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ON q-k-EP MATRICES

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ABSTRACT

The concept of range quaternion k-EP (q-k-EP) matrices is introduced as a special case of quaternion hermitian and generalization of EP matrices. Necessary and sufficient conditions are determined for a matrix to be q-k-EP$_r$ (q-k-EP and rank r). As an application, it is shown that the class of all q-k-EP matrices having the same range space form a group under multiplication.

Key words: *Moore-Penrose Inverse, Quaternion matrix, Rank of matrix, Range hermitian k-EP matrices*

1. INTRODUCTION

The algebra H of real quaternion, which is a four- dimensional non-commutative algebra over real number field R with canonical basis $1, i, j, k$ satisfying the conditions, $i^2 = j^2 = k^2 = ijk = -1$ that implies $ij = -ji = k, jk = -kj = i$ and $ki = -ik = j$.

The elements in H can be written in a unique way as, $\alpha = a + bi + cj + dk$, where a, b, c and d are real numbers, i.e., $H = \{ \alpha = a + bi + cj + dk \mid a, b, c, d \in R\}$.

The conjugate of α is defined as $\bar{\alpha} = a - bi - cj - dk$, and the norm $|\alpha| = \sqrt{\alpha\bar{\alpha}}$ for $0 \neq \alpha \in H$, $\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|^2}$.

We consider K is a permutation matrix associated with the permutation $k(x) = (S_n)$, where S = {1, 2,…,n}.

Also $K^2 = I, \bar{K} = K^T = K^* = K^{-1} = K$.

2. q-k-EP MATRICES

Definition: 2.1 Let $H[x]^{mxn}$ denote the set of all mxn matrices with entries from$H[x]$. For $A \in H[x]^{mxn}$, the conjugate $\bar{A} = \bar{A}_{ij}$. If $A = P + Qj$ with $P, Q \in H[x]^{mxn}$, then $\boldsymbol{\chi}_A = \begin{pmatrix} P & Q \\ -\bar{Q} & \bar{P} \end{pmatrix} \in C[x]^{2mx\,2n}$ denotes the complex adjoint of A.

Moreover, $A^T, A^* \in H[x]^{mxn}$ denotes the transpose and the conjugate transpose of A, respectively.

Definition: 2.2 $A^\dagger \in H[x]^{nxm}$ is called a Moore Penrose inverse of $A \in H[x]^{mxn}$, if it is a solution of the following system of equations, $AXA = A$, $XAX = X$, $(AX)^* = AX$, $(XA)^* = XA$. Note that we require that $A^\dagger$ must be in $H[x]^{nxm}$.'

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Definition: 2.3 A matrix $A \in H[x]^{mxn}$ is said to be q-k-EP if it satisfies the condition $Ax = 0 \Leftrightarrow A^{*}k(x) = 0$ or equivalently$N(A) = N(A^{*}K)$. Moreover, A is said to be k-EP$_r$, if it is k-EP and of rank r.

Definition: 2.4 A k-hermitian matrix A is q-k-EP, for if A is k -hermitian, then by [3, Result 2.1], $A = KA^{*}K$. Hence $N(A) = N(KA^{*}K) = N(A^{*}K)$, which implies A is q-k-EP. However, the converse need not be true.

Theorem: 2.5 For the following are equivalent:

(1) A is q-k-EP
(2) KA is EP
(3) AK is EP
(4) $A^{\dagger}$is q-k-EP
(5) $N(A) = N(A^{\dagger}K)$
(6) $N(A^{*}) = N(AK)$
(7) $R(A) = R(KA^{*})$
(8) $R(A^{*}) = R(KA)$
(9) $KA^{\dagger}K = AA^{\dagger}K$
(10) $A^{\dagger}AK = KAA^{\dagger}$
(11) $A = KA^{*}KH$ for a non singular nxn matrix H.
(12) $A = HKA^{*}K$ for a non singular nxn matrix H.
(13) $A^{*} = HKAK$ for a non singular nxn matrix H.
(14) $A^{*} = KAKH$ for a non singular nxn matrix H.
(15) $C_n = R(A) \oplus N(AK)$.
(16) $C_n = R(KA) \oplus N(A)$.

Proof: The proof for the equivalence of (1), (2) and (3) runs as follows:

A is q-k-EP $\Leftrightarrow N(A) = N(A^{*}K)$ (by Definition 2.3)
$\Leftrightarrow N(KA) = N(KA)^{*}$ [by (P.1)]
$\Leftrightarrow$ KA is EP (by Definition of EP matrix)
$\Leftrightarrow K(KA)K^{*}$ is EP (by [1, Lemma3])
$\Leftrightarrow$ AK is EP [by (P.1)]

Thus (1)$\Rightarrow$ (2) $\Rightarrow$(3) hold.

(2)$\Leftrightarrow$(4): KA is EP $\Leftrightarrow (KA)^{\dagger}$ is EP (by [2, P.163])
$\Leftrightarrow$ $A^{\dagger}K$ is EP [by (P.2)]
$\Leftrightarrow A^{\dagger}$is q-k-EP [by equivalence of (1) and (3) applied to $A^{\dagger}$]

Thus equivalence of (1) and (5) is proved.

