ON q-k-EP MATRICES

K. GUNASEKARAN
Ramanujan Research centre,
PG and Research Department of Mathematics,
Government Arts College (Autonomous), Kumbakonam-612 002, Tamil Nadu, India.

K. GNANABALA*
Ramanujan Research centre,
PG and Research Department of Mathematics,
Government Arts College (Autonomous), Kumbakonam-612 002, Tamil Nadu, India.

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ABSTRACT

The concept of range quaternion k-EP (q-k-EP) matrices is introduced as a special case of quaternion hermitian and generalization of EP matrices. Necessary and sufficient conditions are determined for a matrix to be q-k-EP, (q-k-EP and rank r). As an application, it is shown that the class of all q-k-EP matrices having the same range space form a group under multiplication.

Key words: Moore-Penrose Inverse, Quaternion matrix, Rank of matrix, Range hermitian k-EP matrices

1. INTRODUCTION

The algebra $\mathbb{H}$ of real quaternion, which is a four-dimensional non-commutative algebra over real number field $\mathbb{R}$ with canonical basis $1, i, j, k$ satisfying the conditions, $i^2 = j^2 = k^2 = ijk = -1$ that implies $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

The elements in $\mathbb{H}$ can be written in a unique way as, $\alpha = a + bi + cj + dk$, where $a, b, c$ and $d$ are real numbers, i.e., $\mathbb{H} = \{ a + bi + cj + dk \mid a, b, c, d \in \mathbb{R} \}$.

The conjugate of $\alpha$ is defined as $\bar{\alpha} = a - bi - cj - dk$, and the norm $|\alpha| = \sqrt{\alpha \bar{\alpha}}$ for $0 \neq \alpha \in \mathbb{H}$, $\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2}$.

We consider $K$ is a permutation matrix associated with the permutation $k(x) = (S_n)$, where $S = \{1, 2, \ldots, n\}$.

Also $K^2 = I$, $\bar{K} = K^T = K^* = K^{-1} = K$.

2. q-k-EP MATRICES

Definition: 2.1 Let $H[x]^{m \times n}$ denote the set of all $m \times n$ matrices with entries from $H[x]$. For $A \in H[x]^{m \times n}$, the conjugate $\bar{A} = \bar{A}_{ij}$. If $A = P + Qi$ with $P, Q \in H[x]^{m \times n}$, then $X_A = \left( \begin{array}{c|c} P & Q \\ \hline -Q & P \end{array} \right) \in C[x]^{2mx2n}$ denotes the complex adjoint of $A$.

Moreover, $A^T, A^* \in H[x]^{m \times n}$ denotes the transpose and the conjugate transpose of $A$, respectively.

Definition: 2.2 $A^\dagger \in H[x]^{m \times n}$ is called a Moore Penrose inverse of $A \in H[x]^{m \times n}$, if it is a solution of the following system of equations, $AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA$. Note that we require that $A^\dagger$ must be in $H[x]^{m \times n}$.
**Definition: 2.3** A matrix $A \in H[x]^{m \times n}$ is said to be q-k-EP if it satisfies the condition $Ax = 0 \iff A^k(x) = 0$ or equivalently $N(A) = N(A^kK)$. Moreover, $A$ is said to be k-EP, if it is k-EP and of rank $r$.

**Definition: 2.4** A k-hermitian matrix $A$ is q-k-EP, for if $A$ is k-hermitian, then by [3, Result 2.1], $A = KA^k$. Hence $N(A) = N(KA^kK) = N(A^kK)$, which implies $A$ is q-k-EP. However, the converse need not be true.

**Theorem: 2.5** For the following are equivalent:

1. $A$ is q-k-EP
2. $KA$ is EP
3. $AK$ is EP
4. $A^\dagger$ is q-k-EP
5. $N(A) = N(A^\dagger K)$
6. $N(A^\dagger) = N(AK)$
7. $R(A) = R(KAK)$
8. $R(A^\dagger) = R(KA)$
9. $KA^kK = AA^k$
10. $A^kAK = KAA^\dagger$ for a non singular $nxn$ matrix $H$.
11. $A = KA^kKH$ for a non singular $nxn$ matrix $H$.
12. $A = HKA^kK$ for a non singular $nxn$ matrix $H$.
13. $A^\dagger = KAKH$ for a non singular $nxn$ matrix $H$.
14. $A^\dagger = KAKH$ for a non singular $nxn$ matrix $H$.
15. $C_n = R(A) \oplus N(AK)$.
16. $C_n = R(KA) \oplus N(A)$.

**Proof:** The proof for the equivalence of (1), (2) and (3) runs as follows:

A is q-k-EP $\iff N(A) = N(A^kK)$ (by Definition 2.3)
$\iff N(KA) = N(KA^k)$ [by (P.1)]
$\iff KA$ is EP (by Definition of EP matrix)
$\iff KKA^kK^* = \text{is EP}$ (by [1, Lemma3])
$\iff AK$ is EP [by (P.1)]

Thus (1) $\implies$ (2) $\implies$ (3) hold.

