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SOLUTION OF DIFFUSION EQUATION VIA INFINITESIMAL GROUP METHOD

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ABSTRACT

In this paper, the infinitesimal group transformation method is used to solve diffusion equation. Na & Hansen introduced this infinitesimal transformation method in terms of characteristic function method. With the help of this transformation the diffusion equation can be transformed into an Ordinary Differential Equation. The method presented in this paper is seen to be very general in nature. Basically, the method established the connection between one equation and all other equations under the contact transformation group of transformation, by the procedure of choosing different form of the characteristics function w.

Key Words: *Diffusion Equation, Infinitesimal Group Transformation, Characteristic function.*

1. INTRODUCTION

Characteristic function method is applicable to solve a wider variety of non linear problems. The characteristic function method, in general, is a class of transformation which reduces the number of independent variables in some system of partial differential equation by one. The characteristic function method is more convenient and more systematic since it express the infinitesimals of the group in terms of one or more functions called the characteristic functions, where each function depends only on one of the dependent variable. The characteristic function method is more transparent and requires less effort to arrive the final result.

2. MATHEMATICAL FORMULATION

Joseph Fourier formulated a very complete expansion of the theory of heat conduction. The law that bears his name is

$$q = -k\,\frac{du}{dy} \tag{2.1}$$

The constant, is called the thermal conductivity. We had to deal with a major problem that arises in heat conduction problems. The problem is that Fourier's law involves two dependent variables, u and q. To eliminate q and first solve for u, we introduce the first law of thermodynamics implicitly: Conservation of energy required that q was the same in each metallic slab.

To elimination of q from Fourier's law must now be done in a more general way. Consider a one-dimensional element, as shown, the net heat conduction out of the element during general unsteady heat flow is

$$q_{net}\,A = Q_{net} = -kA\frac{d^2u}{dy^2}\,\delta y \tag{2.2}$$

To eliminate the heat loss Q_{net} in favor of T, we use the general first law statement for closed,

$$Q_{net} = \frac{dU}{dt} = \rho cA\,\frac{d(u-u_{ref})}{dt}\,\delta y = \rho cA\,\frac{du}{dt}\,\delta y \tag{2.3}$$

Where ρ is the density of the slab and c is its specific heat capacity. Equations (2.2) and (2.3) can be combined to give

$$\frac{d^2u}{dy^2} = \frac{\rho c}{k}\,\frac{dT}{dt} = \frac{1}{\alpha}\frac{du}{dt} \tag{2.4}$$

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This is the one dimensional heat diffusion equation.

The heat diffusion equation includes a new property which is as important to transient heat conduction as k is to steady state conduction. This is the thermal diffusivity:

$$\propto = \frac{k}{\rho c} \quad \frac{j}{m.s.k} \; \frac{m^3}{kg} \; \frac{kg.k}{j} = \alpha \frac{m^2}{s} = 1 \tag{2.5}$$

3. PROBLEM SOLUTION

Na & Hansen express the infinitesimals of the groups in terms of characteristics function method, The Heat Equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \tag{3.1}$$

The Equation can be written as,

$$p_1 - p_{22} = 0 \tag{3.2}$$

Where $p_1 = \frac{\partial u}{\partial t}$, $p_2 = \frac{\partial u}{\partial y}$

$$p_{11} = \frac{\partial^2 u}{\partial t^2}, \qquad p_{22} = \frac{\partial^2 u}{\partial y^2}, \qquad p_{12} = \frac{\partial^2 u}{\partial t\, \partial y}$$

Infinitesimal group of transformations are taken as follow;

$$\begin{aligned}
\bar{t} &= t + \varepsilon\xi(t, y, u) + o(\varepsilon^2) \\
\bar{y} &= y + \varepsilon\eta(t, y, u) + o(\varepsilon^2) \\
\bar{u} &= u + \varepsilon\zeta(t, y, u) + o(\varepsilon^2) \\
\overline{p_1} &= p_1 + \varepsilon\pi_1(t, y, u,, p_1, p_2) + o(\varepsilon^2) \\
\overline{p_2} &= p_2 + \varepsilon\pi_2(t, y, u, p_1, p_2) + o(\varepsilon^2) \\
\overline{p_{22}} &= p_{22} + \varepsilon\pi_{22}\, \xi\, (t, y, u, p_1, p_2, p_{11}, p_{22}) + o(\varepsilon^2)
\end{aligned} \tag{3.3}$$