Now we shall prove the equivalence of (1), (6) and (7) using $\rho(A) = \rho(A^{*}) = \rho(A^{*}K) = \rho(AK)$ in the following way:

A is q-k-EP $\Leftrightarrow N(A) = N(A^{*}K)$
$\Leftrightarrow N(A) \subseteq N(A^{*}K)$
$\Leftrightarrow A^{*}K = A^{*}KA^{-}A$ (by [2, P.21])
$\Leftrightarrow A^{*} = A^{*}KA^{-}AK$ (by [P.1])
$\Leftrightarrow A^{*} = A^{*}K^{-1}A^{-}AK$
$\Leftrightarrow A^{*} = A^{*}(AK)^{-}AK$ (by [P.2])
$\Leftrightarrow N(AK) \subseteq N(A^{*})$ (by [2, P.21])
$\Leftrightarrow N(A^{*}) = N(AK)$
$\Leftrightarrow R(A) = R(AK)^{*}$
$\Leftrightarrow R(A) = R(KA)^{*}$ (by [P.1])

Thus (1)$\Rightarrow$ (6) $\Rightarrow$(7) holds.

(1)$\Leftrightarrow$(8):

A is q-k-EP $\Leftrightarrow N(A) = N(A^{*}K)$
$\Leftrightarrow N(A) = N(KA)^{*}$
$\Leftrightarrow R(A^{*}) = R(KA)$

Thus equivalence of (1) and (8) is proved.

(3) ⇔ (9):

AK is EP $\Leftrightarrow (AK)(AK)^{\dagger} = (AK)^{\dagger}(AK)$ (by [2, P.166])

$\Leftrightarrow (AK)(KA^{\dagger}) = (KA^{\dagger})(AK)$ (by [P.2])

$\Leftrightarrow AA^{\dagger} = KA^{\dagger}AK$ (by [P.1])

$\Leftrightarrow AA^{\dagger}K = KA^{\dagger}A$

Thus equivalence of (3) and (9) is proved.

(9) ⇔ (10): Since by the property (P.1), $K^2 = I$, this equivalence follows by pre and post multiplying $KA^{\dagger}A = AA^{\dagger}K$ by K.

(2) ⇔ (11):

KA is EP $\Leftrightarrow (KA)^* = (KA)H_1$, for a non-singular nxn matrix H_1 (by [2, P.166])

$\Leftrightarrow A^*K = KAH_1$

$\Leftrightarrow KA^*K = AH_1$

$\Leftrightarrow A = KA^*KH$ where $H = {H_1}^{-1}$ is a non- singular nxn matrix.

Thus equivalence of (2) and (11) is proved.

(3) ⇔ (12):

AK is EP $\Leftrightarrow (AK)^* = H_1(AK)$, for a non-singular nxn matrix H_1(by [2, P.166])

$\Leftrightarrow KA^* = H_1AK$

$\Leftrightarrow KA^*K = H_1A$

$\Leftrightarrow A = {H_1}^{-1}KA^*K$

$\Leftrightarrow A = HKA^*K$ where $H = {H_1}^{-1}$ is a non- singular nxn matrix.

Thus equivalence of (3) and (12) is proved.

The equivalences (11) ⇔(13) and (12) ⇔(14) follow immediately by taking conjugate transpose and using $K = K^*$.

(13) ⇔ (16): $A^* = HKAK$ for a non singular nxn matrix H.

$\Leftrightarrow A^*A = H(KA)(KA)$

$\Leftrightarrow A^*A = H(KA)^2$

$\Leftrightarrow \rho(A^*A) = \rho(H(KA)^2)$

$\Leftrightarrow \rho(A^*A) = \rho((KA)^2)$

Over the complex field, A^*A and A have the same rank.

Therefore, $\rho((KA)^2) = \rho(A^*A) = \rho(A) = \rho(KA) \Leftrightarrow R(KA) \cap N(KA) = \{0\}$

$\Leftrightarrow R(KA) \cap N(A) = \{0\}$

$\Leftrightarrow H_n = R(KA) \oplus N(A)$.

Thus (13) ⇔ (16) holds.

(14) ⇔ (15): This can be proved along the lines and using $\rho(AA^*) = \rho(A)$. Hence the proof is omitted.

(16) ⇔ (1): If $H_n = R(KA) \oplus N(A)$, then $R(KA) \cap N(A) = \{0\}$.

For $x \in N(A), x \notin R(KA) \Leftrightarrow x \in R(KA)^{\perp} = N(KA)^* = N(A^*K)$.

Hence $N(A) \subseteq N(A^*K)$ and $\rho(A) = \rho(A^*K) \Rightarrow N(A) = N(A^*K) \Rightarrow A$ is q-k-EP.

Thus (1) holds. Similarly, we can prove (15) ⇒(1).

