ON \((m, k)\)-REGULAR FUZZY GRAPHS

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ABSTRACT

In this paper, \((m, k)\) - regular fuzzy graph and totally \((m, k)\)-regular fuzzy graph are introduced and compared through various examples. Also, comparative study between \((m, k)\)-regularity and totally \((m, k)\)-regularity is done. A necessary and sufficient condition under which they are equivalent is provided. \((m, k)\)-regularity on some fuzzy graphs whose underlying crisp graphs are a path and a cycle is studied with some specific membership functions.

Key Words: degree of a vertex in fuzzy graph, regular fuzzy graph, total degree, \(d_m\)-degree of a vertex in fuzzy graph, total \(d_m\)-degree of a vertex in fuzzy graph, \(m\)-neighborly irregular fuzzy graphs.

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1. INTRODUCTION


N. R. Santhi Maheswari and C.Sekar introduced \(d_2\)-degree, total \(d_2\)-degree of a vertex in fuzzy graph, \((2, k)\)-regular fuzzy graph and totally \((2, k)\)-regular fuzzy graphs [7]. (r, 2)-regular fuzzy graph and totally \((r, 2)\)-regular fuzzy graphs [8]. Also they introduced \(d_m\)-degree, total \(d_m\)-degree of a vertex in fuzzy graph, \(m\)-Neighbourly Irregular fuzzy graph and totally \(m\)-Neighbourly totally irregular fuzzy graphs [9]. This is the background to define \((m, k)\)-regular fuzzy graph and totally \((m, k)\)-regular fuzzy graph [10]. Here, \((m, k)\)-regularity on some fuzzy graphs whose underlying crisp graphs are a path and a cycle is studied with some specific membership functions.

2. PRELIMINARIES

We present some known definitions and results for ready reference to go through the work presented in this paper.

**Definition 2.1:** A Fuzzy graph denoted by \(G: (\sigma, \mu)\) on graph \(G^*\) \((V, E)\) is a pair of functions \((\sigma, \mu)\) where \(\sigma: V \rightarrow [0, 1]\) is a fuzzy subset of a non empty set \(V\) and \(\mu: V \times V \rightarrow [0, 1]\) is a symmetric fuzzy relation on \(\sigma\), which satisfies the following condition: for all \(u, v \in V\) the relation \(\mu(u, v) = \mu(uv) = \sigma(u) \wedge \sigma(v)\) is satisfied, where \(\sigma\) and \(\mu\) are called membership function. A fuzzy graph \(G: (\sigma, \mu)\) is complete if \(\mu(u, v) = \mu(uv) = \sigma(u) \wedge \sigma(v)\) for all \(u, v \in V\), where \(uv\) denotes the edge between \(u\) and \(v\). The graph \(G^* : (V, E)\) is called the underlying crisp graph of the fuzzy graph \(G: (\sigma, \mu)\) [2].

**Definition 2.2:** The strength of connectedness between two vertices \(u\) and \(v\) is \(\mu^{\infty}(u, v) = \sup\{ \mu^k(u, v) : k = 1, 2, \ldots \}\) where \(\gamma^k(u, v) = \sup\{ \mu^k(uu_1) \wedge \mu(u_1u_2) \wedge \ldots \wedge \mu(u_{k-1}v) : u, u_1, u_2, \ldots, u_{k-1}, v\}\) is a path connecting \(u\) and \(v\) of length \(k\) [3].
Definition 2.3: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. The degree of a vertex $u$ is $d_G(u) = \sum \mu(uv)$ for $uv \in E$ and $\mu(uv) = 0$, for $uv$ not in $E$, this is equivalent to $d_G(u) = \sum \mu(uv)[4].$

Definition 2.4: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. If $d(v) = k$ for all $v \in V$, then $G$ is said to be an regular fuzzy graph of degree $k[4].$

Definition 2.5: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. The total degree of a vertex $u$ is defined as $td(u) = \sum \mu(u, v) + \sigma(u) = d(u) + \sigma(u), uv \in E$. If each vertex of $G$ has the same total degree $k$, then $G$ is said to be a totally regular fuzzy graph of degree $k$ or $k$-totally regular fuzzy graph [4].

Definition 2.6: Let $G : (\sigma, \mu)$ be a fuzzy graph. The $d_2$-degree of a vertex $u$ in $G$ is $d_2(u) = \sum \mu_2(u, v)$, where $\mu_2(uv) = sup \{ \mu(uu_1, \mu(u_1u_2), \ldots, \mu(u_{m-1}v)) : u, u_1, \ldots, u_{m-1}, v \}$ is the shortest path connecting $u$ and $v$ of length $2$. Also, $\mu(uv) = 0$, for $uv$ not in $E$.