(2)$\iff$(4): $KA$ is EP $\iff (KA)^\dagger$ is EP (by [2, P.163])
$\iff A^kK$ is EP [by (P.2)]
$\iff A^k$ is q-k-EP [by equivalence of (1) and (3) applied to $A^\dagger$]

Thus equivalence of (1) and (5) is proved.

Now we shall prove the equivalence of (1), (6) and (7) using $\rho(A) = \rho(A^\dagger) = \rho(A^kK) = \rho(AK)$ in the following way:

$A$ is q-k-EP $\iff N(A) = N(A^kK)$
$\iff N(A) \subseteq N(A^kK)$
$\iff A^kK = A^kKA^{-}A$ (by [2, P.21])
$\iff A^k = A^kKA^{-}AK$ (by [P.1])
$\iff A^k = A^kK^{-1}A^{-}AK$ (by [2, P.21])
$\iff N(A) \subseteq N(A^k)$ (by [2, P.21])
$\iff N(A^k) = N(A)$
$\iff R(A) = R(KA)^*$ (by [P.1])

Thus (1) $\implies$ (6) $\implies$ (7) holds.

(1)$\iff$(8):

$A$ is q-k-EP $\iff N(A) = N(A^kK)$
$\iff N(A) = N(KA)^*$
$\iff R(A^k) = R(KA)$

Thus equivalence of (1) and (8) is proved.
(3) ⇔ (9):

AK is EP ⇔ (AK)(AK)† = (AK)†(AK) (by [2, P.166])
⇔ (AK)(KA† ) = (KA† )(AK) (by [P.2])
⇔ AA† = KA† AK (by [P.1])
⇔ AA† K = KA† A

Thus equivalence of (3) and (9) is proved.

(9) ⇔ (10): Since by the property (P.1), K² = I, this equivalence follows by pre and post multiplying KA† A = AA† K by K.

(2) ⇔ (11):
KA is EP ⇔ (KA)* = (KA)H₁, for a non-singular nxn matrix H₁ (by [2, P.166])
⇔ A*K = KAH₁
⇔ KA*K = AH₁
⇔ A = KA*KH where H = H₁⁻¹ is a non- singular nxn matrix.

Thus equivalence of (2) and (11) is proved.

(3) ⇔ (12):
AK is EP ⇔ (AK)* = H₁(AK), for a non-singular nxn matrix H₁(by [2, P.166])
⇔ KA*K = H₁A
⇔ A = H₁⁻¹KA*K
⇔ A = H₁⁻¹KA
⇔ A = HKA*K where H = H₁⁻¹ is a non- singular nxn matrix.

Thus equivalence of (3) and (12) is proved.

The equivalences (11) ⇔(13) and (12) ⇔(14) follow immediately by taking conjugate transpose and using K = K*.

(13) ⇔ (16): A* = HKAK for a non singular nxn matrix H.
⇔ A*A = H(KA)(KA)
⇔ A*A = H(KA)²
⇔ ρ(A*A) = ρ(H(KA)²)
⇔ ρ(A*A) = ρ((KA)²)

Over the complex field, A*A and A have the same rank.

Therefore, ρ((KA)²) = ρ(A*A) = ρ(A) = ρ(KA)⇔ R(KA) ∩ N(KA) = {0}
⇔ R(KA) ∩ N(A) = {0}
⇔ Hₙ = R(KA) ⊕ N(A).

Thus (13) ⇔ (16) holds.

(14) ⇔ (15): This can be proved along the lines and using ρ(A*A) = ρ(A). Hence the proof is omitted.

(16) ⇔ (1): If Hₙ = R(KA) ⊕ N(A), then R(KA) ∩ N(A) = {0}.

For x ∈ N(A), x ∉ R(KA) ⇔ x ∈ R(KA)⊥ = N(KA)* = N(A*K).

Hence N(A) ⊆ N(A*K) and ρ(A) = ρ(A*K) ⇒ N(A) = N(A*K) ⇒ A is q-k-EP.

Thus (1) holds. Similarly, we can prove (15) ⇒(1).

Remark: 2.6 [8] Let A ∈ H[x]mxn and B ∈ H[x]nxl. Then
(i) (AB)* = B*A* and AA* = (AA*)*
(ii) If A has a Moore- Penrose Inverse A†, then
(A*)† = (A†)*, A†(A†)*A* = A† = A(A†)A† and A†AA* = A* = A*AA†
(iii) If A has a Moore- Penrose Inverse A*, then A† is unique.
(iv) Let A have the Moore- Penrose Inverse A*. If U ∈ H[x]mxm is a unitary matrix, then (UA)† = A†U*.
For \( x = (x_1, x_2, ..., x_n)^T \in \mathbb{H}[x]^{mxl} \). Let us define the function \( k(x) = (x_{(k(1))}, x_{(k(2))}, ..., x_{(k(n))})^T \in H_n \). Since \( k \) is involutory, it can be verified that the associated permutation matrix \( k \) satisfy the following properties:

\[
K = K^T = K^{-1} \quad \text{and} \quad k(x) = Kx,
\]

\[ (KA)^T = A^T K \quad \text{and} \quad (AK)^T = KA^T \quad \text{for} \quad A \in H[x]^{nn} \quad (\text{by} \ [2, \text{P.182}]) \]