Where

$$\xi = \frac{\partial w}{\partial p_1}$$

$$\eta = \frac{\partial w}{\partial p_2}$$

$$\varsigma = p_1 \frac{\partial w}{\partial p_1} + p_2 \frac{\partial w}{\partial p_2} - w$$

$$\pi_1 = -\frac{\partial w}{\partial t} - p_1 \frac{\partial w}{\partial u}$$

$$\pi_2 = -\frac{\partial w}{\partial y} - p_2 \frac{\partial w}{\partial u}$$

Invariance under the transformations can be written as

$$\pi_1 - \pi_{22} = 0 \tag{3.4}$$

$$\therefore -\frac{\partial w}{\partial t} - p_1 \frac{\partial w}{\partial u} + \frac{\partial^2 w}{\partial y^2} + 2\, p_2 \frac{\partial^2 w}{\partial y \partial u} + p_2^2 \frac{\partial^2 w}{\partial u^2} + 2p_{12}\left(\frac{\partial^2 w}{\partial y \partial p_1} + p_2 \frac{\partial^2 w}{\partial u \partial p_1}\right) + 2p_{22}\left(\frac{\partial^2 w}{\partial y \partial p_2} + p_2 \frac{\partial^2 w}{\partial u \partial p_2}\right)$$

$$+ p_{12}^2 \frac{\partial^2 w}{\partial p_1^2} + 2p_{12}\, p_{22} \frac{\partial^2 w}{\partial p_1 \partial p_2} + p_{22}^2 \frac{\partial^2 w}{\partial p_2^2} + p_{22} \frac{\partial w}{\partial u} = 0 \tag{3.5}$$

w is not a function of p_{12}^2, the co-efficient of p_{12}^2 is zero.

$$\therefore \frac{\partial^2 w}{\partial p_1^2} = 0$$

W is independent of p_1 and it is linear function of p_1.

$$w = w_1(t, y, u, p_2) + p_1\, w_2(t, y, u, p_2) \tag{3.6}$$

The coefficient of p_{12} is zero,

$$2 \frac{\partial^2 w}{\partial y \partial\, p_1} + 2p_2 \frac{\partial^2 w}{\partial u \partial\, p_1} + 2 \frac{\partial^2 w}{\partial p_1 \partial p_2} = 0 \tag{3.7}$$

Using equation (3.5), we have

$$\frac{\partial w_2}{\partial y} + p_2 \frac{\partial w_2}{\partial u} + p_1 \frac{\partial w_2}{\partial p_2} = 0 \tag{3.8}$$

w_2 is not a function of p_1, coefficient of p_1 is must be zero.

$$\frac{\partial w_2}{\partial p_2} = 0 \tag{3.9}$$

$\therefore w_2$ is independent of p_2.

$$w_2 = w_2(t, y, u) \tag{3.10}$$

Similarly w_2 is independent of y and u also.

$$w_2 = w_2(t) \tag{3.11}$$

$$\therefore \; w = w_1(t, y, u, p_2) + p_1 \; w_2(t) \tag{3.12}$$

$$\therefore \; -\frac{\partial w}{\partial t} - p_1 \frac{\partial w}{\partial u} + \frac{\partial^2 w}{\partial y^2} + 2\, p_2 \frac{\partial^2 w}{\partial y \partial u} + p_2^2 \frac{\partial^2 w}{\partial u^2} + 2p_{12} \frac{\partial^2 w}{\partial y \partial p_1} + 2p_{12} p_2 \frac{\partial^2 w}{\partial u \partial p_1} + 2p_{22} \frac{\partial^2 w}{\partial y \partial p_2} + 2p_{22}\; p_2 \frac{\partial^2 w}{\partial u \partial p_2} + p_{12}^2 \frac{\partial^2 w}{\partial p_1^2} + 2p_{12}\; p_{22} \frac{\partial^2 w}{\partial p_1 \partial p_2} + p_{22}^2 \frac{\partial^2 w}{\partial p_2^2} + p_{22} \frac{\partial w}{\partial u} = 0$$

