Some Properties of a Quarter-Symmetric Non-Metric Connexion in a LP- Sasakian Manifold

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(Received on: 23-06-11; Accepted on: 04-07-11)

ABSTRACT

In this paper I have studied a quarter-symmetric non-metric connexion in a Lorentzian para-Sasakian manifold. Some properties of the curvature tensor and the Ricci tensor of the manifold for quarter-symmetric non-metric connexion have been obtained.

Keywords: Quarter- symmetric connexion, LP-Sasakian manifold, curvature tensor, Ricci tensor.

Mathematics subject classification: [53]

1. INTRODUCTION

We consider a $n$–dimensional $C^\infty$-manifold $V_n$. Let there exist in $V_n$, a tensor $F$ of the type (1,1), a vector field $U$, a 1–form $u$ and a Riemannian metric $g$ such that

1. $\bar{X} = X + u(X)U$, 
2. $u(\bar{X}) = 0$, 
3. $g(\bar{X}, \bar{Y}) = g(X, Y) + u(X)u(Y)$, 
4. $g(X, U) = u(X)$, 
5. $(D_\pi F)(Y) = g(X, Y)U + u(Y)X + 2u(X)u(Y)U$, 
6. $D_X U = \bar{X}$

where

for arbitrary vector fields $X, Y$. Then $V_n$ satisfying (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) is called a Lorentzian para–Sasakian manifold [2] (in short LP-Sasakian manifold) while the set $\{F, U, u, g\}$ satisfying (1.1) to (1.6) is called a LP-Sasakian structure. It may be noted that $D$ is the Riemannian connexion with respect to the Riemannian metric $g$.

In a LP-Sasakian manifold it is easy to calculate that

1. $u(U) = -1$, 
2. $U = 0$. 

and

$\text{rank}(F) = n - 1$. 

Let us define a fundamental 2–form $'F$ in a LP-Sasakian manifold as below:

1. $'F(X, Y) \equiv g(\bar{X}, \bar{Y})$. 

Barring $Y$ in (1.3) and using (1.1) and (1.2), we get

$g(\bar{X}, \bar{Y}) = g(X, Y)$

From (1.8) and (1.9), we obtain that

1. $'F(X, Y) = 'F(Y, X)$

Which shows that $'F$ is symmetric in a LP-Sasakian manifold.

Barring $X$ and $Y$ both in (1.8) and using (1.1), (1.2), (1.8) and (1.9), we get

1. $'F(\bar{X}, \bar{Y}) = 'F(X, Y)$

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which implies that \( F \) is hybrid in a LP-Sasakian manifold.

(1.4) implies

\[
u(Y) = g(Y, U).
\]

Taking the covariant derivative of above with respect to the connexion \( D \) along the vector field \( X \) and using (1.4), (1.6) and (1.8), we get

\[
\gamma'(X, Y) = (D_X u)(Y).
\] (1.12)

The Conformal curvature tensor \( Q \), the Conharmonic curvature tensor \( L \), the Concircular curvature tensor \( C \) and the Projective curvature tensor \( P \) in \( V_n \) are given by [3]

\[
\begin{align*}
Q(X, Y, Z) &= K(X, Y, Z) - \frac{1}{n-2}[Ric(Y, Z)X - Ric(X, Z)Y + g(Y, Z)RX - g(X, Z)RY] \\
&+ \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y],
\end{align*}
\] (1.13)

\[
L(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-2}[Ric(Y, Z)X - RX(Y, Z)Y + g(Y, Z)RX - g(X, Z)RY],
\] (1.14)

\[
C(X, Y, Z) = K(X, Y, Z) - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]
\] (1.15)

and

\[
P(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-1}[Ric(Y, Z)X - Ric(X, Z)Y]
\] (1.16)

Agreement (1.1): A LP-Sasakian manifold will always be denoted by \( V_n \).