Remark: 2.6 [8] Let $A \in H[x]^{mxn}$ and $B \in H[x]^{nxl}$. Then

(i) $(AB)^* = B^*A^*$ and $AA^* = (AA^*)^*$

(ii) If A has a Moore- Penrose Inverse $A^{\dagger}$, then $(A^*)^{\dagger} = (A^{\dagger})^*$, $A^{\dagger}(A^{\dagger})^*A^* = A^{\dagger} = A^*(A^{\dagger})^*A^{\dagger}$ and $A^{\dagger}AA^* = A^* = A^*AA^{\dagger}$

(iii) If A has a Moore- Penrose Inverse $A^{\dagger}$, then $A^{\dagger}$ is unique.

(iv) Let A have the Moore- Penrose Inverse $A^{\dagger}$. If $U \in H[x]^{mxm}$ is a unitary matrix, then $(UA)^{\dagger} = A^{\dagger}U^*$.

For $x = (x_1, x_2, \dots, x_n)^T \in H[x]^{nxl}$. Let us define the function $k(x) = (x_{k(1)}, x_{k(2)}, \dots, x_{k(n)})^T \in H_n$. Since k is involutory, it can be verified that the associated permutation matrix k satisfy the following properties:

$K = K^T = K^{-1}$ and $k(x) = Kx$, (P.1)

$(KA)^\dagger = A^\dagger K$ and $(AK)^\dagger = KA^\dagger$ for $A \in H[x]^{nxn}$ (by [2, P.182]) (P.2)

Theorem: 2.7 Let $A \in H[x]^{nxn}$. Then any two of the following conditions imply the other one:

(1) A is EP
(2) A is q-k-EP
(3) $R(A) = R(KA)$

Proof: First we prove that whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds, then by [1, Theorem.1], A is EP implies $R(A) = R(A^*)$. Now by Theorem 2.5, A is q-k-EP $\Leftrightarrow R(A^*) = R(KA)$. Therefore, A is q-k-EP $\Leftrightarrow R(A) = R(KA)$.

This completes the proof of [(1) and (2)] $\Rightarrow$(3) and [(1) and (3)] $\Rightarrow$(2).

Now let us prove [(2) and (3)] $\Rightarrow$(1): Since A is q-k-EP, by Theorem 2.5, KA is EP. Hence, $R(KA) = R(KA)^*$. By using (3), we have $R(A) = R(KA) = R(KA)^* = R(A^*K) = R(A^*)$.

Again by [1, Theorem 1], A is EP. Thus (1) holds.

Note: 2.8 [8] Let $A \in H[x]^{mxn}$ have the Moore- Penrose Inverse $A^\dagger$. Consider A has a homomorphism from $H[x]^{nxl}$ to $H[x]^{mxl}$. Then Image (A) = Image (AA^*) = Image $(AA^\dagger)$ and Image (A^*) = Image (A^*A) = Image $(A^\dagger A)$.

Lemma: 2.9 [8] If $E \in H[x]^{mxm}$ is a symmetric projection, that is , $E = E^2 - E^*$, then $E \in H[x]^{mxm}$.

Proof: Let $f_1, f_2, \dots f_m$ be the entries on the first row of E. From, $E - E^*$, we may assume that $f_1 = \overline{f_1} \neq 0$. Then $E = E^2$ we have

$$f_1 = f_1\overline{f}_1 + \textstyle\sum_{i=2}^m f_i\overline{f}_i = {f_1}^2 + \sum_{i=2}^m f_i\overline{f}_i$$

Since $f_1 = \overline{f_1}$ the leading co-efficient of ${f_1}^2$ is a positive real number. Note that the leading co-efficient of $\sum_{i=2}^m f_i\overline{f}_i$ is also a positive real number. Thus,

$$\deg({f_1}^2) \geq \deg(f_1) = \deg\left({f_1}^2 + \sum_{i=2}^m f_i\overline{f}_i\right)$$
$$= \max\{\deg({f_1}^2), \deg(\textstyle\sum_{i=2}^m f_i\overline{f}_i)\} \geq \deg({f_1}^2)$$

This shows that $f_1 \in H$. Further more, 0=$\deg(f_1) = \deg(\sum_{i=2}^m f_i\overline{f}_i)$ and the leading co-efficient of $\{f_i\overline{f}_i\}\{\overline{f}_i \neq 0\}$ are positive reals imply that $f_i \in H$ for all $1 \leq i \leq m$. The same discussions can be done for the other rows of E. Therefore, $E \in H[x]^{mxm}$.

Lemma: 2.10 If $A \in H[x]^{nxn}$ is normal and AA^* is q-k-EP, Then A is q-k-EP.

Proof: Since A is normal, A is EP and AA^* is q-k-EP $\Leftrightarrow$ R $(AA^*) = R(KAA^*)$ implies $R(A) = R(KA)$. That A is q-k- EP Then follows from Theorem 2.7.

Lemma: 2.11 Let $E = E^* = E^2 \in H[x]^{nxn}$ be a hermitian idempotent that commutes with k, the permutation matrix associated with a fixed product of disjoint transpositions k is S_n . Then, $H_k(E) = \{A: A$ is q-k-EP and R $(A) = R(E)\}$ and forms a maximal subgroup of H_{nxn} containing E as identity.