The minimum $d_2$-degree of $G$ is $\delta_2(G) = \lambda/\lambda(d_2(v) : v \in V).$

The maximum $d_2$-degree of $G$ is $\Delta_2(G) = \nu(d_2(v) : v \in V)[7].$

Definition 2.7: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. If $d_2(v) = k$ for all $v \in V$, then $G$ is said to be a $(2, k)$-regular fuzzy graph [7].

Definition 2.8: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. If $d_2(v) = r$ and $d_2(v) = k$ for all $v \in V$, then $G$ is said to be an $(r, 2, k)$-regular fuzzy graph[8].

Definition 2.9: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. The total $d_2$-degree of a vertex $u \in V$ is defined as $td_2(u) = \sum \mu_2(u, v) + \sigma(u) = d_2(u) + \sigma(u)$.

The minimum $td_2$-degree of $G$ is $\delta_2(G) = \lambda/\lambda(td_2(v) : v \in V).$

The maximum $td_2$-degree of $G$ is $\Delta_2(G) = \nu(td_2(v) : v \in V)[7].$

Definition 2.10: If each vertex of a fuzzy graph $G$ has the same total $d_2$-degree $k$, then $G$ is said to be a totally $(2, k)$-regular fuzzy graph[7].

Definition 2.11: If each vertex of a fuzzy graph $G$ has the same degree $r$ and same total $d_2$-degree $k$, then $G$ is said to be a totally $(r, 2, k)$-regular fuzzy graph [8].

Definition 2.12: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. The $d_m$-degree of a vertex $u$ in $G$ is $d_m(u) = \sum \mu_m(uv)$, where $\mu_m(uv) = sup \{ \mu(uu_1, A \mu(u_1u_2), A \ldots A \mu(u_{m-1}v)) : u, u_1, u_2, \ldots, u_{m-1}, v \}$ is the shortest path connecting $u$ and $v$ of length $m$. Also, $\mu(uv) = 0$, for $uv$ not in $E$.

The minimum $d_m$-degree of $G$ is $\delta_m(G) = \lambda/\lambda(d_m(v) : v \in V)\}.$

The maximum $d_m$-degree of $G$ is $\Delta_m(G) = \nu(d_m(v) : v \in V)[9].$

Definition 2.13: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. The total $d_m$-degree of a vertex $u \in V$ is defined as $td_m(u) = \sum \mu_m(uv) + \sigma(u) = d_m(u) + \sigma(u)$.

The minimum $td_m$-degree of $G$ is $\delta_m(G) = \lambda/\lambda(td_m(v) : v \in V)\}.$

The maximum $td_m$-degree of $G$ is $\Delta_m(G) = \nu(td_m(v) : v \in V)[9].$

3. $(m, k)$-REGULAR FUZZY GRAPHS

In this section, we define $(m, k)$-Regular Fuzzy Graphs and illustrates this with $(3, k)$-regular graph.

Definition 3.1: Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^r : (V,E)$. If $d_m(v) = k$ for all $v \in V$, then $G$ is said to be an $(m, k)$-regular fuzzy graph.
**Example 3.2:** Consider $G^*$: $(V,E)$ where $V = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $E = \{u_1u_2, u_3u_4, u_5u_6, u_6u_7, u_7u_1\}$. Define $G: (\sigma, \mu)$ by $\sigma(u_1) = 0.5$, $\sigma(u_2) = 0.4$, $\sigma(u_3) = 0.4$, $\sigma(u_4) = 0.5$, $\sigma(u_5) = 0.4$, $\sigma(u_6) = 0.4$, $\sigma(u_7) = 0.5$ and $\mu(u_1u_2) = 0.2$, $\mu(u_2u_3) = 0.3$, $\mu(u_3u_4) = 0.2$, $\mu(u_4u_5) = 0.3$, $\mu(u_5u_6) = 0.2$, $\mu(u_6u_7) = 0.2$, $\mu(u_7u_1) = 0.2$.