2. CERTAIN PROPERTIES ON \( V_n \)

**Theorem (2.1):** In \( V_n \), we have

\[
(D_X' F)(Y, U) = g(\bar{X}, \bar{Y}).
\] (2.1)

\[
(D_X' F)(\bar{Y}, Z) + (D_X' F)(Y, \bar{Z}) = u(Z)(D_X u)(Y) + u(Y)(D_X u)(Z).
\] (2.2)

\[
(D_X' F)(\bar{Y}, \bar{Z}) + (D_X' F)(Y, Z) = u(Y)g(\bar{X}, \bar{Z}) - u(Z)g(\bar{X}, \bar{Y}).
\] (2.3)

**Proof:** In view of (1.7 b) and (1.8), we have

\[
\gamma'(F, Y, U) = 0
\] (2.4)

Taking the covariant derivative of (2.4) with respect to the connexion \( D \) along the vector field \( X \) and using (1.8) and (2.4), we get (2.1).

We know that

\[
(D_X F)(Y, Z) = g((D_X F)(Y), Z)
\] (2.5)

which implies

\[
(D_X' F)(\bar{Y}, Z) = g((D_X F)(\bar{Y}), Z).
\] (2.6)

Since

\[
F\bar{Y} = F^2 Y.
\]

Therefore taking the covariant derivative of above with respect to the connexion \( D \) along the vector field \( X \) and using (1.1), we get

\[
(D_X F)(\bar{Y}) + (D_X F)(\bar{Y}) = u(Y)D_X U + (D_X u)(Y)U.
\]

Operating \( g \) on both the sides of above and using (1.6) and (2.5), we get (2.2).

Barring \( Z \) in (2.2) and using (1.1), (1.2), (1.4), we get (2.3).

**Theorem (2.2):** In \( V_n \), we have

\[
K(X, Y, Z, T) = g(Y, Z)g(X, T) - g(X, Z)g(Y, T),
\] (2.7)

\[
Ric(Y, Z) = (n-1)g(Y, Z),
\] (2.8)

\[
RY = (n-1)Y,
\] (2.9)

\[
r = n(n-1).
\] (2.10)
Proof: From (1.12), we have
\[ F(Y, Z) = (D_Y u)(Z). \]  
(2.11)

Taking the covariant derivative of above with respect to the connexion \( D \) along the vector field \( X \) and using (2.11), we get
\[ (D_X F)(Y, Z) = (D_X D_Y u)(Z) - (D_{[X,Y]} u)(Z). \]  
(2.12)

Interchanging \( X \) and \( Y \) in above, we get
\[ (D_Y F)(X, Z) = (D_Y D_X u)(Z) - (D_{[X,Y]} u)(Z). \]  
(2.13)

Subtracting (2.13) from (2.12), we get
\[ (D_X F)(Y, Z) - (D_Y F)(X, Z) = (D_X D_Y u)(Z) - (D_Y D_X u)(Z) - (D_{[X,Y]} u)(Z). \]  
(2.14)

From (1.14), we have
\[ u(Z) = g(Z, U). \]  
(2.15)

Taking the covariant derivative of (2.15) with respect to the connexion \( D \) along the vector field \( Y \) and using (2.15), we get
\[ (D_Y u)(Z) = g(Z, D_Y U). \]  
(2.16)

Taking the covariant derivative of above with respect to the connexion \( D \) along the vector field \( X \) and using (2.16), we get
\[ (D_X D_Y u)(Z) = g(Z, D_Y D_X U). \]  
(2.17)

Interchanging \( X \) and \( Y \) in above, we get
\[ (D_Y D_X u)(Z) = g(Z, D_Y D_X U). \]  
(2.18)

Further (2.16) yields
\[ (D_{[X,Y]} u)(Z) = g(Z, D_{[X,Y]} U). \]  
(2.19)

Subtracting (2.18) and (2.19) from (2.17) and using (2.14), we get
\[ (D_X F)(Y, Z) - (D_Y F)(X, Z) = g(Z, K(X, Y, U)). \]  
(2.20)

From (1.5), we have
\[ (D_X F)(Y, Z) = g(X, Y) u(Z) + u(Y) g(X, Z) + 2u(X) u(Y) u(Z). \]  
(2.21)

Using (2.21) in (2.20), we get
\[ g(Z, K(X, Y, U)) = u(Y) g(X, Z) - u(X) g(Y, Z). \]  
Which is equivalent to
\[ 'K(X, Y, U, Z) = u(Y) g(X, Z) - u(X) g(Y, Z) \]  
(2.22)

where
\[ 'K(X, Y, U, Z) \equiv g(K(X, Y, U), Z). \]  
(2.22)

which is equivalent to
\[ 'K(X, Y, Z, U) = u(X) g(Y, Z) - u(Y) g(X, Z) \]  
(2.23)

(2.23) is equivalent to (2.7).

