DUALITY FOR MULTIOBJECTIVE PROGRAMMING INVOLVING $(\Phi, \rho)$-UNIVEXITY

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The concepts of $(\Phi, \rho)$-invexity have been given by Carsiti, Ferrara and Stefanescu [20]. We consider a dual model associated to a multiobjective programming problem involving support functions and a weak duality result is established under appropriate $(\Phi, \rho)$-univexity conditions.

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1. INTRODUCTION

For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual [6, 23] is well known. While studying duality under generalized convexity, Mond and Weir [29] proposed a number of different duals for nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

For $\Phi(x,a,(y,r)) = F(x,a) + rd^2(x,a)$, where $F(x,a)$ is sublinear on $\mathbb{R}^n$, the definition of $(\Phi, \rho)$-invexity reduces to the definition of $(F, \rho)$-convexity introduced by Preda [17], which in turn Jeyakumar [18] generalizes the concepts of F-convexity and $\rho$-invexity. For more details reader may consult [1,2,3,4,5,7,9,17,18,24,27,29].

The more recent literature, Mishra [22], Xu [11], Ojha [12], Ojha and Mukherjee [15] for duality under generalized $(F, \rho)$-convexity, Liang et al. [13] and Hachimi [14] for optimality criteria and duality involving $(F, \alpha, \rho, d)$-convexity or generalized $(F, \alpha, \rho, d)$-type functions. The $(F, \rho)$-convexity was recently generalized to $(\Phi, \rho)$-invexity by Caristi, Ferrara and Stefanescu [20], and here we will use this concept to extend some theoretical results of multiobjective programming.

Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity, too. This is not the case in multiobjective programming; Ferrara and Stefanescu [16] showed that sufficiency Kuhn-Tucker condition can be proved under $(\Phi, \rho)$-invexity, even if Hanson’s invexity is not satisfied, Puglisi [21]. Therefore, the results of this paper are real extensions of the similar results known in the literature.

In Section 2 we define the $(\Phi, \rho)$-univexity. In Section 3 we consider a class of Multiobjective programming problems and for the dual model we prove a weak duality result and strong duality.

2. NOTATIONS AND PRELIMINARIES

We consider $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^q$, are differential functions and $X \subset \mathbb{R}^n$ is an open set. We define the following multiobjective programming problem:

$$(P) \text{ minimize } f(x) = (f_1(x),...,f_p(x))$$
subject to $g(x) \geq 0, x \in X$
Let \( X_0 \) be the set of all feasible solutions of (P) that is, \( X_0 = \{ x \in X \mid g(x) \geq 0 \} \).

We quote some definitions and also give some new ones.

**Definition 2.1:** A vector \( a \in X_0 \) is said to be an efficient solution of problem (P) if there exist no \( x \in X_0 \) such that \( f(a) - f(x) \in R^p \setminus \{0\} \) i.e., \( f_i(x) \leq f_i(a) \) for all \( i \in \{1, \ldots, p\} \), and for at least one \( j \in \{1, \ldots, p\} \) we have \( f_j(x) < f_j(a) \).

**Definition 2.2:** A point \( a \in X_0 \) is said to be a weak efficient solution of problem (VP) if there is no \( x \in X \) such that \( f(x) < f(a) \).

**Definition 2.3:** A point \( a \in X_0 \) is said to be a properly efficient solution of (VP) if it is efficient and there exist \( a \in \mathbb{R}^n \) and \( \rho \in \mathbb{R}^+ \) with respect to the gradient vector at \( \{ (a, \rho) \} \) be a \( (a, \rho) \)-univex at \( (a, \rho) \), then this is \( \{ (a, \rho) \} \)-univexity from \([1]\), Let \( f \in \mathbb{R}^n \) defined on an open set of \( \mathbb{R}^n \) that is, \( f : \mathbb{R}^n \to \mathbb{R} \). A point \( \phi \in \mathbb{R}^n \) such that \( \phi \) is a real number, we have \( \psi_0(\alpha) = -\psi_0(-\alpha) \) and \( \psi_1(-\alpha) = -\psi_1(\alpha) \).