**Theorem 2.7** Let \( A \in H[x]^{nn} \). Then any two of the following conditions imply the other one:

1. \( A \) is EP
2. \( A \) is q-k-EP
3. \( R(A) = R(KA) \)

**Proof:** First we prove that whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds, then by [1, Theorem 1], \( A \) is EP implies \( R(A) = R(A^*) \). Now by Theorem 2.5, \( A \) is q-k-EP \( \iff R(A^*) = R(KA) \). Therefore, \( A \) is q-k-EP \( \iff R(A) = R(KA) \).

This completes the proof of \((1) \Rightarrow (2) \) and \((1) \Rightarrow (3) \).

Now let us prove \((2) \Rightarrow (1) \): Since \( A \) is q-k-EP, by Theorem 2.5, \( KA \) is EP. Hence, \( R(KA) = R(KA^*) \). By using (3), we have \( R(A) = R(KA) \Rightarrow R(A) = R(A^*) \). Again by [1, Theorem 1], \( A \) is EP. Thus (1) holds.

**Note 2.8** [8] Let \( A \in H[x]^{nn} \) have the Moore- Penrose Inverse \( A^+ \). Consider \( A \) has a homomorphism from \( H[x]^{mxn} \) to \( H[x]^{mxn} \). Then Image (\( A \)) = Image (\( AA^+ \)) = Image (\( AA \)) and Image (\( A^+ \)) = Image (\( A^+ A \)) = Image (\( A^+ A \)).

**Lemma 2.9** [8] If \( E \in H[x]^{mxm} \) is a symmetric projection, that is \( E = E^2 - E^* \), then \( E \in H[x]^{mxm} \).

**Proof:** Let \( f_1, f_2, ... , f_m \) be the entries on the first row of \( E \). From, \( E = E^* \), we may assume that \( f_1 = \overline{f}_1 \neq 0 \). Then \( E = E^2 \) we have

\[
f_1 = f_1 \overline{f}_1 + \sum_{i=2}^{m} f_i \overline{f}_i = f_1^2 + \sum_{i=2}^{m} f_i \overline{f}_i
\]

Since \( f_1 = \overline{f}_1 \) the leading co-efficient of \( f_1^2 \) is a positive real number. Note that the leading co-efficient of \( \sum_{i=2}^{m} f_i \overline{f}_i \) is also a positive real number. Thus,

\[
\deg(f_1^2) \geq \deg(f_1) = \deg(f_1^2 + \sum_{i=2}^{m} f_i \overline{f}_i) = \max\{\deg(f_1^2), \deg(\sum_{i=2}^{m} f_i \overline{f}_i)\} \geq \deg(f_1^2)
\]

This shows that \( f_1 \in H \). Further more, \( 0 \neq \deg(f_1^2) = \deg(\sum_{i=2}^{m} f_i \overline{f}_i) \) and the leading co-efficient of \( \{f_i \overline{f}_i \} \{f_i \neq 0 \} \) are positive reals imply that \( f_i \in H \) for all \( 1 \leq i \leq m \). The same discussions can be done for the other rows of \( E \). Therefore, \( E \in H[x]^{mxm} \).

**Lemma 2.10** If \( A \in H[x]^{nn} \) is normal and \( AA^+ \) is q-k-EP, then \( A \) is q-k-EP.

**Proof:** Since \( A \) is normal, \( A \) is EP and \( AA^+ \) is q-k-EP \( \iff R(AA^+) = R(KAA^+) \) implies \( R(A) = R(KA) \). That \( A \) is q-k-EP then follows from Theorem 2.7.

**Lemma 2.11** Let \( E = E^* = E^2 \in H[x]^{nn} \) be a hermitian idempotent that commutes with \( k \), the permutation matrix associated with a fixed product of disjoint transpositions \( k \) is \( S_n \). Then, \( H_k(E) = \{ A: A \) is q-k-EP and \( R(A) = R(E) \} \) and forms a maximal subgroup of \( H_{mn} \) containing \( E \) as identity.

**Proof:** Since \( E K = KE \), by (P.1) and (P.2) we have \( E = EKE \) and \( EE^+ = E^2 = E \Rightarrow (KE)(KE) = (KE)(KE)^t \);

Hence \( R(E) = R(KE) \).

Since \( E \) is hermitian it is automatically EP. And by Theorem 2.7, \( E \) is K-EP and \( R(A) = R(KE) \Rightarrow \{AA^+ = EE^+ = E\} \)
also \( A = KE \Rightarrow (KE)(KE)^t = KEE^+K^t = KAA^tK^t = (KA)(KA)^t \).