Contracting \( X \) in (2.23), we get (2.8).

(2.8) implies
\[ g(RY, Z) = (n - 1) g(Y, Z) \]  
which is equivalent to (2.9).

Contracting \( Y \) in (2.9), we get (2.10).

Corollary (2.1): In \( V_n \), we have
\[ 'K(\bar{X}, \bar{Y}, Z, T) = 'K(X, Y, \bar{Z}, \bar{T}) \]  
and
\[ 'K(\bar{X}, \bar{Y}, Z, T) = 'K(X, Y, Z, T) + u(T)(u(X) g(Y, Z) - u(Y) g(X, Z)) + u(Z)(u(Y) g(X, T) - u(X) g(Y, Z)). \]  

The proof is obvious from (1.3), (1.9) and (2.7).

Corollary (2.2): \( V_n \) is conformally flat.

Proof: Using equation (2.7), (2.8), (2.9) and (2.10) in (1.13), we get
\[ Q(X, Y, Z) = 0 \]  
(2.24)

which proves the corollary.

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Corollary (2.3): In $V_n$, we have
\[ L(X,Y,Z) = \frac{n}{2-n}(g(Y,Z)X - g(X,Z)Y). \] (2.25)

**Proof:** using equations (2.7), (2.8), (2.9) in (1.14), we get (2.25).

Corollary (2.4): $V_n$ is concircularly flat.

**Proof:** Using equations (2.7) and (2.10) in (1.15), we get
\[ C(X,Y,Z) = 0 \] (2.26)
which proves the corollary.

Corollary (2.5): $V_n$ is projectively flat.

**Proof:** Using equations (2.7) and (2.8) in (1.16), we get
\[ P(X,Y,Z) = 0 \] (2.27)
which proves the corollary.

3. QUARTER-SYMMETRIC NON-METRIC CONNEXION IN $V_n$

We consider a quarter-symmetric non-metric connexion $E$ [5] defined by
\[ E_{\alpha}Y = D_{\alpha}Y + u(Y)\bar{X}. \] (3.1)

Let $R$ and $K$ be the curvature tensor with respect to $E$ and $D$ respectively. Then, it is easy to calculate that
\[ R(X,Y,Z) = K(X,Y,Z) + g(\bar{X},Z)\bar{Y} - g(\bar{Y},Z)\bar{X} + u(Z)[u(Y)X - u(X)Y] \] (3.2)
where
\[ R(X,Y,Z) = E_{\alpha}E_{\beta}Z - E_{\beta}E_{\alpha}Z - E_{[\alpha\beta]}Z \]
and
\[ K(X,Y,Z) = D_{\alpha}D_{\beta}Z - D_{\beta}D_{\alpha}Z - D_{[\alpha\beta]}Z. \]

Contracting $X$ in (3.2), we get
\[ \bar{R}(Y,Z) = Ric(Y,Z) + g(Y,Z) + nu(Y)u(Z) \] (3.3)
where
\[ \bar{R}(Y,Z) = C_{[\alpha\beta]}R(X,Y,Z) \]
and (3.3) implies
\[ Ric(Y,Z) = C_{[\alpha\beta]}K(X,Y,Z). \]

Contracting $Y$ in above, we get
\[ \bar{R}(Y) = R(Y) + Y + nu(Y)u. \] (3.5)

Further (3.6) shows that the scalar curvatures of $V_n$ with respect to $E$ and $D$ are equal.