For convenience, let us write the definition \( \nabla \psi(a, b) \) as of \((\phi, \rho)\)-univexity from \([1]\). Let \( \phi : X_0 \to R \) be a differentiable function \((X_0 \subseteq R^n, X \subseteq X_0, a \in X_0) \). An element of all \((n+1)\)-dimensional Euclidean Space \( R^{n+1} \) is represented as the ordered pair \((z, r) \) with \( z \in R^n \) and \( r \in R, \rho \) is a real number and \( \Phi \) is a real valued function defined on \( X_0 \times X_0 \times R^{n+1} \), such that \( \Phi(x, a, \rho) \) is convex on \( R^{n+1} \) and \( \Phi(x, a, (0, r)) \geq 0 \), for every \((x, a) \in X_0 \times X_0 \) and \( r \in R_+ \). Let \( b(x, a) = \lim_{\lambda \to 0} \frac{b(x, a, \lambda)}{\lambda} \) \( b(x, a) \geq 0 \), and \( b \) does not depend upon \( \lambda \) if the corresponding functions are differentiable. \( \psi_0 : R \to R \) is an \( n \)-dimensional vector-valued function.

We assume that \( \psi_0, \psi_1 : R \to R \) satisfying \( u \leq 0 \Rightarrow \psi_0(u) \leq 0 \) and \( u \geq 0 \Rightarrow \psi_1(u) \leq 0 \), and \( b(x, a) > 0 \) and \( b(x, a) \geq 0 \). and \( \psi_0(\alpha) = -\psi_0(\alpha) \) and \( \psi_1(-\alpha) = -\psi_1(\alpha) \).

**Example 2.1:**

\[
\begin{align*}
\min f(x) & = x - 1 \\
g(x) & = -x - 1 \leq 0, x \in X_0 \in [1, \infty) \\
\Phi(x, a; (y, r)) & = 2(2' - 1)|x - a| + \langle y, x - a \rangle \\
\end{align*}
\]

for \( \psi_0(x) = x, \psi_1(x) = -x, \rho = \frac{1}{2} \) (for \( f \) ) and \( \rho = 1 \) (for \( g \)), then this is \((\phi, \rho)\)-univex but it is not \((\phi, \rho)\)-invex.

**Definition 2.4:** A real-valued twice differentiable function \( f(., y) : X \times X \to R \) is said to be \((\Phi, \rho)\)-univex at \( u \in X \) with respect to \( p \in R^r \), if for all \( b : X \times X \to R_+, \Phi : X \times X \times R^{r+1} \to R \), \( \rho \) is a real number, we have

\[
b(x, u)[\psi[f_j(x, y) - f_j(u, y)]] \geq \Phi(x, u; (\nabla f(u, y), \rho))
\]  

\( (2.1) \)
Definition 2.5:
A real-valued twice differentiable function \( f(y, \lambda) : X \times X \to R \) is said to be \( (\Phi, \rho) \)-pseudounivex at \( a \in X \) with respect to \( \rho \in R^n \), if for all \( b : X \times X \to R_+, \Phi : X \times X \times R^{n+1} \to R \), \( \rho \) is a real number, we have
\[
\Phi(x, u; (\nabla f(y, \lambda), \rho)) \geq 0 \Rightarrow b(x, u) [\psi[f_x(x, y) - f_y(u, y)]] \geq 0 \tag{2.2}
\]

Definition 2.6:
A real-valued twice differentiable function \( f(y, \lambda) : X \times X \to R \) is said to be \( (\Phi, \rho) \)-quasiunivex at \( a \in X \) with respect to \( \rho \), if for all \( b : X \times X \to R_+, \Phi : X \times X \times R^{n+1} \to R \), \( \rho \) is a real number, we have
\[
\Phi(x, u; (\nabla f(y, \lambda), \rho)) \leq 0 \Rightarrow b(x, u) [\psi[f_x(x, y) - f_y(u, y)]] \leq 0 \tag{2.3}
\]

Remark 2.1:
(i) If we consider the case \( b = 1, \psi \equiv I \), \( \Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u)) \) (with \( F \) is sublinear in third argument, and then the above definition reduce to \( F \)-convexity.

(ii) When \( \Phi(x, u; (\nabla f(u), \rho)) = F(x, u; \nabla f(u)) = \eta(x, u)^T \nabla f(u) \), where \( \eta : X \times X \to R^n \), \( b = 1, \psi \equiv I \) the above definition reduce to \( \eta \) - (pseudo/quasi)-convexity.

A real valued twice differentiable function \( f \) is \( (\Phi, \rho) \)-pseudoconcave if \( -f \) is \( (\Phi, \rho) \)-pseudoconvex.

