

**AN EXTENSION OF BANACH CONTRACTION PRINCIPLE  
THROUGH RATIONAL EXPRESSION**

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**ABSTRACT**

**I**n This Paper we prove An Extension of Banach contraction principle through rational expression satisfying Three continuous mappings. Some result with S. banach (1922). Our Result include the well known result of R.Kannan(1968), L.B.Ciric (1971), B.Fisher (1971), K.Isoki, S.S.Rajput and P.L.sharma(1982) and include the result of An Extension of Banach contraction principle (1988) as special case with a different and constructive method.

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**MAIN RESULT**

**Theorem 1.1:** Let E, F and T are three continuous mappings of a Banach space satisfying the following conditions:

$$ET = TE, FT = TF, E(X) \subset T(X) \text{ and } F(X) \subset T(X). \quad (1.1)$$

$$\|(Ex - Fy)\| \leq \alpha \frac{\|Ty - Fy\| [1 + \|Tx - Ex\|]}{1 + \|Tx - Ty\|} + \beta [\|Tx - Ex\| + \|Ty - Fy\|] + \gamma \left[ \frac{\|Tx - Fy\|}{1 + \|Ty - Ex\|} + \delta \|Tx - Ty\| \right] \quad (1.2)$$

For all x, y in X where  $\alpha, \beta, \gamma, \delta \geq 0, \alpha + 2\beta + 2\gamma + \delta < 1$ . then E, F and T have a common fixed point in X.

**Proof:** Let  $x_0$  be an arbitrary element of X and let  $\{Tx_n\}$  be defined as

$$Tx_{2n+1} = Ex_{2n}, Tx_{2n+2} = Fx_{2n+1} \text{ for } n = 1, 2, 3, 4, \dots \quad (1.3)$$

We can do this since  $E(X) \subset T(X)$  and  $F(X) \subset T(X)$ .

From (1.2) we have

$$\|Tx_{2n+1} - Tx_{2n+2}\| = \|Ex_{2n} - Fx_{2n+1}\|$$

$$\begin{aligned} &\leq \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}\| [1 + \|Tx_{2n} - Tx_{2n+1}\|]}{1 + \|Tx_{2n} - Tx_{2n+1}\|} + \beta [\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] \\ &\quad + \gamma [\|Tx_{2n} - Tx_{2n+2}\| + \|Tx_{2n+1} - Tx_{2n+1}\|] + \delta \|Tx_{2n} - Tx_{2n+1}\| \\ &\leq \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}\| [1 + \|Tx_{2n} - Tx_{2n+1}\|]}{1 + \|Tx_{2n} - Tx_{2n+1}\|} + \beta [\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] \\ &\quad + \gamma [\|Tx_{2n} - Tx_{2n+2}\| + 0] + \delta \|Tx_{2n} - Tx_{2n+1}\| \\ &\leq \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}\| [1 + \|Tx_{2n} - Tx_{2n+1}\|]}{1 + \|Tx_{2n} - Tx_{2n+1}\|} + \beta [\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] \\ &\quad + \gamma [\|Tx_{2n} - Tx_{2n+2}\| + \|Tx_{2n+1} - Tx_{2n+1}\|] + \delta \|Tx_{2n} - Tx_{2n+1}\| \\ &\leq \alpha \frac{\|Tx_{2n+1} - Tx_{2n+2}\| [1 + \|Tx_{2n} - Tx_{2n+1}\|]}{1 + \|Tx_{2n} - Tx_{2n+1}\|} + \beta [\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] \\ &\quad + \gamma [\|Tx_{2n} - Tx_{2n+1}\| + \|Tx_{2n+1} - Tx_{2n+2}\|] + \delta \|Tx_{2n} - Tx_{2n+1}\| \end{aligned}$$

[using triangle inequality]

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$$\|Tx_{2n+1} - Tx_{2n+2}\| \leq \alpha \|Tx_{2n+1} - Tx_{2n+2}\| + \beta \|Tx_{2n} - Tx_{2n+1}\| + \beta \|Tx_{2n+1} - Tx_{2n+2}\| + \gamma \|Tx_{2n} - Tx_{2n+1}\| + \gamma \|Tx_{2n+1} - Tx_{2n+2}\| + \delta \|Tx_{2n} - Tx_{2n+1}\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}\|(1 - \alpha - \beta - \gamma) \leq (\beta + \gamma + \delta)\|Tx_{2n} - Tx_{2n+1}\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}\| \leq \frac{(\beta + \gamma + \delta)}{(1 - \alpha - \beta - \gamma)} \|Tx_{2n} - Tx_{2n+1}\|$$

$$\|Tx_{2n+1} - Tx_{2n+2}\| \leq h \|Tx_{2n} - Tx_{2n+1}\|$$

$$\text{Where } h = \frac{(\beta + \gamma + \delta)}{(1 - \alpha - \beta - \gamma)} < 1$$