Proof: Since E K=K E, by (P.1) and (P.2) we have E=KEK and $EE^\dagger = E^2 = E = (KE)(EK) = (KE)(KE)^\dagger$;

Hence R(E)=R(KE).

Since E is hermitian it is automatically EP. And by Theorem 2.7, E is K-EP and R(A) =R(E) =R(KE) $\Rightarrow$ [$AA^\dagger = EE^\dagger$=E] also $A^\dagger = E = (KE)(KE)^\dagger = KEE^\dagger K^\dagger = KAA^\dagger K^\dagger = (KA)(KA)^\dagger$.

Therefore R(A)=R(K A). Hence by Theorem 2.7, A is EP and $H_k(E) = H(E) = \{A: A \text{ is EP and } R(A) = R(E)\}$.

By [5, Theorem 2.1], $H_k(E)$ forms a maximal subgroup $H[x]^{nxn}$ containing E as identity.

3. EIGEN VALUES

Definition: 3.1 $A \in H[x]^{nxn}$ is hermitian, that is $A = AA^*$, if and only if there exists a unitary matrix $U \in H[x]^{nxn}$ such that $U^*AU = \text{diag}(d_1, d_2, \dots d_n)$, where d_i are the eigen values of A.

Lemma: 3.2 For $A \in H[x]^{nxn}$, A is k-EP $\Leftrightarrow$ $N(A) \subset N(P)$, where P is k-hermitian part of A.

Proof: If A is k-EP, then by Theorem2.5, KA is EP.

Since K is non-singular, $N(A) = N(KA) = N(KA)^* = N(A^*K) = N(KA^*K)$.

Then for $x \in N(A)$, both $Ax = 0$ and $KA^*Kx = 0$, which implies that $Px = \frac{1}{2}(A + KA^*K)x = 0$. Thus $N(A) \subseteq N(P)$. Conversely, $N(A) \subseteq N(P)$; Then Ax=0 implies Px=0 and hence Qx=0. Therefore, $N(A) \subseteq N(Q)$.

Thus $N(A) \subseteq N(P) \cap N(Q)$.

Since both P and Q are k-hermitian, and by [3, Result 2.1],

We have, $P=KP^*K$ and $Q=KQ^*K$.

Hence $N(P) = N(KP^*K) = N(P^*K)$ and $N(Q)=N(KQ^*K) = N(Q^*K)$.

Now $N(A) \subseteq N(P) \cap N(Q) = N(P^*K) \cap N(Q^*K) \subseteq N(P^* - iQ^*)K$.

Therefore, $N(A) \subseteq N(A^*K)$ and $\rho(A) = \rho(A^*K)$.

Hence, $N(A)=N(A^*K)$. Therefore, A is q-k-EP. Hence the theorem.

Lemma: 3.3 [8] Let $A \in H[x]^{mxn}$ .Then A has the Moore-Penrose inverse $A^\dagger$ if and only if $A = U\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ with $U \in H^{mxm}$ unitary and $A_1 A_1^* + A_2 A_2^*$, a unit in $H[x]^{rxr}$ with $r \leq \min\{m, n\}$.

Moreover, $A^\dagger = \begin{pmatrix} A_1^*(A_1 A_1^* + A_2 A_2^*)^{-1} & 0 \\ A_2^*(A_1 A_1^* + A_2 A_2^*)^{-1} & 0 \end{pmatrix} U^*$

Proof: If A has the Moore- Penrose Inverse $A^\dagger$, then $AA^\dagger = AA^\dagger AA^\dagger = (AA^\dagger)^2 = (AA^\dagger)^*$.

By Lemma 2.9, $AA^\dagger \in H^{mxm}$. $AA^\dagger$ is hermitian and hence, by Lemma 3.2, there exists a unitary matrix $U \in H^{mxm}$ such that $U^*AA^\dagger U = D$, where D is diagonal. Since, $D^2 = (U^* AA^\dagger U)(U^* AA^\dagger U) = U^* AA^\dagger AA^\dagger U = U^* AA^\dagger U = D$, the diagonal entries of D are either 1 or 0. Therefore, we can re arrange the rows of U so that $D = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ with $r \leq \min\{m, n\}$.

Set $A' = U^*A$. By Lemma 2.6, A' has its own generalized inverse $A'^\dagger$ and $A'A'^\dagger = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Set $A' = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, for arbitrary quaternion polynomial matrices $A_1 \in H[x]^{rxr}$, $A_2 \in H[x]^{rx(n-r)}$, $A_3 \in H[x]^{(m-r)xr}$ and $A_4 \in H[x]^{(m-r)x(n-r)}$. Since $A' = A' A'^\dagger A' = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$, we must have $A' = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ and therefore $A' A'^* = \begin{bmatrix} A_1 A_1^* + A_2 A_2^* & 0 \\ 0 & 0 \end{bmatrix}$

Similarly, $A'^\dagger = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}$, for some B_1 and B_2.