3. TWO WOLFE TYPE SYMMETRIC DUALITY

We consider here the following pair of multiobjective mathematical programs and establish weak, strong duality theorems.

\[
\text{(MP)} \quad \text{Minimize} \quad f_i(x, y) \\
\text{subject to} \quad \sum_{i=1}^r \lambda_i \nabla_y f_i(x, y) \leq 0 \tag{3.1} \\
\lambda > 0, \quad \lambda^T e = 1 \tag{3.2} \\
x \geq 0 \tag{3.3}
\]

\[
\text{(MD)} \quad \text{Maximize} \quad f_i(u, v) \\
\text{subject to} \quad \sum_{i=1}^r \lambda_i \nabla_u f_i(u, v) \geq 0 \tag{3.4} \\
\lambda > 0, \quad \lambda^T e = 1 \tag{3.5} \\
v \geq 0 \tag{3.6}
\]

Here, \( e(1, 1, 1, \ldots, 1)^T \in R, \lambda_i \in R, i = 1, 2, \ldots, r \) and \( f_i, i = 1, 2, \ldots, r \) are twice differentiable function from \( R^n \to R^n \to R, \) and \( \lambda_i = R^n \times R^n \times R^n \to R_+ \)

Theorem 3.1 (Weak duality):
Let \( (x, y, \lambda, z_1, z_2, \ldots, z_r) \) be a feasible solution of (MP) and \( (u, v, \lambda, w_1, w_2, \ldots, w_r) \) a feasible solution of (MD) and

(i) \( \sum_{i=1}^r \lambda_i f_i(y, v) \) is \( (\Phi_0, \rho) \)-univex at \( u \) for fixed \( v \),

(ii) \( \sum_{i=1}^r \lambda_i f_i(x, v) \) is \( (\Phi_1, \rho) \)-unicave at \( y \) for fixed \( x \).
(iii) \( \Phi_0(x,u;(\xi,\rho)) + u^T\xi \geq 0 \), where \( \xi = \sum_{i=1}^{r} \lambda_i \nabla f_i(u,v) \), and

(iv) \( \Phi_1(v,y;(\xi,\rho)) + y^T\xi \leq 0 \), for \( \xi = \sum_{i=1}^{r} \lambda_i \nabla f_i(x,y) \).

Then, \( f_i(x,y) \not\leq f_i(u,v) \).

**Proof:** Since \( \sum_{i=1}^{r} \lambda_i f_i(.,v) \) be \( (\Phi_0,\rho) \)-univex at \( u \) for fixed \( v \), for \( \lambda > 0 \),

\[
\sum_{i=1}^{r} \lambda_i [b_0(x,y,u,v)\psi(f(x,v) - f_i(u,v))] \geq \Phi_0(x,u;(\nabla f_i(u,v),\rho))
\]

We get facilitate, with the help of hypothesis (iii) and (3.4), with property of \( b_0 \) and \( \psi \)

\[
\sum_{i=1}^{r} \lambda_i f_i(x,v) \geq \sum_{i=1}^{r} \lambda_i (f_i(u,v))
\]

(3.7)

Now, \( f_i(x,.) \) be \( (\Phi_1,\rho) \) pseudounicavity assumption at \( y \) for fixed \( x \), for \( \lambda > 0 \), we have, we have, \( \sum_{i=1}^{r} \lambda_i b_i(x,y,u,v)\psi[f_i(x,v) - f_i(x,y)] \leq \Phi_1(x,y;(\nabla f_i(x,y),\rho_i)) \)

and hypothesis (iv), (3.1), with property of \( b_0 \) and \( \psi \), it implies that

\[
\sum_{i=1}^{r} \lambda_i f_i(x,y) \leq \sum_{i=1}^{r} \lambda_i (f_i(u,v))
\]

(3.8)

Combining (3.7) and (3.8), we get

\[
\sum_{i=1}^{r} \lambda_i f_i(x,y) \geq \sum_{i=1}^{r} \lambda_i f_i(u,v)
\]

3.1 MOND-WEIR TYPE SYMMETRIC DUALITY

We consider here the following pair of multiobjective mathematical programs and establish weak, strong and converse duality theorems.