$$-\frac{\partial w_1}{\partial t} - p_1 \frac{\partial w_2}{\partial t} - p_1 \frac{\partial w_1}{\partial u} - p_1^2 \frac{\partial w_2}{\partial u} + \frac{\partial^2 w_1}{\partial y^2} + p_1 \frac{\partial^2 w_2}{\partial y^2} + 2p_2 \frac{\partial^2 w_1}{\partial y\, \partial u} + 2p_1\, p_2 \frac{\partial^2 w_2}{\partial y\, \partial u} + p_2^2 \frac{\partial^2 w_1}{\partial u^2} + p_1\, p_2^2 \frac{\partial^2 w_2}{\partial u^2} + 2\, p_{12} \frac{\partial w_2}{\partial y} + 2\, p_{12}\, p_2 \frac{\partial w_2}{\partial u} + 2\, p_{22} \frac{\partial^2 w_1}{\partial y \partial p_2} + 2\, p_{22}\, p_1 \frac{\partial^2 w_2}{\partial y\, \partial p_2} + 2\; p_{22}\, p_2 \frac{\partial^2 w_1}{\partial u \partial p_2} + 2\; p_{22}\, p_2\, p_1 \frac{\partial^2 w_2}{\partial u \partial p_2} + p_{12}^2 \frac{\partial w_2}{\partial p_1} + 2\, p_{12}\; p_{22} \frac{\partial w_2}{\partial p_2} + p_{22}^2 \frac{\partial^2 w_1}{\partial p_2^2} + p_{22}^2\, p_1 \frac{\partial^2 w_2}{\partial p_2^2} + p_{22} \frac{\partial w_1}{\partial u} + p_1\; p_{22} \frac{\partial^2 w_2}{\partial u^2} = 0 \tag{3.13}$$

Taking coefficients of p,

$$p_1^0: \; -\frac{\partial w_1}{\partial t} + \frac{\partial^2 w_1}{\partial y^2} + 2p_2 \frac{\partial^2 w_1}{\partial y\, \partial u} + p_2^2 \frac{\partial^2 w_1}{\partial u^2} = 0 \tag{3.14}$$

$$p_1^1: \; -\frac{\partial w_2}{\partial t} - \frac{\partial w_1}{\partial u} + \frac{\partial^2 w_2}{\partial y^2} = 0 \tag{3.15}$$

$$p_1^2: \; \frac{\partial^2 w_2}{\partial u^2} = 0 \tag{3.16}$$

$$p_1\, p_2: \; \frac{\partial^2 w_2}{\partial y\, \partial u} = 0 \tag{3.17}$$

$$p_{22}: \; 2\frac{\partial^2 w_1}{\partial y \partial\, p_2} + \frac{\partial^2 w_1}{\partial u} = 0 \tag{3.18}$$

$$p_{22}^2: \; \frac{\partial^2 w_1}{\partial p_2^2} = 0 \tag{3.19}$$

From equation (3.19), we have

$$\therefore \; w_1 = w_{11} + p_2\; w_{12} \tag{3.20}$$

From equation (3.18), we have

$$2\frac{\partial^2 w_1}{\partial y \partial\, p_2} + \frac{\partial^2 w_1}{\partial u} = 0 \tag{3.21}$$

$$2\, \frac{\partial w_{12}}{\partial y} + \frac{\partial^2 w_1}{\partial u} = 0 \tag{3.22}$$

W is depends only on y.

$$\therefore \; w_{12} = w_{121} + w_{122}\, y \tag{3.23}$$

But
$$w = w_1(t, y, u, p_2) + p_1\; w_2(t) \tag{3.24}$$

$$w = w_{11} + p_2\; [w_{121} + w_{122}\, y] + p_1\; w_2 \tag{3.25}$$

Now from equation (3.14), the coefficients of p_2 are;

$$p_2^0: \; -\frac{\partial w_1}{\partial t} + \frac{\partial^2 w_1}{\partial y^2} = 0 \tag{3.26}$$

$$p_2^1: \; 2\, \frac{\partial^2 w_1}{\partial y\, \partial u} = 0 \tag{3.27}$$

$$p_2^2: \; \frac{\partial^2 w_1}{\partial u^2} = 0 \tag{3.28}$$

Now
$$p_2^2: \; \frac{\partial^2 w_1}{\partial u^2} = 0 \tag{3.29}$$

$$\therefore \; w_{11} = w_{111} + w_{112}\, u \tag{3.30}$$

Also,
$$p_2^2: \; \frac{\partial^2 w_1}{\partial u^2} = 0$$

$$\therefore \; \frac{\partial^2 w_{11}}{\partial u^2} = 0 \tag{3.31}$$

$\therefore$ w_{11} is independent of u.