By Lemma 2.8, Image $(A' A'^*) = $ Image (A') = Image$(A' A' A'^\dagger)$= Image$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

This implies the surjectivity of $A_1 A_1^* + A_2 A_2^*$ on $H[x]^{rxl}$.

Therefore $A_1 A_1^* + A_2 A_2^*$ is a unit in $H[x]^{rxr}$ and $A = UA' = U\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$

Next we have that,

$$A'^{\dagger} = A'^{\dagger}(A'^{\dagger})^{*}A'^{*} = A'^{\dagger}(A'^{*})^{\dagger}A'^{*} = A'^{*}\left(A'A'^{*}\right)^{\dagger}$$
$$= \begin{bmatrix} A_1^{*} & 0 \\ A_2^{*} & 0 \end{bmatrix}\begin{bmatrix} (A_1 A_1^{*} + A_2 A_2^{*})^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A_1^{*}(A_1 A_1^{*} + A_2 A_2^{*})^{-1} & 0 \\ A_2^{*}(A_1 A_1^{*} + A_2 A_2^{*})^{-1} & 0 \end{bmatrix},$$

which gives $A^{\dagger} = \begin{bmatrix} A_1^{*}(A_1 A_1^{*} + A_2 A_2^{*})^{-1} & 0 \\ A_2^{*}(A_1 A_1^{*} + A_2 A_2^{*})^{-1} & 0 \end{bmatrix} U^{*}$. The converse can be proved by direct computation.

Lemma: 3.4 Let $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, □ is an rxr non-singular matrix. Then the following are equivalent.

1. B is k-EP_r
2. $R(KB) = R(B)$.
3. BB^* is k-EP_r
4. $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, where K_1, K_2 are permutation matrices of order r and $n - r$ respectively.
5. $K = K_1K_2$ where K_1 is the product of disjoint transpositions on $S_n = \{1,2,\dots,n\}$ leaving (r+1, r+2, …, n) fixed, and K_2 is the product of the disjoint transpositions leaving (1,2,…, r) fixed.

Proof: Since B is EP_r , the equivalence of (1) and (2) follows from Theorem 2.7.

(2) ⇔ (3): follows from Theorem 2.5.

(2) ⇔ (4): Let us partition, $K = \begin{bmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{bmatrix}$, Where K_1 is rxr.

Then R(KB)=R(B) $\Leftrightarrow (KB)(KB)^{\dagger} = BB^{\dagger}$
$$\Leftrightarrow KBB^{\dagger}K = BB^{\dagger}$$
$$\Leftrightarrow KBB^{\dagger} = BB^{\dagger}K$$
$$\Leftrightarrow K\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}K$$
$$\Leftrightarrow \begin{bmatrix} K_1 & 0 \\ K_3^T & 0 \end{bmatrix} = \begin{bmatrix} K_1 & K_3 \\ 0 & 0 \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = K$$

Thus equivalence of (2) and (4) holds. The equivalence of (4) and (5) is clear *from* the definition of K.

Lemma: 3.5 A matrix $A \in H[x]^{nxn}$ if q-k-EP_r if and only if there exists a unitary matrix U and an rxr nonsingular matrix F such that A =KU $\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} U^*$.

Proof: Let us assume that A is q-k-EP_r .Then by Theorem 2.5, $H_{n=}$ R(KA) ⊕ N(A). Choose an orthonormal basis $\{x_1, x_2, \dots, x_n\}$ of R(KA)= $R(A^*)$, and extend it to a basis$\{x_1, x_2, \dots x_r, x_{r+1} \dots, x_n\}$ of H_n where $\{x_{r+1}, \dots, x_n\}$ is an orthonormal basis of N(A).

If (u, v) denotes the usual inner product on H_n and $1 \leq i \leq r < j \leq n$ it follows that $x_1 \in R(KA) = R(A^*) \Rightarrow x_1A^*y$.

Therefore, $(x_i, x_j) = (A^*y, x_j) = (y, Ax_j) = 0$ [Since $x_j \in N(A)$]. Hence $\{x_1, x_2, \dots, x_n\}$ is an ortho normal basis of H_n. If we consider KA as the matrix of a linear transformation relative to any ortho normal basis of H_n, then $U^*KAU = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$, Where F is rxr nonsingular matrix, whence $A = KU\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}U^*$.

Conversely, if $A = KU\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}U^*$, $U^*KAU = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$.

But N(KA)= $N(KA)^*$,which implies KA is EP_r, and by Theorem 2.5, A is q-k-EP_r.

Lemma: 3.6 Let $A \in H_{nxn}$, Then A is q-k-EP_r with $K = K_1K_2$ (where K_1 and K_2 are as in Lemma 3.4) if and only if A is Unitarily q-k-similar to a diagonal block q-k-EP_r matrix $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where D is an rxr non-singular matrix.

Proof: Since A is q-k-EP_r by Lemma 3.5, there exists a unitary matrix U and an rxr non singular matrix F such that $A = (KUK)K \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} U^*$.

Since $K = K_1K_2$, the associated permutation matrix is $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$.

Hence, $A = (KUK)K \begin{bmatrix} K_1F & 0 \\ 0 & 0 \end{bmatrix} U^* = (KUK) \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$, where $D=K_1F$.