$d_3(u_1) = (0.2 \times 0.3 \times 0.2) + (0.2 \times 0.3 \times 0.2) = 0.2 + 0.2 = 0.4$

$d_3(u_2) = (0.2 \times 0.3 \times 0.2) + (0.2 \times 0.3 \times 0.3) = 0.2 + 0.2 = 0.4$

$d_3(u_3) = (0.3 \times 0.2 \times 0.2) + (0.2 \times 0.3 \times 0.2) = 0.2 + 0.2 = 0.4$

$d_3(u_4) = (0.2 \times 0.3 \times 0.2) + (0.3 \times 0.2 \times 0.2) = 0.2 + 0.2 = 0.4$

$d_3(u_5) = (0.3 \times 0.2 \times 0.3) + (0.2 \times 0.3 \times 0.2) = 0.2 + 0.2 = 0.4$

$d_3(u_6) = (0.2 \times 0.3 \times 0.2) + (0.3 \times 0.2 \times 0.3) = 0.2 + 0.2 = 0.4$

$d_3(u_7) = (0.2 \times 0.2 \times 0.3) + (0.2 \times 0.3 \times 0.2) = 0.2 + 0.2 = 0.4$

It is noted that $d_3(u_1) = 0.4$, $d_3(u_2) = 0.4$, $d_3(u_3) = 0.4$, $d_3(u_4) = 0.4$, $d_3(u_5) = 0.4$, $d_3(u_6) = 0.4$, $d_3(u_7) = 0.4$. Each vertex has the same $d_3$-degree 0.4. Hence $G$ is a $(3, 0.4)$-regular fuzzy graph.

**Example 3.3:** Consider $G^*$: $(V,E)$, where $V = \{u, v, w, x, y, z\}$ and $E = \{uv, vw, wx, xy, yz, zu\}$

**4. TOTALLY $(m, k)$-REGULAR FUZZY GRAPHS**

In this section, we define totally $(m, k)$-regular fuzzy graphs and the necessary and sufficient condition under which they are equivalent is provided.

**Definition 4.1:** If each vertex of $G$ has the same total $d_m$ - degree $k$, then $G$ is said to be totally $(m, k)$-regular fuzzy graph.

**Example 4.2:** From Example 3.2, we have

$d_3(u_1) = d_3(u_2) = 0.2 + 0.2 + 0.2 = 0.6$

$d_3(u_3) = d_3(u_4) = 0.2 + 0.2 + 0.2 = 0.6$

$d_3(u_5) = d_3(u_6) = 0.2 + 0.2 + 0.2 = 0.6$

$d_3(u_7) = d_3(u_8) = 0.2 + 0.2 + 0.2 = 0.6$

It is noted that each vertex has the same total $d_3$-degree 0.6.

Hence $G$ is a totally $(3, 0.6)$-regular fuzzy graph. This fuzzy graph is a $(3, 0.2)$-regular fuzzy graph which is a totally $(3, 0.6)$-regular fuzzy graph.

**Remark 3.3:** The $(3, 0.4)$-regular fuzzy graph given in example 3.2 is not totally $(3, k)$-regular. Since $td_3(u_1) = 0.9$, $td_3(u_2) = 0.8$, $td_3(u_3) = 0.8$, $td_3(u_4) = 0.9$, $td_3(u_5) = 0.8$, $td_3(u_6) = 0.8$, $td_3(u_7) = 0.9$. It is noted that $td_3(u_1) = td_3(u_2) = td_3(u_3) = td_3(u_4) = td_3(u_5) = td_3(u_6) = td_3(u_7) = 0.9$. %
Proof: Suppose that \( \sigma(u) = c \), for all \( u \in V \). Assume that \( G : (\sigma, \mu) \) is \((m, k)\)-regular fuzzy graph.

Then \( d_m(u) = k \), for all \( u \in V \). So, \( td_m(u) = d_m(u) + \sigma(u) \), for all \( u \in V \).

\[ \Rightarrow td_m(u) = k + c, \text{ for all } u \in V. \]

Hence \( G : (\sigma, \mu) \) is totally \((m, k + c)\)-regular fuzzy graph.

Thus (1) \( \Rightarrow \) (2) is proved.

Suppose \( G : (\sigma, \mu) \) is totally \((m, k + c)\)-regular fuzzy graph.

\[ \Rightarrow td_m(u) = k + c, \text{ for all } u \in V \]

\[ \Rightarrow d_m(u) + \sigma(u) = k + c, \text{ for all } u \in V \]

\[ \Rightarrow d_m(u) + c = k + c, \text{ for all } u \in V \]

\[ \Rightarrow d_m(u) = k, \text{ for all } u \in V \]

Hence \( G : (\sigma, \mu) \) is \((m, k)\)-regular fuzzy graphs. Hence (1) and (2) are equivalent.