Therefore \( R(A) = R(KA) \). Hence by Theorem 2.7, \( A \) is EP and \( H_k(E) = H(E) = \{ A : A \) is EP and \( R(A) = R(E) \} \).

By [5, Theorem 2.1], \( H_k(E) \) forms a maximal subgroup \( H[x]^{nn} \) containing \( E \) as identity.

3. EIGEN VALUES
Definition: 3.1 A ∈ H[x]^{mn} is hermitian, that is A = AA^*, if and only if there exists a unitary matrix U ∈ H[x]^{mn} such that U^*AU = diag(d_1, d_2, ..., d_n), where d_i are the eigen values of A.

Lemma: 3.2 For A ∈ H[x]^{mn}, A is k-EP ⇔ N(A) ⊆ N(P), where P is k-hermitian part of A.

Proof: If A is k-EP, then by Theorem 2.5, KA is EP. Since K is non-singular, N(A) = N(KA) = N(KA)^* = N(A^*K) = N(KA^*K).

Then for x ∈ N(A), both Ax = 0 and KA’Kx = 0, which implies that Px = \frac{1}{2}(A + KA^*K)x = 0. Thus N(A) ⊆ N(P).

Conversely, N(A) ⊆ N(P); Then Ax=0 implies Px=0 and hence Qx=0. Therefore, N(A) ⊆ N(Q).

Thus N(A) ⊆ N(P) ∩ N(Q).

Since both P and Q are k-hermitian, and by [3, Result 2.1], we have, P=KP^*K and Q=KQ^*K.

Hence N(P) = N(KP^*K)= N(P^*K) and N(Q)=N(KQ^*K) = N(Q^*K).

Now N(A) ⊆ N(P) ∩ N(Q) = N(P^*K) ∩ N(Q^*K) ⊆ N(P^* - iQ^*)K.

Therefore, N(A) ⊆ N(A^*K) and ρ (A) = ρ (A^*K).

Hence, N(A) = N(A^*K). Therefore, A is q-k-EP. Hence the theorem.

Lemma: 3.3 [8] Let A ∈ H[x]^{mn}. Then A has the Moore-Penrose inverse A^ if and only if A = U \[\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\] with U ∈ H^{mxm} unitary and A_1 A_1^* + A_2 A_2^* a unit in H[x]^{nxr} with r ≤ min(m, n).

Moreover, A^ = \[\begin{bmatrix} (A_1 A_1^* + A_2 A_2^*)^{-1} & 0 \\ A_2^* A_1 A_1^* + A_2 A_2^* \end{bmatrix}\] U^*

Proof: If A has the Moore-Penrose Inverse A^, then AA^ = AA^AA^ = (AA^)^2 = (AA^)^*.

By Lemma 2.9, AA^ ∈ H^{mn}. AA^ is hermitian and hence, by Lemma 3.2, there exists a unitary matrix U ∈ H^{mxm} such that U^*AA^U = D, where D is diagonal. Since, D^2 = (U^* AA^U)(U^* AA^U) = U^* AA^ AA^U = U^* AA^ U = D, the diagonal entries of D are either 1 or 0. Therefore, we can arrange the rows of U so that D=\[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\] with r ≤ min(m, n).

Set A'= U^* A. By Lemma 2.6, A' has its own generalized inverse A'^+ and A'A'^+ = \[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\]. Set A' = \[\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\], for arbitrary quaternion polynomial matrices A_1 ∈ H[x]^{nxr}, A_2 ∈ H[x]^{ix(n-r)}, A_3 ∈ H[x]^{(m-r)xr} and A_4 ∈ H[x]^{(m-r)x(n-r)}. Since A'^+A'^A' = \[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\] \[\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\] \[\begin{bmatrix} A_1^* & A_2 \\ 0 & 0 \end{bmatrix}\], we must have A'= \[\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\] and therefore A'^+ = \[\begin{bmatrix} A_1 A_1^* + A_2 A_2^* & 0 \\ 0 & 0 \end{bmatrix}\].

Similarly, A'^+ = \[\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}\], for some B_1 and B_2.

By Lemma 2.8, Image (A'^+ A'^+) = Image (A') = Image (A'A'^+) = Image \[\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\].

This implies the surjectivity of A_1 A_1^* + A_2 A_2^* on H[x]^{ml}.

Therefore A_1 A_1^* + A_2 A_2^* is a unit in H[x]^{nxr} and A= UA' = U \[\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\].
\[ A^* = A^{\dagger} (A^{\dagger})^* A^* = A^{\dagger} (A^{\dagger})^* A^{\dagger} (A^{\dagger})^* = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix} \begin{bmatrix} (A_1 A_1^* + A_2 A_2^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \]

which gives \[ A^* = \begin{bmatrix} A_1^* (A_1 A_1^* + A_2 A_2^*)^{-1} & 0 \\ A_2^* (A_1 A_1^* + A_2 A_2^*)^{-1} & 0 \end{bmatrix} U^*. \] The converse can be proved by direct computation.

