde{A}, d)$ is a metric space by lemma 1.15 and let f be an isodistance from $(\widetilde{A}, d)$ onto a dense fuzzy subset of $(\widehat{\widetilde{A}},\hat{d})$. Then the fuzzy distance $(\widehat{\widetilde{A}},\widehat{\widetilde{D}}_{\hat{d}},*)$ is given by :$\widehat{\widetilde{D}}_{\hat{d}}(\widehat{x_\alpha}, \widehat{y_\beta}) = \frac{t}{t+\hat{d}(\widehat{x_\alpha},\widehat{y_\beta})}$ for all $\widehat{x_\alpha}, \widehat{y_\beta} \in \widehat{X}$. $t=\alpha\wedge\beta$

Since by Theorem 2.8 $(\widehat{\widetilde{A}},\hat{d})$ is a metric space then by Proposition 1.17 $(\widehat{\widetilde{A}},\widehat{\widetilde{M}}_{\hat{d}},*)$ is a fuzzy distance space .

Theorem 3.10: Suppose that (X,d) is a metric space and let $\widetilde{A}$ be a fuzzy set in X. let $(\widetilde{A},\widetilde{D}_d,*)$be the fuzzy distance space induced by $(\widetilde{A}$,d). Then the fuzzy completion of $(\widetilde{A},\widetilde{D}_d,*)$ is the fuzzy distance space $(\widehat{X},\widehat{\widetilde{D}}_{\hat{d}},*)$ where $\widehat{\widetilde{D}}_{\hat{d}}(\hat{x}_\alpha, \hat{y}_\beta) = \frac{t}{t+\hat{d}(\hat{x}_\alpha,\hat{y}_\beta)}$ for all $\hat{x}_\alpha, \hat{y}_\beta \in \widehat{X}$ where $t =\alpha\wedge\beta$.

Proof: $(\widetilde{A},\hat{d})$ is a fuzzy complete metric space then $(\hat{\widetilde{A}},\hat{\widetilde{D}}_{\hat{d}},*)$ is a fuzzy complete fuzzy metric space by Theorem 1.46. Hence $(\hat{\widetilde{A}},\hat{\widetilde{D}}_{\hat{d}},*)$ is the fuzzy completion of the fuzzy distance space $(\widetilde{A},\widetilde{D}_d,*)$.

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