Conversely assume that (1) and (2) are equivalent. Let \( G : (\sigma, \mu) \) is a totally \((m, k+)\)-regular fuzzy graph and an \((m, k)\)-regular fuzzy graph.

\[ \Rightarrow td_m(u) = k + c \text{ and } d_m(u) = k, \text{ for all } u \in V \]

\[ \Rightarrow d_m(u) + \sigma(u) = k + c \text{ and } d_m(u) = k, \text{ for all } u \in V \]

\[ \Rightarrow d_m(u) + \sigma(u) = k + c \text{ and } d_m(u) = k, \text{ for all } u \in V \Rightarrow \sigma(u) = c, \text{ for all } u \in V. \]

Hence \( \sigma(u) = c, \text{ for all } u \in V. \)

Remark 4.5: Any graph with diameter less than \( m \) is \((m, 0)\) - regular fuzzy graph.

5. \((m, k)\)-REGULARITY ON A PATH ON 2\( m \) VERTICES WITH SOME SPECIFIC MEMBERSHIP FUNCTIONS

In this section, \((m, k)\)-regularity on a path is studied with some specific membership functions.

Theorem 5.1: Let \( G : (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \), a path on \( 2m \) vertices. \( G : (\sigma, \mu) \) is \((m, k)\)-regular fuzzy graph if \( \mu(uv) = k \), for all \( uv \in E \).

Proof: Suppose that \( \mu \) is constant function say \( \mu(uv) = k \), for all \( uv \in E \), then \( d_m(v) = k \), for all \( v \in V \). Hence \( G \) is an \((m, k)\)-regular fuzzy graph.

Remark 5.2: Converse of the above theorem need not be true.

Theorem 5.3: Let \( G : (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \), a path on \( 2m \) vertices. If the alternate edges have the same membership values, then \( G \) is an \((m, k)\)-regular fuzzy graph, where \( k = \min \{c_1, c_2\} \).

Proof: Let \( G : (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \), a path on \( 2m \) vertices. Let \( e_1, e_2, e_3, \ldots, e_{2m-1} \) be the edges of the path \( G^* \) in that order. If the alternate edges have the same membership values, then

\[
\mu(e_i) = \begin{cases} 
  c_1 & \text{if } i \text{ is odd} \\
  c_2 & \text{if } i \text{ is even}
\end{cases}
\]

For, \( i = 1, 2, 3, 4, 5, \ldots, m \),

\[ d_m(v_i) = \{\mu(e_1) + A \mu(e_{2i-1}) A \ldots \mu(e_{2i-1}) A \mu(e_{2i-1}) + A \mu(e_{2i}) A \mu(e_{2i+1}) A \mu(e_{2i+1}) + A \mu(e_{2i+1}) + A \mu(e_{2i+2}) + A \mu(e_{2i+2}) \}
\]

\[ = \{c_1 A c_2 A \ldots c_2 A c_1\}
\]

\[ = k, \text{ where } k = \min \{c_1, c_2\} \]
For, \( i = 1, 2, 3, 4, 5, \ldots, m \),
\[
d_{an}(v_i) = \{ \mu(e_i) A \mu(e_{i+1}) A \ldots A \mu(e_{m-1}) A \mu(e_m) \}
\]
\[
= \{ c_1 A c_2 A \ldots A c_m \}
\]
\[
= k, \text{ where } k = \min \{ c_1, c_2 \}
\]

Hence \( G \) is an \((m, k)\)-regular fuzzy graph.

**Theorem 5.4:** Let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^*: (V, E) \), a path on \( 2m \) vertices. If middle edge has membership value less than the membership values of the remaining edges, then \( G \) is an \((m, k)\)-regular fuzzy graph, where \( k \) = membership value of the middle edge.

**Proof:** Let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^*: (V, E) \), a path on \( 2m \) vertices. Let \( e_1, e_2, e_3, \ldots, e_{2m-1} \) be the edges of the path \( G^*: (V, E) \) in that order. Let the membership values of the edges \( e_1, e_2, e_3, \ldots, e_{2m-1} \) be \( c_1, c_2, c_3, \ldots, c_{m-1}, c_m, c_{m+1}, \ldots, c_{2m-1} \) such that \( c_m = k \leq c_1, c_2, c_3, \ldots, c_{2m-1} \).

For, \( i = 1, 2, 3, 4, 5, \ldots, m \),
\[
d_{an}(v_i) = \{ \mu(e_i) A \mu(e_{i+1}) A \ldots A \mu(e_{m-1}) A \mu(e_m) \}
\]
\[
= \{ c_i A c_{i+1} A \ldots A c_{m-1} A c_m \}
\]
\[
= c_m = k.
\]

For, \( i = 1, 2, 3, 4, 5, \ldots, m \),
\[
d_{an}(v_i) = \{ \mu(e_i) A \mu(e_{i+1}) A \ldots A \mu(e_{m-1}) A \mu(e_m) \}
\]
\[
= \{ c_i A c_{i+1} A \ldots A c_{m-1} A c_m \}
\]
\[
= c_m = k.
\]

Hence \( G \) is an \((m, k)\)-regular fuzzy graph.

