THE EDGE VERSION OF INVERSE SUM INDEG INDEX OF CONNECTED GRAPH

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ABSTRACT

The inverse sum indeg index is defined as \( I_{SSI}(G) = \sum_{uv \in E(G)} \frac{1}{d_u + d_v} = \sum_{uv \in E(G)} \frac{d_u \times d_v}{d_u + d_v} \) in which degree of vertex \( v \) is denoted by \( d_v \). Now we define a new version of ISI index as \( I_{SSI_e}(G) = \sum_{ef \in E(L(G))} \frac{d_e \times d_f}{d_e + d_f} \), where \( d_e \) denotes the degree of an edge \( e \) in \( G \). The goal of this paper is to the study of the \( I_{SSI_e} \) index of graphs.

Key words: sum indeg index, Edge version, circumcronone series, Benzenoid.

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1. INTRODUCTION

A graph is a collection of points and lines connecting them. The points and lines of a graph are also called vertices and edges respectively. If \( e \) is an edge of \( G \), connecting the vertices \( u \) and \( v \), then we write \( e = uv \) and say "\( u \) and \( v \) are adjacent". A connected graph is a graph such that there is a path between all pairs of vertices. The distance \( d(u,v) \) between two vertices \( u \) and \( v \) is the length of the shortest path between \( u \) and \( v \) in \( G \). A simple graph is an unweighted, undirected graph without loops or multiple edges. A single number that can be used to characterize some property of the graph is called a Topological Index for that graph. Obviously, the number of vertices and the number of edges are topological indices. The Wiener index is the first graph invariant reported (distance based) topological index and is defined as a half sum of the distances between all the pairs of vertices in a graph [1]. Also, the edge version of Wiener index which were based on distance between edges introduced by A. Iranmanesh \( et \) al. in 2008 [2]. These topological indices are formulated as follows:

\[ W_v(G) = \sum_{[u,v] \in V(G)} d(u,v) \]  \hspace{1cm} (1)

\[ W_e(G) = \sum_{[e,f] \in E(G)} d(e,f) \]  \hspace{1cm} (2)

in which degree of vertex \( v \) and edge \( e \) are denoted by \( d_v \) and \( d_e \). The degree of a vertex \( v \) is the number of vertices joining to \( v \). Also, the degree of an edge \( e \in E(G) \) is the number of its adjacent vertices in \( L(G) \), where the line graph \( L(G) \) of a graph \( G \) is defined to be the graph whose vertices are the edges of \( G \), with two vertices being adjacent if the corresponding edges share a vertex in \( G \).

\[ ISI_e(G) = \sum_{uv \in E(G)} \frac{d_u \times d_v}{d_u + d_v} \]  \hspace{1cm} (3)

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Here, we define the new member (edge version of inverse sum indeg index) of this class on the ground of the end-vertex degree $d_e$ and $d_f$ of edges $e$ and $f$ in a line graph of $G$ as follows:

$$ISI_e (G) = \sum_{ef \in E(L(G))} \frac{d_e \times d_f}{d_e + d_f}$$  \hspace{1cm} (4)

where $d_e$ denotes the degree of the edge $e$ in $G$. The aim of this paper is to study the $ISI_e$ index for some families of graphs.

2. MAIN RESULT

The goal of this section is to study and computing the $ISI_e$ index of the complete graph $K_n$, path $P_n$, cycle $C_n$ and star graph $S_n$.

Also we obtain a closed formula of this index for a famous molecular graph that is Circumcoronene Series of Benzenoid $H_k$. The circumcoronene homologous series of benzenoid is family of molecular graph, which consist several copy of benzene $C_6$ on circumference. The first terms of this series are $H_1 =$ benzene, $H_2 =$ coronene, $H_3 =$ circumcoronene $H_4 =$ circumcircumcoronene, see Figure 1 and Figure 2, where they are shown.

These molecular graphs are presented in many papers, (see the paper series [18-21]). For every positive integer number $k$, the general form of circum- coronene series of benzenoid $H_k$ is shown in Figure 1. Also, its line graph is shown in Figure 3. For more information of this family, see the paper series [3, 6, 9-15].