Theorem (3.1): In $V_n$, we have
\[ (E_{X}F)Y = g(\bar{X},\bar{Y})U, \] (3.7)
\[ E_{\alpha}U = 0, \] (3.8)
\[ (E_{X}u)Y = g(\bar{X},Y), \] (3.9)
\[ 'R(X,Y,Z,T) = 'K(X,Y,Z,T) + g(\bar{X},Z)g(\bar{Y},T) - g(\bar{Y},Z)g(\bar{X},T) + u(Z)[u(Y)g(X,T) - u(X)g(Y,T)]. \] (3.10)

**Proof:** We known that
\[ (E_{X}F)Y = E_{X}\bar{Y} - \bar{E}_{X}\bar{Y}. \]

Using (1.3) and (3.1) in above, we get (3.7).
Replacing $Y$ by $U$ in (3.1), we get

$$E_X U = D_X U + u(U)\bar{X}.$$  

Using (1.6) and (1.7a) in above, we get (3.8).

We know that

$$(E_X u) Y = E_X(u(Y)) - u(E_X Y).$$

Using (1.8) and (3.1) in above, we get (3.9).

Operating $g$ on both the sides of (3.2) and using

$$'R(X, Y, Z, T) \equiv g(R(X, Y, Z), T)$$

and

$$'K(X, Y, Z, T) \equiv g(K(X, Y, Z), T)$$

we get (3.10).

**Theorem (3.2):** In $V_n$, we have

$$R(X, Y, Z) + R(Y, Z, X) + R(Z, X, Y) = 0$$  

(3.11)

**Proof:** Using equation (3.2) and Bianchi first identity with respect to Levi-Civita connexion $D$, we get the result.

**Theorem (3.3):** In $V_n$, the conformal curvature tensor $\tilde{Q}$ with respect to the quarter-symmetric non-metric connexion $E$ is given by

$$\tilde{Q}(X, Y, Z) = g(\bar{X}, Z)\bar{Y} - g(\bar{Y}, Z)\bar{X} - \frac{2}{n-2} u(Z)[u(Y)X - u(X)Y]$$

$$- \frac{2}{n-2}[g(Y, Z)X - g(X, Z)Y] - \frac{n}{n-2} g(Y, Z)u(X) - g(X, Z)u(Y) U.$$  

(3.12)

**Proof:** In view of (1.13) $\tilde{Q}$ in $V_n$ is given by

$$\tilde{Q}(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-2} [\text{Ric}(X, Z)Y - \text{Ric}(X, Z)Y + g(Y, Z)\tilde{R}X - g(X, Z)\tilde{R}Y]$$

$$+ \frac{2}{(n-1)(n-2)} g(Y, Z)X - g(X, Z)Y.$$  

(3.13)

which is equivalent to

$$'\tilde{Q}(X, Y, Z, T) = 'R(X, Y, Z, T) - \frac{1}{n-2} [\text{Ric}(X, Z)g(X, T) - \text{Ric}(X, Z)g(Y, T)]$$

$$+ g(Y, Z)\text{Ric}(X, T) - g(X, Z)\text{Ric}(Y, T) + \frac{2}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y]$$

(3.14)

where

$$'\tilde{Q}(X, Y, Z, T) = g'(\tilde{Q}(X, Y, Z), T).$$

Now using equation (1.13), (3.2), (3.3) and (3.6) in the above equation, we get

$$'\tilde{Q}(X, Y, Z, T) = 'Q(X, Y, Z, T) + g(\bar{X}, Z)\bar{Y} - g(\bar{Y}, Z)\bar{X} - \frac{2}{n-2} u(Z)[u(Y)g(X, T) - u(X)g(Y, T)]$$

$$- \frac{2}{n-2}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)] - \frac{n}{n-2} [g(Y, Z)u(X) - g(X, Z)u(Y)]u(T).$$

(3.15)

Using (2.24) in above, we get (3.12).

**Theorem (3.4):** The conharmonic curvature tensors $\tilde{L}$ with respect to quarter-symmetric non-metric connexion $E$ in $V_n$ is given by

$$\tilde{L}(X, Y, Z) = g(\bar{X}, Z)\bar{Y} - g(\bar{Y}, Z)\bar{X} - \frac{2}{n-2} u(Z)[u(Y)X - u(X)Y]$$

$$- \frac{n+2}{n-2} [g(Y, Z)X - g(X, Z)Y] - \frac{n}{n-2} g(Y, Z)u(X) - g(X, Z)u(Y) U.$$  

(3.16)