**Remark 5.5:** If \( \sigma \) is not constant function, then the \((m, k)\)-regular fuzzy graph in the above theorems 6.1, 6.3 and 6.4 are not totally \((m, k)\)-regular fuzzy graphs.

## 6. \((m, k)\)-Regularity on a Cycle with Some Specific Membership Functions

In this section, \((m, k)\)-regularity on a cycle is studied with some specific membership functions.

**Theorem 6.1:** For any \( m \geq 2 \), let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^*: (V, E) \), a cycle of length \( \geq 2m + 1 \). If \( \mu \) is a constant function, then \( G \) is an \((m, k)\)-regular fuzzy graph, where \( k = \mu(uv) \).

**Theorem 6.2:** For any \( m \geq 1 \), let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^*: (V, E) \), a cycle of length \( \geq 2m + 1 \). If \( \mu \) is a constant function, then \( G \) is an \((m, k)\)-regular fuzzy graph, where \( k = 2\mu(uv) \).

**Theorem 6.3:** For any \( m \geq 1 \), let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^*: (V, E) \), an even cycle of length \( \geq 2m + 2 \). If the alternate edges have the same membership values, then \( G \) is an \((m, k)\)-regular fuzzy graph.

**Proof:** If the alternate edges have the same membership values, then
\[
\mu(e_i) = \begin{cases} 
  c_1 & \text{if } i \text{ is odd} \\
  c_2 & \text{if } i \text{ is even}
\end{cases}
\]

If \( c_1 = c_2 \), then \( \mu \) is a constant function. So, \( G \) is an \((m, 2c_1)\)-regular fuzzy graph. If \( c_1 < c_2 \), then
\[
d_{an}(v) = \min \{ c_1, c_2 \} + \min \{ c_1, c_2 \} = c_1 + c_1 = 2c_1, \text{ for all } v \in V.
\]

Hence \( G \) is an \((m, 2c_1)\)-regular fuzzy graph.

If \( c_1 > c_2 \), \( d_{an}(v) = \min \{ c_1, c_2 \} + \min \{ c_1, c_2 \} = c_2 + c_2 = 2c_2, \text{ for all } v \in V.
\]

Hence \( G \) is an \((m, 2c_2)\)-regular fuzzy graph.

**Remark 6.4:** Even if the alternate edges of a fuzzy graph whose underlying graph is an even cycle of length \( \geq 2m + 2 \) have the same membership values, then \( G \) need not be totally \((m, k)\)-regular fuzzy graph, since if \( \sigma \) is not constant function then \( G \) is not totally \((m, k)\)-regular fuzzy graph, for any \( m \geq 1 \).
Theorem 6.5: For any \( m > 1 \), let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \), a cycle of length \( \geq 2m + 1 \).

Let \( \mu(e_i) = \begin{cases} c_1 & \text{if } i \text{ is odd} \\ c_2 \geq c_1 & \text{if } i \text{ is even} \end{cases} \)

then \( G \) is an \((m, k)\)-regular fuzzy graph.

**Proof:** Let \( \mu(e_i) = \begin{cases} c_1 & \text{if } i \text{ is odd} \\ c_2 \geq c_1 & \text{if } i \text{ is even} \end{cases} \)

**Case-1:** Let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \) an even cycle of length \( \geq 2m + 2 \). Then by theorem 6.3, \( G \) is an \((m, 2c_1)\)-regular fuzzy graph.

**Case-2:** Let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \) an odd cycle of length \( \geq 2m + 1 \).

Even if \( \mu(e_i) = \begin{cases} c_1 & \text{if } i \text{ is odd} \\ c_2 \geq c_1 & \text{if } i \text{ is even} \end{cases} \)

then \( G \) need not be totally \((m, k)\)-regular fuzzy graph, since if \( \sigma \) is not constant function then \( G \) is not totally \((m, k)\)-regular fuzzy graph.

Theorem 6.6: Let \( G: (\sigma, \mu) \) be a fuzzy graph on \( G^* : (V, E) \), a cycle of length \( \geq 2m + 1 \).

Even if \( \mu(e_i) = \begin{cases} c_1 & \text{if } i \text{ is odd} \\ c_2 \geq c_1 & \text{if } i \text{ is even} \end{cases} \)

then \( G \) need not be totally \((m, k)\)-regular fuzzy graph, since if \( \sigma \) is not constant function then \( G \) is not totally \((m, k)\)-regular fuzzy graph.

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