(MP)

Minimize \( f_i(x,y) \)

subject to

\[
\sum_{i=1}^{r} \lambda_i \nabla f_i(x,y) \leq 0
\]

(3.9)

\[
y^T \sum_{i=1}^{r} \lambda_i [\nabla f_i(x,y)] \geq 0
\]

(3.10)

\[
\lambda > 0, \quad \lambda^T e = 1
\]

(3.11)

\[
x \geq 0
\]

(3.12)

(MD)

Maximize \( f_i(u,v) \)

subject to

\[
\sum_{i=1}^{r} \lambda_i \nabla u f_i(u,v) \geq 0
\]

(3.13)

\[
u^T \sum_{i=1}^{r} \lambda_i \nabla u f_i(u,v) \leq 0
\]

(3.14)
Here, \( e(1,1,\ldots,1)T \in R, \lambda_i \in R, i = 1,2,\ldots,r \) and \( f_i, i = 1,2,\ldots,r \) are twice differentiable function from \( R^r \times R^m \rightarrow R \), and \( b_i = R^r \times R^m \rightarrow R_+ \).

**Theorem 3.1 (Weak duality):**
Let \( (x, y, \lambda, z_1, z_2, \ldots, z_r) \) be a feasible solution of (MP) and \( (u, v, \lambda, w_1, w_2, \ldots, w_r) \) a feasible solution of (MD) and

(i) \( \sum_{i=1}^{r} \lambda_i f_i (\cdot, v) \) is \((\Phi_0, \rho)\) -pseudounivex at \( u \) for fixed \( v \),

(ii) \( \sum_{i=1}^{r} \lambda_i f_i (\cdot, \cdot) \) is \((\Phi_1, \rho)\) -pseudounicave at \( y \) for fixed \( x \),

(iii) \( \Phi_0 (x, u; (\zeta, \rho)) + u^T \zeta \geq 0 \), where \( \zeta = \sum_{i=1}^{r} \lambda_i \nabla_y f_i (u, v) \), and

(iv) \( \Phi_1 (v, y; (\zeta, \rho)) + y^T \zeta \leq 0 \), for \( \zeta = \sum_{i=1}^{r} \lambda_i \nabla_y f_i (x, y) \).

Then, \( f_i (x, y) \not\geq f_i (u, v) \).

**Proof:** Since \( \sum_{i=1}^{r} \lambda_i f_i (\cdot, \cdot) \) be \((\Phi_0, \rho)\) -pseudounivex at \( u \) for fixed \( v \), for \( \lambda > 0 \), we get facilitate with the help of hypothesis (iii) and (3.14) with property of \( b_0 \) and \( \psi \),

\[
\sum_{i=1}^{r} \lambda_i f_i (x, v) - \sum_{i=1}^{r} \lambda_i f_i (u, v) \geq 0 \text{ for all } i = 1,2,\ldots,r \tag{3.17}
\]

\[
\sum_{i=1}^{r} \lambda_i f_i (x, v) \geq \sum_{i=1}^{r} \lambda_i f_i (u, v) \tag{3.18}
\]

Now, \( f_i (\cdot, \cdot) \) be \((\Phi_1, \rho)\) pseudounicavity assumption at \( y \) for fixed \( x \), for \( \lambda > 0 \), we have,

we have, hypothesis (iv)and (3.10), with property of \( b_0 \) and \( \psi \),it implies that

\[
\sum_{i=1}^{r} \lambda_i f_i (x, v) - \sum_{i=1}^{r} \lambda_i f_i (x, y) \leq 0 \tag{3.19}
\]

\[
\sum_{i=1}^{r} \lambda_i f_i (x, v) \leq \sum_{i=1}^{r} \lambda_i f_i (x, y) \tag{3.20}
\]

Combining (3.18) and (3.20), we get

\[
\sum_{i=1}^{r} \lambda_i f_i (x, y) \geq \sum_{i=1}^{r} \lambda_i f_i (u, v)
\]

**Theorem 3.2 (Strong duality):**
Let \((x', y', \lambda')\) be a weak efficient solution for (MP) for fixed \( \lambda = \lambda' \) in (MD), assume that (i) The set \( \sum_{i=1}^{r} \lambda_i [\nabla_y f_i] \) is +ive or –ive definite for all \( i = 1,2,\ldots,r \);