Similarly we can see

$$\|Tx_{2n} - Tx_{2n+1}\| \leq h \|Tx_{2n-1} - Tx_{2n}\|$$

Proceeding in this way, we have

$$\begin{aligned} \|Tx_{2n+1} - Tx_{2n+2}\| &\leq h \|Tx_{2n} - Tx_{2n+1}\| \\ &\leq h^2 \|Tx_{2n-1} - Tx_{2n}\| \\ &\dots \\ &\leq h^{2n+1} \|Tx_0 - Tx_1\| \end{aligned}$$

By routine calculation the following inequalities hold for  $k > n$

$$\begin{aligned} \|Tx_n - Tx_{n+k}\| &\leq \sum_{i=1}^k \|Tx_{n+i-1} - Tx_{n+i}\| \\ &\leq \sum_{i=1}^k h^{n+1-i} \|Tx_0 - Tx_1\| \\ &\leq \frac{h^n}{1-h} \|Tx_0 - Tx_1\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence  $\{Tx_n\}$  is a Cauchy sequence by the completeness of  $X$ .  $\{Tx_n\}$  converges to a point  $u$  in  $X$ . It follows from (1.3) that  $\{Ex_{2n}\}$  and  $\{Fx_{2n+1}\}$  also converges to  $u$ . since  $E, F$  and  $T$  are continuous we have

$$E(Tx_{2n}) \rightarrow Eu, F(Tx_{2n+1}) \rightarrow Fu \quad (1.4)$$

From (1.1)  $t$  commutes with  $E$  and  $F$  therefore

$$E(Tx_{2n}) = T(Ex_{2n}), F(Tx_{2n+1}) = T(Fx_{2n+1}) \text{ for all } n = 0, 1, 2, 3, \dots$$

Taking  $n \rightarrow \infty$  we have

$$Eu = Tu = Fu \quad (1.5)$$

and

$$T(Tu = T(Eu)) = E(Tu) = E(Fu) = F(Eu) = T(Fu) = F(Tu) = F(Eu) = F(Fu). \quad (1.6)$$

By (1.2), (1.5) and (1.6). if  $Eu \neq F(Eu)$  we have

$$\begin{aligned} \|Eu - F(Eu)\| &\leq \alpha \frac{\|T(Eu) - F(Eu)\| [1 + \|Tu - Eu\|]}{[1 + \|Tu - T(Eu)\|]} + \beta [\|Tu - Eu\| + \|T(Eu) - F(Eu)\|] \\ &\quad + \gamma [\|Tu - F(Eu)\| + \|T(Eu) - Eu\|] + \delta \|Tu - T(Eu)\| \\ &\leq (2\gamma + \delta) \|Eu - F(Eu)\| \\ &< \|Eu - F(Eu)\| \quad [:(2\gamma + \delta) < 1] \end{aligned} \quad (1.7)$$

Leading to a contradiction. Hence

$Eu = F(Eu)$ . Using (1.6) and (1.7) we get

$$Eu = F(Eu) = T(Eu) = E(Eu)$$

Which shows that  $Eu$  is the common fixed of  $E, F$  and  $T$ .

Let  $z$  and  $w$  ( $z \neq w$ ) be two points in  $X$  such that

$Ez = Fz = Tz = z$  and  $Ew = Fw = Tw = w$ . Then by (1.2) we have

$$\begin{aligned}
 \|z - w\| &= \|Ez - Fw\| \\
 &\leq \alpha \frac{\|Tw - Fw\| [1 + \|Tz - Ez\|]}{\|1 + \|Tz - Fw\|\|} + \beta [\|Tz - Ez\| + \|Tw - Fw\|] + \gamma [\|Tz - Fw\| + \|Tw - Ez\|] + \delta \|Tz - Tw\| \\
 &\leq \alpha \cdot 0 + \beta \cdot 0 + \gamma [\|Tz - Fw\| + \|Tw - Ez\|] + \delta \|Tz - Tw\| \\
 &\leq (2\gamma + \delta) \|Ez - Fw\| \\
 &< \|Ez - Fw\| \quad [ \because (2\gamma + \delta) < 1 ]
 \end{aligned}$$

Leading to a contraction. Hence  $z = w$ . This implies the uniqueness of common fixed point for E, F and T. This completes the proof of the theorem.

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