**Lemma 3.4** Let \( B = [D \ 0] \), \( D \) is an \( r \times r \) non-singular matrix. Then the following are equivalent.

1. \( B \) is \( k \)-EP,
2. \( R(KB) = R(B) \).
3. \( BB^* \) is \( k \)-EP,
4. \( K = [K_1 \ 0 \ K_2] \), where \( K_1, K_2 \) are permutation matrices of order \( r \) and \( n - r \) respectively.
5. \( K = K_1K_2 \) where \( K_1 \) is the product of disjoint transpositions on \( S_n = \{1, 2, \ldots, n\} \) leaving \( (r+1, r+2, \ldots, n) \) fixed, and \( K_2 \) is the product of the disjoint transpositions leaving \( (1, 2, \ldots, r) \) fixed.

**Proof:** Since \( B \) is EP, the equivalence of (1) and (2) follows from Theorem 2.7.

**Lemma 3.5** A matrix \( A \in H[x]^{\text{ran}} \) if q-k-EP, if and only if there exists a unitary matrix \( U \) and an \( r \times r \) nonsingular matrix \( F \) such that \( A = KU[\begin{bmatrix} F \\ 0 \end{bmatrix} 0 U^* \). 

**Proof:** Let us assume that \( A \) is q-k-EP. Then by Theorem 2.5, \( H_n = R(KA) \oplus N(A) \). Choose an orthonormal basis \( \{x_1, x_2, \ldots, x_n\} \) of \( R(KA) = R(A^*) \) and extend it to a basis \( \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_r\} \) of \( H_n \) where \( \{x_{n+1}, \ldots, x_r\} \) is an orthonormal basis of \( N(A) \).

If \((u, v)\) denotes the usual inner product on \( H_n \) and \( 1 \leq i \leq r \leq j \leq n \) it follows that \( x_j \in R(KA) = R(A^*) \Rightarrow x_jA^*y \).

Therefore, \( (x_j, x_i) = (A^*y, x_i) = (y, Ax_i) = 0 \) (Since \( x_j \in N(A) \)). Hence \( \{x_1, x_2, \ldots, x_n\} \) is an orthonormal basis of \( H_n \). If we consider \( KA \) as the matrix of a linear transformation relative to any orthonormal basis of \( H_n \), then \( U^*KAU = \begin{bmatrix} F \\ 0 \end{bmatrix} \). Where \( F \) is \( r \times r \) nonsingular matrix, whence \( A = KU[\begin{bmatrix} F \\ 0 \end{bmatrix} 0 U^* \).

Conversely, if \( A = KU[\begin{bmatrix} F \\ 0 \end{bmatrix} 0 U^* \), then \( U^*KAU = \begin{bmatrix} F \\ 0 \end{bmatrix} \).

But \( N(KA) = N(KA)^* \), which implies \( KA \) is EP, and by Theorem 2.5, \( A \) is q-k-EP.
**Lemma: 3.6**  
Let $A \in H_{nn}$. Then $A$ is q-k-EP, with $K = K_1K_2$ (where $K_1$ and $K_2$ are as in Lemma 3.4) if and only if $A$ is Unitarily q-k-similar to a diagonal block q-k-EP matrix $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where $D$ is an rxr non-singular matrix.

**Proof:** Since $A$ is q-k-EP, by Lemma 3.5, there exists a unitary matrix $U$ and an rrxr non-singular matrix $F$ such that $A = (KUK)\begin{bmatrix} F \\ 0 \\ 0 \\ 0 \end{bmatrix}U^*$. Since $K = K_1K_2$, the associated permutation matrix is $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$. Hence, $A = (KUK)\begin{bmatrix} K_1F \\ 0 \\ 0 \\ 0 \end{bmatrix}U^* = (KUK)\begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}U^*$, where $D=K_1F$. Thus, $A$ is Unitarily q-k-similar to a diagonal block q-k-EP matrix $B = \begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}$ where $D$ is an rrxr non-singular matrix.

Now, That $B$ is q-k-EP follows from Theorem 3.4, $K = K_1K_2$ and $K = \begin{bmatrix} K_1 \\ 0 \\ K_2 \end{bmatrix}$. Since, $A$ is Unitarily q-k-similar to $B = \begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}$, there exists a unitary matrix $U$ such that $A = (KUK)\begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}U^*$. Since $B$ is q-k-EP, by Theorem 2.5, $KB=K\begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix} = U^*KAU$ is EPr. By [1, Lemma 2], $KA$ is EPr. Now, $A$ is q-k-EP follows from Theorem 2.5 and $\rho(A) = r$. Hence $A$ is q-k-EP.

The proof is complete.

**Lemma: 3.7**  
Let $A \in H_{nxn}$. Then eigen values of $AA^*$ are real.

**Proof:** Let $B=AA^*$ and $\lambda \in H$ be an eigen value of $B$ with corresponding eigen vector $X= (x_1, x_2, ..., x_m)^T \neq 0$ such that $BX=X\lambda$. Then $B^*X = X^*\lambda^*$. Note that $B=B^*$. We have that $X^*BX=\lambda^* X^*X$.