Thus, A is Unitarily q-k-similar to a diagonal block q-k-EP_r matrix $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where D is an rxr non-singular matrix.

Now, That B is q-k-EP_r follows from Theorem 3.4, $K = K_1K_2$ and $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$

Since, A is Unitarily q-k-similar to $B = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, there exists a unitary matrix U such that $A = KUK \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$. Since B is q-k-$EP_r$,

By Theorem 2.5, $KB=K\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = U^*KAU$ is EP_r.

By [1, Lemma 2], KA is EP_r. Now, A is q-k-EP follows from Theorem 2.5 and $\rho(A) = r$. Hence A is q-k-EP_r. The proof is complete.

Lemma: 3.7 Let $A \in H[x]^{nxn}$. Then eigen values of AA^* are real.

Proof: Let $B=AA^*$ and $\lambda \in H$ be an eigen value of B with corresponding eigen vector $X= (x_1, x_2, \dots, x_m)^T \neq 0$ such that $BX=X\lambda$. Then $X^*BX= X^*X\lambda$.

Note that $B=B^*$. We have that $X^*BX=\lambda^* X^*X$.

Thus, $X^*X\lambda = \lambda^* X^*X= (X^*X\lambda)^*$.

$$(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \lambda = (\sum \bar{x}_i x_i)\lambda$$
$$= ((\sum \bar{x}_i x_i)\lambda)^*$$
$$= \lambda^* (\sum \bar{x}_i x_i)^*$$
$$= \lambda^*(\sum \bar{x}_i x_i).$$

By a known lemma, $0\neq \sum \bar{x}_i x_i \in R[x]$.

The above equation gives $\lambda = \lambda^*$ which implies $\lambda \in R$.

Lemma: 3.8 If a is k-EP, then (λ, x) is a (k-eigen value, k-eigen vector) pair for A if and only if $(1/\lambda, k(x))$ is a (k-eigen value, k-eigen vector) pair for $A^\dagger$.

Proof: (λ, x) is a (k-eigen value, k-eigen vector) pair for A

$\Leftrightarrow Ax=\lambda kx$ (by [3, P.22])

$\Leftrightarrow KAx=\lambda x$ (by P.1)

$\Leftrightarrow (KA)^\dagger x=\frac{1}{\lambda}x$ (by [2, P.161])

$\Leftrightarrow A^\dagger Kx=\frac{1}{\lambda}x$ (by P.2)

$\Leftrightarrow A^\dagger k(x)=\frac{1}{\lambda}K(k(x))$

$\Leftrightarrow (1/\lambda, k(x))$ is a (k-eigen value, k-eigen vector) pair for $A^\dagger$.

Definition: 3.9 For $A \in H[x]^{mxn}$, let $B=AA^*$ and χ_B be its complex adjoint. Then $f_B(\lambda) = \det(\lambda I_{2m} - \chi_B)$ is called the characteristic polynomial of A.

Lemma: 3.10 Let $A \in H[x]^{mxn}$ and B=AA* . Then Then $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (R[x])[\lambda]$

Proof: We first show that $f_B(\lambda) \in (R[x])[\lambda]$. Note that B=AA*, we have
$\det((\lambda I_{2m} - \chi_B)^T) = \det(\lambda I_{2m} - \chi_B) = \det((\lambda I_{2m} - \chi_B)^*)$,

Thus $\det(\lambda I_{2m} - \chi_B) = \det(\overline{\lambda I_{2m} - \chi_B})$.

Therefore, $\det(\lambda I_{2m} - \chi_B) = f_B(\lambda) \in (R[x])[\lambda]$.

Next we show that $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (C[x])[\lambda]$.

Let B = P + Q j. For any fixed $1 \le i, j \le m$,

We have $B_{ij} = a + bi + cj + dk$, where a, b, c and $d \in R[x]$.

Since B is hermitian, $B_{ji} = a - bi - cj - dk$ and therefore $P_{ij} = a + bi, P_{ji} = a - bi$ and $Q_{ij} = c + di$, $Q_{ji} = c - di$.

So $P^T = \overline{P}$ and $Q = -Q^T$.

Therefore, $\chi_B = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} P & P \\ -\overline{Q} & P^T \end{pmatrix} \Rightarrow \lambda I_{2m} - \chi_B = \begin{pmatrix} \lambda I_m - P & Q \\ -\overline{Q} & \lambda I_m - P^T \end{pmatrix}$.

Next, we have $\begin{pmatrix} I_m & -I_m \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_m & 0 \\ I_m & I_m \end{pmatrix} \begin{pmatrix} I_m & -I_m \\ 0 & I_m \end{pmatrix} \begin{pmatrix} \lambda I_m - P & Q \\ -\overline{Q} & \lambda I_m - P^T \end{pmatrix} = \begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix}$.

Therefore, $f_B(\lambda) = \det \begin{pmatrix} \lambda I_m - P & Q \\ -\overline{Q} & \lambda I_m - P^T \end{pmatrix} = \det \begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix}$

Note that, $\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix}^T = -\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix}$ which implies that $\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix}$ is skew symmetric.