(ii) and the set \( [\nabla_y f_1, \nabla_y f_2, \ldots, \nabla_y f_r] \) for all \( i = 1,2,\ldots,r \) is linearly independent, such that \((x', y', \lambda')\) is a feasible solution of (MD), \( b_i (x', y', u', \lambda') > 0, i = 1,2,\ldots,r \), and the two objectives have the same values. Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (MP) and (MD), then \((x', y', \lambda')\) is an efficient solution for (MD).
Proof: Let \((x', y', \lambda')\) is a weak efficient solution for (MP), then it is weakly efficient solution. Hence, there exist \(\alpha \in \mathbb{R}^r, \beta \in \mathbb{R}^r, \gamma \in \mathbb{R}^r, \mu \in \mathbb{R}^r\) and \(n \in \mathbb{R}\) not all zero, \(i = 1, 2, \ldots, r\) such that the following Fritz john optimality condition (28) are satisfied at \((x', y', \lambda')\).

\[
\alpha \nabla_x f_i + (\beta - ny')^T \lambda (\nabla_y f_i) = s
\]  
\[\sum_{i=1}^r (\alpha_i - n \lambda_i) \nabla_y f_i + (\beta - ny')^T \lambda' (\nabla_y f_i) = 0, \text{ for all } i = 1, 2, \ldots, r.\]  
\[(\beta - ny')\nabla_y f_i - \mu_i = 0, \text{ for all } i = 1, 2, \ldots, r.\]  
\[\beta^T \sum_{i=1}^r \lambda \nabla_y f_i = 0\]  
\[\Rightarrow ny' \sum_{i=1}^r \lambda \nabla_y f_i = 0\]  
\[\mu^T \lambda' = 0\]  
\[(\alpha, \beta, s, \lambda, \mu, n) \geq 0, \]  
\[(\alpha, \beta, s, \lambda, \mu, n) \neq 0, \lambda' > 0\]  

and \(\mu \geq 0\). (3.26) implies \(\mu = 0\). Consequently, (3.23) gives

\[
(\beta - ny') \nabla_y f_i = 0
\]
\[(\beta - ny') \nabla [\nabla_y f_i] (\beta - ny') = 0\]

hence, in the view of (i), \(\beta = ny'\).  

From (3.22) and (3.31), \(\sum_{i=1}^r (\alpha_i - n \lambda_i) \nabla_y f_i = 0\).  

assumption (ii) and (3.32), gives

\[\alpha_i = n \lambda_i \text{ for all } i = 1, 2, \ldots, r.\]  

If, \(n = 0 \Rightarrow \alpha_i = 0, \beta = 0, \mu_i = 0, s = 0, \text{ for all } i = 1, 2, \ldots, r.\)  
\[(\alpha, \beta, s, \lambda, \mu, n) = 0, n > 0.\]

Then we obtain \((\alpha, \beta, s, \lambda, \mu, n) = 0\), which contradicts (3.28), hence \(n > 0\).

from (3.33) \(\lambda' > 0\) we have \(\alpha_i > 0, \text{ for all } i = 1, 2, \ldots, r.\) From (3.21), (3.31) and (3.33) we get,

\[\alpha_i \nabla_x f_i = s/n \geq 0\]  

By (3.27) and (3.31)

Since, \(\eta > 0\), we have,

\[y' = \beta/n \geq 0\]  

From (3.34), it follows that \(\alpha_i \nabla_x f_i = 0\) 

From ((3.34)-(3.36)), we know that \((x', y', \lambda')\) is feasible for (MD).

\[f_i(x', y') = f_i(x', y')\]  

and the objective values of (MD) and (MP) are equal.

We claim that \((x', y', \lambda')\) is an efficient solution for (MD) for if it is not true, then there would exist
(u, v, \lambda') feasible for (MD) such that \( f_i(u, v) \not\leq f_i(x', y'), i = 1, 2, \ldots, r \) (3.38)

Using equality (3.38) a contradiction (Weak Duality Theorem 3.2) is obtained.

If \((x', y', \lambda')\) is improperly efficient, then for every scalar \(M > 0\), there exist a feasible solution \((u', v', \lambda')\) in (MD) and an index \(i\) such that
\[
f_i(u, v) - f_i(x', y') > M [f_j(x', y') - f_j(u, v)] \quad \text{for all } j \text{ satisfying}
\]
\[
f_j(x', y') > f_j(u, v)
\]
(3.39)

Whenever \(f_i(u, v) > f_i(x', y')\) (3.40)

It implies that
\[
f_i(u, v) > f_i(x', y')\]

can be made arbitrarily large and hence for \(\lambda'\) with \(\lambda' > 0\), we have
\[
\sum_{i=1}^{r} \lambda' f_i(u, v) > \sum_{i=1}^{r} \lambda' f_i(x', y')
\]
(3.42)

Which contradicts the weak duality theorem 3.2.

REFERENCES


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