**Lemma 1:** Let $K_n$ be the complete graph on $n$ vertices. Then $L(K_n)$ will be a $2(n-1)$ regular graph and for every $e \in E(K_n)$ \ ($e \in V(L(K_n))$) \ $d_v = 2(n-1)$ So \ $|E(L(K_n))| = \frac{1}{2} |E(K_n)| 2(n-1) = \frac{n(n-1)^2}{2}$. This implies that

$$ISI_e (K_n) = \sum_{ef \in E(L(K_n))} \frac{d_e \times d_f}{d_e + d_f} = \frac{|E(L(K_n))|}{2(n-1)+2(n-1)} = \frac{n(n-1)^3}{2}.$$ \hspace{1cm} (5)

**Lemma 2:** Let $C_n$ be the cycle of length $n$: Then one can see that $L(C_n) = C_n$ and for every $v \in V(c_n)$ and $e \in V(L(C_n)) d_v = d_e = 2$ , so $ISI_e (C_n) = |E(L(C_n))| = \frac{2 \times 2}{2+2} = n$  \hspace{1cm} (6)
Figure-2: The Circumcoronene Series of Benzenoid $H_k(k \geq 1)$ with dges

**Lemma 3:** Let $P_n$ be a path of length $n$. Then $L(P_n) = P_{n-1}$ and for all vertices of $L(P_n)$: $d_v = 2$, (except first and end vertices on path, that are as degree one), thus

$$ ISI_e (P_n) = (n - 3) \times \frac{2 \times 2}{2 + 2} + 2 \times \left(\frac{2 \times 1}{2 + 1}\right) $$

$$ = \frac{3n-5}{3}. \quad (7) $$

**Lemma 4:** Let $S_n$ be a star graph with $n + 1$ vertices. Then $L(S_n)$ will be a star graph with $n - 1$ regular graph (or a complete graph on $n$ vertices) and for every $e \in E(S_n)$ or ($e \in V(L(S_n))$) $d_v = (n - 1)$. So $|E(L(S_n))| = |E(K_n)| = \frac{1}{2}n(n - 1)$. This implies that

$$ ISI_e (S_n) = |E(L(S_n))| \frac{(n-1)(n-1)}{(n-1)+(n-1)} = \frac{n(n-1)^2}{4} \quad (8) $$

In addition, to achieve our aims we need the following definition.

**Definition I[16]:** Let $G$ be the molecular graph and $d_v$ is degree of vertex $v \in V(G)$ (Obviously $1 \leq \delta \leq d_v \leq \Delta \leq n - 1$, where

$$ \delta = \min\{d_v | v \in V(G)\} \quad \text{and} \quad \Delta = \max\{d_v | v \in V(G)\}. $$

Let $G = H_k$, we divide edge set $E(G)$ and vertex set $V(G)$ of graph $G$ to several partitions, as follow: for $i \in [2\delta, 2\Delta]$, define $E_i = \{e = uv \in | d_u + d_v = i\}$, for $j \in [\delta^2, \Delta^2]$ , define $E_j = \{e = uv \in | d_u \times d_v = j\}$, for $k \in [\delta, \Delta]$, define $V_k = \{v \in V(G) | d_v = k\}$.

**Theorem 4:** Consider the graph $G = H_k(k \geq 2)$ is circumcoronene series. Then the Inverse sum indeg index of $H_k$ given by

$$ ISI(H_k) = \frac{1}{10}(135k^2 - 81k - 54). \quad (9) $$

**Proof:** Let $G = H_k(k \geq 2)$ be the circumcoronene series of benzenoid. Thus, this graph has $6k^2$ vertices and $3 \times 6k(k-1) + 2 \times 6k = 9k^2 - 3k$ edges. From the structure of $H_k$, one can see that there are two partitions $V_3 = \{v \in V(H_k) | d_v = 3\}$ and $V_2 = \{v \in V(H_k) | d_v = 2\}$ with size $6k(k-1)$ and $6k$, respectively. The edge set of graph $H_k$ can be divide in to three partitions $E_4$, $E_5$ and $E_6$ . In other words, For every $e = uv$ belong to $E_4$, $d_u = d_v = 3$. Similarly, for every $e = uv$ belong to $E_5$, $d_u = d_v = 2$. Finally, if $e = uv$ be an edge of $E_6$ then $d_u = 2$ and $d_v = 3$. We mark the members of $E_4$ by red color, the members of $E_5$ by green color and the members of $E_6$ by black color, in Figure 2. Thus, we have following computations:

Clearly $|E_4| = 4$, $|E_5| = 12(k - 1)$ and $|E_6| = 9k^2 - 15k + 6$. Hence we have

$$ ISI(H_k) = \sum_{e=uv \in E_4} \frac{d_u \times d_v}{d_u + d_v} + \sum_{e=uv \in E_5} \frac{d_u \times d_v}{d_u + d_v} + \sum_{e=uv \in E_6} \frac{d_u \times d_v}{d_u + d_v} $$
Theorem 5: Let \( G \) be the graphs from the circumcoronene series of benzenoid \( H_k \) \( \forall k \geq 1 \) with \( 6k^2 \) vertices and \( 9k^2 - 3k \) edges then

\[
ISI_e(H_k) = \frac{(1260k^2 - 1170k + 99)}{35}
\]

(10)