By [9], the determinant of $\begin{pmatrix} \overline{Q} & P^T - \lambda I_m \\ \lambda I_m - P & Q \end{pmatrix}$ also called its P fattian, can be written as the square of a polynomial in its entries.

Therefore, $f_B(\lambda) = g(\lambda)^2$, where $g(\lambda) \in (C[x])[\lambda]$.

Finally, we show that $g(\lambda) \in (R[x])[\lambda]$.

Suppose, otherwise, then $g(\lambda) = a(\lambda) + b(\lambda)i$, where $a(\lambda)$ and $b(\lambda) \in (R[x])[\lambda]$ with $b(\lambda) \neq 0$.

By (1), $g(\lambda)^2 = a(\lambda)^2 - b(\lambda)^2 + 2\, a(\lambda)b(\lambda)i \in (R[x])[\lambda]$.

Thus $a(\lambda) = 0$ and $f_B(\lambda) = (b(\lambda)i)^2 = b(\lambda)^2$,where $b(\lambda) \in (R[x])[\lambda]$.

For a fixed $x \in R$, Let $\lambda' I_{2m} - \chi_B \in H^{2mx2m}$ is diagonally dominant with non-negative diagonal entries and that $(b(x))(\lambda') \neq 0$.

Since, $\lambda' I_{2m} - \chi_B$ is also hermitian, $\lambda' I_{2m} - \chi_B$ is positive definite [10]. But $\det(\lambda' I_{2m} - \chi_B) = -(b(x))(\lambda')^2 < 0$, a contradiction. Therefore, b=0 and thus $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (R[x])[\lambda]$.

Lemma: 3.11 Let $A \in H[x]^{mxn}$, B=AA* and $f_B(\lambda) = g(\lambda)^2$. Then g(B)=0. We will call $g(\lambda)$ the generalized characteristic polynomial of A.

Proof: Note that $g(\lambda) \in (R[x])[\lambda]$, by Theorem 3.10.

Then $\chi_g(B) = g(\chi_B)$. Next $f_B(\chi_B) = 0$ by the Cayley Hamilton theorem for complex polynomial matrices [9].

Therefore, $g(\chi_B)$=0, and $0 = g(\chi_B) = \chi_g(B)$, that is g(B)=0.

Lemma: 3.12 Let $A \in H[x]^{mxn}$ has the Moore Penrose inverse $A^\dagger$. Set B=AA*. Then

(i) $B^{\dagger} = (A^*)^{\dagger} A^{\dagger}$ and $B^{\dagger}B = AA^{\dagger}$
(ii) $B^{\dagger}B = BB^{\dagger}$ and $(B^{\dagger}B)^2 = B^{\dagger}B$
(iii) $(B^{\dagger})^k = (B^k)^{\dagger}$ and $(B^{n-k})^{\dagger}(B^{n-k}) = B^{\dagger}B$ for any $k \in N$

Lemma: 3.13 Let $A \in H[x]^{mxn}$, $B \in H[x]^{pxq}$ and $A \in H[x]^{mxq}$. If $A^{\dagger}, B^{\dagger}$ both exists, then the quaternion polynomial matrix equation $AXB = C$ has a solution in $H[x]^{nxp}$ if and only if $AA^{\dagger} \subset B^{\dagger}B = C$, in which case the general solution is $X = A^{\dagger} \subset B^{\dagger} + Y - A^{\dagger}AYBB^{\dagger}$, where $Y \in H[x]^{nxp}$ is arbitrary.

Lemma: 3.14 Let $A \in H[x]^{mxn}$ has the Moore Penrose inverse $A^{\dagger}$ and $B=AA^*$. Suppose the generalized characteristic polynomial of A is: $g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \ldots + a_{m-1}\lambda + a_m$, where $a_i \in R[x]$. If k is the largest number such that $a_k \neq 0$, then the generalized inverse of A is given by $A^{\dagger} = -\frac{1}{a_k}A^*[B^{k-1} + a_1B^{k-2} + \cdots + a_{k-1}I]$. If $a_i = 0$, for all $1 \leq i \leq m$, then $A^{\dagger} = 0$.

Lemma: 3.15 Let $A \in H[x]^{mxn}$ has the Moore Penrose inverse $A^{\dagger}$ and Set $B=AA^*$. Then for $1 \leq k \leq m$, $tr[B^k + a_1B^{k-1} + \cdots + a_{k-1}B = -ka_k$, where the a_i arise from the generalized characteristic polynomial of A:
$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \ldots + a_{m-1}\lambda + a_m$

Proof: Let Y=yI where $y \in R$. We can write,
$$g(Y) = g(Y) - g(B)$$
$$= (Y-B)[Y^{m-1} + (B + a_1I)Y^{m-2} + \cdots + (B^{m-1} + a_1B^{m-2} + \cdots + a_mI)].$$