Thus, $X^*X\lambda = \lambda^* X^*X = (X^*X)\lambda^*$.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \lambda = (\sum x_i x_i)^* \lambda = \lambda^* (\sum x_i x_i)^* = \lambda^* \sum x_i x_i.)$$

By a known lemma, $0 \neq \sum x_i x_i \in \mathbb{R}[x]$. The above equation gives $\lambda = \lambda^*$ which implies $\lambda \in \mathbb{R}$.

**Lemma: 3.8**  
If $a$ is k-EP, then $(\lambda, x)$ is a (k-eigen value, k-eigen vector) pair for $A$ if and only if $(1/\lambda, k(x))$ is a (k-eigen value, k-eigen vector) pair for $A^\dagger$.

**Proof:** $(\lambda, x)$ is a (k-eigen value, k-eigen vector) pair for $A$  
\[ \Rightarrow Ax = \lambda kx \]  
(by [3, P.22])  
\[ \Rightarrow KA = \lambda Kx \]  
(by P.1)  
\[ \Rightarrow (KA)^* x = \frac{1}{\lambda} x \]  
(by [2, P.161])  
\[ \Rightarrow A^\dagger Kx = \frac{1}{\lambda} x \]  
(by P.2)  
\[ \Rightarrow A^\dagger k(x) = \frac{1}{\lambda} K(k(x)) \]  
\[ \Rightarrow (1/\lambda, k(x)) \text{ is a (k-eigen value, k-eigen vector) pair for } A^\dagger. \]

**Definition: 3.9**  
For $A \in H_{nn}$, let $B=AA^*$ and $\chi_B$ be its complex adjoint. Then $f_B(\lambda) = \det(\lambda I_{2m} - \chi_B)$ is called the characteristic polynomial of $A$. © 2016, IUMA. All Rights Reserved
Let $A \in H[x]^{m\times n}$ and $B=AA^*$. Then $f_{\bar{B}}(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (R[x])[\lambda]$

**Proof:** We first show that $f_{\bar{B}}(\lambda) \in (R[x])[\lambda]$. Note that $B=AA^*$, we have 
\[
\det((\lambda I_{2m} - \chi_{B})^T) = \det(\lambda I_{2m} - \chi_{B}) = \det((\lambda I_{2m} - \chi_{B})^*),
\]
Thus $\det(\lambda I_{2m} - \chi_{B}) = \det(\lambda I_{2m} - \chi_{B})$.

Therefore, $\det(\lambda I_{2m} - \chi_{B}) = f_{\bar{B}}(\lambda) \in (R[x])[\lambda]$. 

Next we show that $f_{\bar{B}}(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (C[x])[\lambda]$.

Let $B = P + Q$ j. For any fixed $1 \leq i, j \leq m$,

We have $B_{ij} = a + bi + cj + dk$, where $a, b, c$ and $d \in R[x]$. 

Since $B$ is hermitian, $B_{ij} = a - bi - cj - dk$ and therefore $P_{ij} = a + bi, P_{ji} = a - bi$ and $Q_{ij} = c + di, Q_{ji} = c - di$.

So $P^T = \bar{P}$ and $Q = -Q^T$.

Therefore, $\chi_{B} = \left(\begin{array}{cc} P & Q \\ \bar{Q} & \bar{P} \end{array}\right) = \left(\begin{array}{cc} P & P^T \\ -Q & P^T \end{array}\right) \Rightarrow \lambda I_{2m} - \chi_{B} = \left(\begin{array}{cc} \lambda I_{m} - P & Q \\ -\bar{Q} & \lambda I_{m} - P^T \end{array}\right)$.

Next, we have 
\[
\left(\begin{array}{cc} I_{m} & -I_{m} \\ 0 & I_{m} \end{array}\right) \left(\begin{array}{cc} I_{m} & 0 \\ 0 & I_{m} \end{array}\right) \left(\begin{array}{cc} \lambda I_{m} - P & Q \\ -\bar{Q} & \lambda I_{m} - P^T \end{array}\right) = \left(\begin{array}{cc} \bar{Q} & \bar{P}^{T} - \lambda I_{m} \\ Q & \lambda I_{m} \end{array}\right).
\]

Therefore, $f_{\bar{B}}(\lambda) = \det\left(\begin{array}{cc} \lambda I_{m} - P & Q \\ -\bar{Q} & \lambda I_{m} - P^{T} \end{array}\right) = \det\left(\begin{array}{cc} \bar{Q} & \bar{P}^{T} - \lambda I_{m} \\ Q & \lambda I_{m} \end{array}\right)$ which implies that $\left(\begin{array}{cc} \bar{Q} & \bar{P}^{T} - \lambda I_{m} \\ Q & \lambda I_{m} \end{array}\right)$ is skew symmetric.