**Proof:** Consider the circumcoronene series of benzenoid \( H_k \) for all positive integer number \( k \). By theorem 4 the edge version of Inverse sum indeg index of \( H_k \) is equivalent with Inverse sum indeg index of its line graph. At first, we partition the vertex set and edge set of \( H_k \) as follow:

\[
V_3 = \{ v \in V(H_k) | d_v = 3 \} \Rightarrow |V_3| = 6k(k - 1)
\]

\[
V_2 = \{ v \in V(H_k) | d_v = 2 \} \Rightarrow |V_2| = 6k
\]

\[
E_4 = \{ e = uv \in E(H_k) | d_u = d_v = 2 \} \Rightarrow |E_4| = 6
\]

\[
E_5 = \{ e = uv \in E(H_k) | d_u = 3 \text{ and } d_v = 2 \} \Rightarrow |E_5| = 12(k - 1)
\]

\[
E_6 = \{ e = uv \in E(H_k) | d_u = d_v = 3 \} \Rightarrow |E_6| = 9k^2 - 15k + 6
\]

In Figure 1, all edges belong to \( E_4, E_5 \) and \( E_6 \), marked by red, green and black colors, respectively. It is very clear that number of vertices in \( L(H_k) \) = number of edges in \( H_k = 9k^2 - 15k + 6 + 12k - 12 + 6 = 9k^2 - 3 \) and number of edges in \( L(H_k) \) = sum of the degrees of the edges of \( H_k = \frac{4(9k^2 - 15k + 6)}{2} + 3 \times 12(k - 1) + 2 \times 6 \)

\[= 18k^2 - 12k \]. Alternatively, we can partition the vertex set and edge set of \( L(H_k) \) by using the results of [11] as follow:

\[
V_{L_2} = \{ e \in E(L(H_k)) | d_e = 2 \} \Rightarrow |V_{L_2}| = |E_4| = 6
\]

\[
V_{L_3} = \{ e \in E(L(H_k)) | d_e = 3 \} \Rightarrow |V_{L_3}| = |E_5| = 12(k - 1)
\]

\[
V_{L_4} = \{ e \in V(L(H_k)) \text{ or } e \in E(H_k) | d_e = 4 \} \Rightarrow |V_{L_4}| = |E_6| = 9k^2 - 15k + 6
\]

\[
E_{L_5} = \{ \mu = ef \in E(L(H_k)) | d_e = 2, d_f = 3 \} \Rightarrow |E_{L_5}| = 2 |V_{L_2}| = 12
\]

\[
E_{L_6} = \{ \mu = ef \in E(L(H_k)) | d_e = 3, d_f = 3 \} \Rightarrow |E_{L_6}| = |V_{L_3}| - |V_{L_2}| = 6(2k - 3)
\]

\[
E_{L_7} = \{ \mu = ef \in E(L(H_k)) | d_e = 3, d_f = 3 \} \Rightarrow |E_{L_7}| = |V_{L_4}| - |V_{L_2}| = 12(k - 1)
\]

\[
E_{L_8} = \{ \mu = ef \in E(L(H_k)) | d_e = d_f = 4 \} \Rightarrow |E_{L_8}| = E(L(H_k)) - |E_{L_7}| - |E_{L_6}| - |E_{L_5}| = 18k^2 - 36k + 18
\]

\[= 18(k - 1)^2\]

In Figure 3 all edges belong to \( E_{L_5}, E_{L_6}, E_{L_7} \) and \( E_{L_8} \) are marked by red, green, and blue black colors, respectively.

Hence

\[
ISI_e(H_k) = \sum_{e = uv \in E(L(H_k))} \frac{d_u \times d_v}{d_u + d_v} + \sum_{e \in E_{L_5}} \frac{2 \times 3}{2 + 3} + \sum_{e \in E_{L_6}} \frac{3 \times 3}{3 + 3} + \sum_{e \in E_{L_7}} \frac{3 \times 4}{3 + 4} + \sum_{e \in E_{L_8}} \frac{4 \times 4}{4 + 4}
\]
\[ \frac{1}{5} |EL_5| + \frac{9}{6} |EL_6| + \frac{12}{7} |EL_7| + \frac{16}{8} |EL_8| \]
\[ = \frac{1}{6} (12) + \frac{9}{2} (12k - 18) + \frac{12}{7} (12k - 12) + 2 (18k^2 - 36k + 18) \]

\[ ISI_e (H_k) = \left( \frac{1260k^2 - 1170k + 99}{35} \right) \]

**Example 1:** Let \( H_3 \) be the Circumcoronene. Then the number of edges \( e_5, e_6, e_7 \) and \( e_8 \) in line graph \( H_3 \) are equal to 12, 18, 24 and 72, respectively (see Figure 4). So \( ISI_e \) index of \( H_3 \) is equal to

\[ ISI_e (H_k) = ISI(L(H_3)) = 226.5428 \]

**REFERENCES**