As long as y is not an eigen value of B, (yI-B) =Y-B is non-singular, so we can write:
$$(Y - B)^{-1}g(Y) = [Y^{m-1} + (B + a_1I)Y^{m-2} + \left(B^2 + a_1B + a_2I\right)Y^{m-3} \ldots + B^{m-1} + a_1B^{m-2} + \cdots + a_mI)].$$

Taking the traces gives:
$$tr[(Y - B)^{-1}g(Y)] = mY^{m-1} + tr(B + a_1I)Y^{m-2} + tr\left(B^2 + a_1B + a_2I\right)Y^{m-3} + \cdots + tr(B^{m-1} + a_1B^{m-2} + \cdots + a_mI)].$$

Let $C=(Y - B)^{-1}g(Y)$. Since $g(Y)=g(yI)=g(y)I$, $C= g(y)(Y - B)^{-1}$.
Therefore, tr $C= g(y)\, tr[(Y - B)^{-1}]$.

Let $\lambda_1, \ldots, \lambda_{m'}$ where $m' \leq m$, be all the non zero eigen values of B. $tr[(Y - B)^{-1}]$ is the sum of the eigen value of $[(Y - B)]^{-1}$.

We will show that these eigen values are the fractions $\frac{1}{y-\lambda_1}, \ldots, \frac{1}{y-\lambda_{m'}}$

Let ς be an eigen value of $(Y - B)^{-1}$ with corresponding eigen vector z such that: $(Y - B)^{-1}Z = Z\varsigma$, ς is real (by Lemma 3.7) and hence $(Y - B)Z = Z\frac{1}{\varsigma} \Rightarrow BZ = Z\left(Y - \frac{1}{\varsigma}\right)$.

Therefore, $Y = \frac{1}{\varsigma} = \lambda_i \Rightarrow \varsigma = \frac{1}{y-\lambda_i}$ for some $1 \leq i \leq m'$.

Since $g(y) = (y-\lambda_1)(y - \lambda_2) \ldots (y - \lambda_{m'})$. We have that $g'(y) = g(y)\left(\frac{1}{y-\lambda_1} + \cdots + \frac{1}{y-\lambda_{m'}}\right)$ and tr $C= g'(y)$. The derivative of g is also equal to $g'(y) = mY^{m-1} + a_1(m - 1)Y^{m-2} + \cdots + a_{m-1}$.

Therefore,
$$mY^{m-1} + a_1(m - 1)Y^{m-2} + \cdots + a_{m-1} = mY^{m-1} + tr(B + a_1I)Y^{m-2} + \cdots + tr(B^{m-1} + a_1B^{m-2} + \cdots + a_mI).$$

Comparing the co-efficient of Y^{m-k-1} on both sides, we obtain
$$a_k(m - k) = tr\left(B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B + a_kI\right)$$
$$= tr\left(B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B\right) + tr(a_kI)$$
And then $-ka_k = tr\left(B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B\right)$.

Lemma: 3.16 Let $A \in H[x]^{mxn}$ has the Moore Penrose inverse $A^{\dagger}$ and $B=AA^*$. Suppose the generalized characteristic polynomial of A:

$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \ldots + a_{m-1}\lambda + a_m$, where $a_m \in R[x]$.

Define, $a_0 = 1$. If P is the largest integer such that $a_P \neq 0$ and we construct the sequence $A_0, \ldots, A_P$ as follows:

$A_0 = 0$	$-1 = q_0$	$B_0 = I$
$A_1 = AA^*B$	$\frac{\text{tr } A_1}{1} = q_1$	$B_1 = A_1 - q_1 I$
$\vdots$	$\vdots$	$\vdots$
$A_{p-1} = AA^*B_{p-2}$	$\frac{\text{tr } A_{p-1}}{p-1} = q_{p-1}$	$B_{p-1} = A_{p-1} - q_{p-1} I$
$A_p = AA^*B_{p-1}$	$\frac{\text{tr } A_p}{p} = q_p$	$B_p = A_p - q_p I$

Then $q_i(x) = -a_i(x), i = 0, \ldots, P$.

Proof: We will show $q_i(x) = -a_i(x), i = 0, \ldots, P$ by mathematical induction. By the definition clearly, $q_0 = -a_0$ holds.

Now we assume that $q_i(x) = -a_i(x)$ holds for all $1 \leq i \leq k-1$. Then

$$\begin{aligned} A_k &= AA^*B_{k-1} \\ &= BB_{k-1} \\ &= B(A_{k-1} - q_{k-1}I) \\ &= B((B(A_{k-2} - q_{k-2}I) - q_{k-1}I) \\ &\vdots \\ &= B^k + q_1B^{k-1} + q_2B^{k-2} + \cdots + q_{k-1}B \\ &= B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B \quad . \end{aligned}$$

And thus $\text{tr}(A_k) = \text{tr}(B^k + a_1B^{k-1} + a_2B^{k-2} + \cdots + a_{k-1}B)$,

Which by Lemma 3.15 is equal to $-ka_k$. So, $q_k = \frac{\text{Tr}(A_k)}{k} = -a_k$.

Therefore, $q_i(x) = -a_i(x)$ for all $p \geq i \geq 0$.

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