By [9], the determinant of $\left(\begin{array}{cc} \bar{Q} & \bar{P}^{T} - \lambda I_{m} \\ Q & \lambda I_{m} \end{array}\right)$ also called its P fattian, can be written as the square of a polynomial in its entries.

Therefore, $f_{\bar{B}}(\lambda) = g(\lambda)^2$, where $g(\lambda) \in (C[x])[\lambda]$.

Finally, we show that $g(\lambda) \in (R[x])[\lambda]$.

Suppose, otherwise, then $g(\lambda) = a(\lambda) + b(\lambda)i$, where $a(\lambda)$ and $b(\lambda) \in (R[x])[\lambda]$ with $b(\lambda) \neq 0$.

By (1), $g(\lambda)^2 = a(\lambda)^2 - b(\lambda)^2 + 2a(\lambda)b(\lambda)i \in (R[x])[\lambda]$.

Thus $a(\lambda) = 0$ and $f_{\bar{B}}(\lambda) = (b(\lambda)i)^2 = b(\lambda)^2$, where $b(\lambda) \in (R[x])[\lambda]$.

For a fixed $x \in R$, let $\lambda I_{2m} - \chi_{B} \in H^{2m\times 2m}$ is diagonally dominant with non-negative diagonal entries and that 
\[(b(x))(\lambda) \neq 0.
\]

Since, $\lambda I_{2m} - \chi_{B}$ is also hermitian, $\lambda I_{2m} - \chi_{B}$ is positive definite [10]. But $\det(\lambda I_{2m} - \chi_{B}) = -(b(x))(\lambda)^2 < 0$, a contradiction. Therefore, $b=0$ and thus $f_{\bar{B}}(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (R[x])[\lambda]$.

**Lemma:** Let $A \in H[x]^{m\times n}$, $B=AA^*$ and $f_{\bar{B}}(\lambda) = g(\lambda)^2$. Then $g(B)=0$. We will call $g(\lambda)$ the generalized characteristic polynomial of $A$.

**Proof:** Note that $g(\lambda) \in (R[x])[\lambda]$, by Theorem 3.10.

Then $g_{\bar{B}}(B) = g(\chi_{B})$. Next $f_{\bar{B}}(\chi_{B}) = 0$ by the Cayley Hamilton theorem for complex polynomial matrices [9].

Therefore, $g(\chi_{B})=0$, and $0 = g(\chi_{B}) = g_{\bar{B}}(B)$, that is $g(B)=0$.

**Lemma:** Let $A \in H[x]^{m\times n}$ has the Moore Penrose inverse $A^\dagger$. Set $B=AA^*$. Then
(i) $B^\dagger = (A^\dagger)^\dagger A^\dagger$ and $B^\dagger B = A A^\dagger$
(ii) $B^\dagger B = B B^\dagger$ and $(B^\dagger B)^2 = B^\dagger B$
(iii) $(B^\dagger)^\dagger = (B^\dagger)^\dagger$ and $(B^{m-k})^\dagger (B^{k-m}) = B^\dagger B$ for any $k \in N$

**Lemma 3.13** Let $A \in H[x]^{m \times m}$, $B \in H[x]^{m \times n}$ and $A \in H[x]^{m \times q}$. If $A^\dagger$, $B^\dagger$ both exists, then the quaternion polynomial matrix equation $AXB = C$ has a solution in $H[x]^{m \times mp}$ if and only if $AA^\dagger \subseteq B^\dagger B = C$, in which case the general solution is $X = A^\dagger C + Y - A^\dagger AYB B^\dagger$, where $Y \in H[x]^{m \times np}$ is arbitrary.

**Lemma 3.14** Let $A \in H[x]^{m \times m}$ has the Moore Penrose inverse $A^\dagger$ and $B = AA^\dagger$. Suppose the generalized characteristic polynomial of $A$ is: $g(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m$, where $a_i \in R[x]$. If $k$ is the largest number such that $a_k \neq 0$, then the generalized inverse of $A$ is given by $A^\dagger = -\frac{1}{a_k} A^\dagger [B^{k-1} + a_1 B^{k-2} + \cdots + a_k I]$. If $a_k = 0$, for all $1 \leq i \leq m$, then $A^\dagger = 0$.

**Lemma 3.15** Let $A \in H[x]^{m \times m}$ has the Moore Penrose inverse $A^\dagger$ and $B = AA^\dagger$. Then for $1 \leq k \leq m$, $\text{tr}(B^k + a_1 B^{k-1} + \cdots + a_k I) = -ka_k$, where the $a_i$ arise from the generalized characteristic polynomial of $A$: $g(\lambda) = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_{m-1} \lambda + a_m$.

**Proof:** Let $Y = yI$ where $y \in R$. We can write, $g(Y) = g(Y) - g(B)$

$= (Y - B) [Y^{m-1} + (B + a_1 I) Y^{m-2} + \cdots + (B^{m-1} + a_1 B^{m-2} + \cdots + a_m I)]$.