**Proof:** In view of (1.14), we have

$$\tilde{L}(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-2} [\text{Ric}(X, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)\tilde{R}X - g(X, Z)\tilde{R}Y]$$

which implies

$$'\tilde{L}(X, Y, Z, T) = 'R(X, Y, Z, T) - \frac{1}{n-2} [\text{Ric}(X, Z)g(X, T) - \text{Ric}(X, Z)g(Y, T)]$$

$$+ g(Y, Z)\text{Ric}(X, T) - g(X, Z)\text{Ric}(Y, T)]$$

(3.17)

where

$$'\tilde{L}(X, Y, Z, T) \equiv g'(\tilde{L}(X, Y, Z), T).$$

Now using equation (1.14), (3.2) and (3.3) in (3.17), we get

$$'\tilde{L}(X, Y, Z, T) = 'L(X, Y, Z, T) + g(\bar{X}, Z)\bar{Y} - g(\bar{Y}, Z)\bar{X} - \frac{2}{n-2} u(Z)[u(Y)g(X, T) - u(X)g(Y, T)]$$

$$- \frac{2}{n-2}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)] - \frac{n}{n-2} [g(Y, Z)u(X) - g(X, Z)u(Y)]u(T).$$
Using (2.25) in above, we get (3.15).

**Theorem (3.5):** The concircular curvature tensor \( \tilde{\mathcal{C}} \) with respect to the quarter-symmetric non-metric connexion \( E \) is given by
\[
\tilde{\mathcal{C}}(X, Y, Z) = g(\tilde{X}, Z)\tilde{Y} - g(\tilde{Y}, Z)\tilde{X} + u(Z)[u(Y)X - u(X)Y].
\] (3.18)

**Proof:** In view of (1.15), \( \tilde{\mathcal{C}} \) is given by
\[
\tilde{\mathcal{C}}(X, Y, Z) = R(X, Y, Z) - \frac{\varphi}{n(n-1)}[g(Y, Z)X - g(X, Z)Y].
\] (3.19)

Which is equivalent to
\[
'\tilde{\mathcal{C}}(X, Y, Z, T) = 'R(X, Y, Z, T) - \frac{\varphi}{n(n-1)}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)]
\] (3.20)

where
\[
'\tilde{\mathcal{C}}(X, Y, Z, T) = g(\tilde{\mathcal{C}}(X, Y, Z), T).
\]

Now using equation (1.15), (2.26), (3.2) and (3.6) in the above equation, we get (3.18).

**Theorem (3.6):** The projective curvature tensor \( \bar{P} \) in \( V_n \) with respect to quarter-symmetric non-metric connexion \( E \) is given by
\[
\bar{P}(X, Y, Z) = g(\bar{X}, Z)\bar{Y} - g(\bar{Y}, Z)\bar{X} - \frac{1}{n-1}[g(Y, Z)X - g(X, Z)Y] - \frac{1}{n-1}[u(Y)X - u(X)Y]u(Z).
\] (3.21)

**Proof:** In view of (1.16), the projective curvature tensor in \( V_n \) with respect to \( E \) is given by
\[
\bar{P}(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}[\bar{Ric}(Y, Z)X - \bar{Ric}(X, Z)Y]
\]

which is equivalent to
\[
'\bar{P}(X, Y, Z, T) = 'R(X, Y, Z, T) - \frac{1}{n-1}[\bar{Ric}(Y, Z)g(X, T) - \bar{Ric}(X, Z)g(Y, T)]
\] (3.22)

where
\[
'\bar{P}(X, Y, Z, T) = g(\bar{P}(X, Y, Z), T).
\]

Using equations (1.16), (3.2), (3.3) in (3.22), we get
\[
'\bar{P}(X, Y, Z, T) = 'P(X, Y, Z, T) + g(\bar{X}, Z)g(\bar{Y}, T) - g(\bar{Y}, Z)g(\bar{X}, T)
\]
\[
- \frac{1}{n-1}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)] - \frac{1}{n-1}[u(Y)g(X, T) - u(X)g(Y, T)]u(Z).
\] (3.23)

Using (2.27) in above, we get (3.21).

**Acknowledgement:** The author is thankful to the Head, Department of Mathematics for his good wishes.

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