As long as $y$ is not an eigen value of $B$, $(yI - B)$ is non-singular, so we can write:

$(Y - B)^{-1} g(Y) = [Y^{m-1} + (B + a_1 I) Y^{m-2} + \cdots + (B^{m-1} + a_1 B^{m-2} + \cdots + a_m I)]$.

Taking the traces gives:

$\text{tr}[(Y - B)^{-1} g(Y)] = m Y^{m-1} + \text{tr}(B + a_1 I) Y^{m-2} + \text{tr}(B^2 + a_1 B + a_2 I) Y^{m-3} + \cdots + \text{tr}(B^{m-1} + a_1 B^{m-2} + \cdots + a_m I)]$.

Let $C = (Y - B)^{-1} g(Y)$. Since $g(Y) = g(yI) = g(y)I$, $C = g(y)(Y - B)^{-1}$. Therefore, $\text{tr} C = g(y) \text{tr}[(Y - B)^{-1}]$.

Let $\lambda_1, \ldots, \lambda_m$ be the non zero eigen values of $B$. $\text{tr}[(Y - B)^{-1}]$ is the sum of the eigen value of $[Y - B]^{-1}$.

We will show that these eigen values are the fractions $\frac{1}{y - \lambda_1}, \ldots, \frac{1}{y - \lambda_m}$.

Let $\zeta$ be an eigen value of $(Y - B)^{-1}$ with corresponding eigen vector $z$ such that: $(Y - B)^{-1} z = Z \zeta$. $\zeta$ is real (by Lemma 3.7) and hence $(Y - B) Z = Z \frac{1}{\zeta}$.

Therefore, $Y = \frac{1}{\zeta} = \lambda_1 \Rightarrow \zeta = \frac{1}{y - \lambda_i}$ for some $1 \leq i \leq m$.

Since $g(y) = (y - \lambda_1)(y - \lambda_2) \cdots (y - \lambda_m)$. We have that $g'(y) = g(y) \left(\frac{1}{y - \lambda_1} + \cdots + \frac{1}{y - \lambda_m}\right)$ and $\text{tr} C = g'(y)$. The derivative of $g$ is also equal to $g'(y) = m Y^{m-1} + a_1 (m - 1) Y^{m-2} + \cdots + a_{m-1}$.

Therefore, $m Y^{m-1} + a_1 (m - 1) Y^{m-2} + \cdots + a_{m-1} = m Y^{m-1} + \text{tr}(B + a_1 I) Y^{m-2} + \cdots + \text{tr}(B^{m-1} + a_1 B^{m-2} + \cdots + a_m I)$.

Comparing the co-efficient of $Y^{m-k-1}$ on both sides, we obtain

$a_k(m - k) = \text{tr} \left(B^k + a_1 B^{k-1} + a_2 B^{k-2} + \cdots + a_k I + B + a_k I\right)$

$= \text{tr} \left(B^k + a_1 B^{k-1} + a_2 B^{k-2} + \cdots + a_k I \right) + \text{tr}(a_k I)$

And then $-a_k = \text{tr} \left(B^k + a_1 B^{k-1} + a_2 B^{k-2} + \cdots + a_k I \right)$.

**Lemma 3.16** Let $A \in H[x]^{m \times m}$ has the Moore Penrose inverse $A^\dagger$ and $B = AA^\dagger$. Suppose the generalized characteristic polynomial of $A$:

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\[ g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m, \text{ where } a_m \in \mathbb{R}[x]. \]

Define, \( a_0 = 1 \). If \( P \) is the largest integer such that \( a_P \neq 0 \) and we construct the sequence \( A_0, \ldots, A_P \) as follows:

\[
\begin{align*}
A_0 &= 0 \\
A_1 &= \text{AA}'B \\
B_0 &= I \\
A_p &= \text{AA}'B_{p-1} \\
B_p &= A_p - q_{p-1}I
\end{align*}
\]

Then \( q_i(x) = -a_i(x), i = 0, \ldots, P \).

**Proof:** We will show \( q_i(x) = -a_i(x), i = 0, \ldots, P \) by mathematical induction. By the definition clearly, \( q_0 = -a_0 \) holds.

Now we assume that \( q_i(x) = -a_i(x) \) holds for all \( 1 \leq i < k \). Then

\[
A_k = \text{AA}'B_{k-1}
\]

\[
= BB_{k-1} = B(A_{k-1} - q_{k-1}I)
\]

\[
= B((B(A_{k-2} - q_{k-2}I) - q_{k-1}I)
\]

\[
= \cdots
\]

\[
= B^k + q_1B^{k-1} + q_2B^{k-2} + \cdots + q_{k-1}B
\]

\[
= B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B
\]

And thus \( \text{tr}(A_k) = \text{tr}(B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B) \).

Which by Lemma 3.15 is equal to \( -ka_k \). So, \( q_k = \frac{\text{tr}(A_k)}{k} = -a_k \).

Therefore, \( q_i(x) = -a_i(x) \) for all \( p \geq i \geq 0 \).

**REFERENCES**


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