$$\therefore \; w_{11} = w_{111} + w_{112}\, u \tag{3.32}$$

For
$$p_2^1: \; -\frac{dw_{121}}{dt} - \frac{dw_{122}}{dt} y + 2\, \frac{\partial^2 w_{11}}{\partial y\, \partial u} = 0 \tag{3.33}$$

Here $\frac{\partial^2 w_1}{\partial y\ \partial u} = \frac{\partial w_{112}}{\partial y}$ (3.34)

$\therefore$ w_{112} can be written as,

$$\therefore\ w_{112} = w_{1121} + w_{1122}\, y + w_{1123}\, y^2 \tag{3.35}$$

$$-\frac{dw_{121}}{dt} - \frac{dw_{122}}{dt} y + 2\, w_{1122} + 4y w_{1123} = 0 \tag{3.36}$$

All the W's are independent of y,

$$\therefore\ -\frac{dw_{121}}{dt} + w_{1122} = 0 \tag{3.37}$$

$$\therefore\ -\frac{dw_{122}}{dt} + 4w_{1123} = 0 \tag{3.38}$$

Now from equation (3.29) and (3.30),

$$-\frac{\partial}{\partial t}\ (w_{111} + w_{112}\ u) + \frac{\partial^2}{\partial y^2}\ (w_{111} + w_{112}\ u) = 0 \tag{3.39}$$

$$-\frac{\partial w_{111}}{\partial t} - \left(\frac{\partial}{\partial t}\ w_{112}\right) u + \frac{\partial^2 w_{111}}{\partial y^2} + \left(\frac{\partial^2}{\partial y^2} w_{112}\right) u = 0$$

$$\left(-\frac{\partial w_{111}}{\partial t} + \frac{\partial^2 w_{111}}{\partial y^2}\right) - \left(\frac{\partial w_{1121}}{\partial t} - 2\, w_{1123}\right) u - \frac{\partial w_{1122}}{\partial t}\ yu - \frac{\partial w_{1123}}{\partial t} y^2 u = 0 \tag{3.40}$$

The coefficient of u^0, u^1, uy, uy^2 are zero.

$$\frac{\partial w_{1122}}{\partial t} = 0 \quad \Rightarrow \quad w_{1122} = c_1 \tag{3.41}$$

$$\frac{\partial w_{1123}}{\partial t} = 0 \quad \Rightarrow \quad w_{1123} = c_2 \tag{3.42}$$

$$\frac{\partial w_{1121}}{\partial t} - 2\, w_{1123} = 0 \quad \Rightarrow \quad w_{1121} = 2c_2 t + c_3 \tag{3.43}$$

From equation (3.37),

$$w_{121} = 2c_1 t + c_4 \tag{3.44}$$

From equation (3.38),

$$w_{122} = 4c_2 t + c_5 \tag{3.45}$$

Now $-\frac{\partial w_2}{\partial t} + 2w_{122} = 0$

$$w_2 = 2w_{122} t + c_6$$

$$w_2 = 8\, c_2 t^2 + 2t\, c_5 + c_6 \tag{3.46}$$

$$w = w_{111} + [2c_2 t + c_3 + c_1 y + c_2 y^2]u + [2c_1 t + c_4 + (4c_2 t + c_5)y]\, p_2 + [8\, c_2 t^2 + 2t\, c_5 + c_6]p_1 \tag{3.47}$$

The characteristic equation is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = \frac{du}{\zeta} \tag{3.48}$$

$$\frac{dt}{8\, c_2 t^2 + 2t\, c_5 + c_6} = \frac{dy}{2c_1 t + c_4 + (4c_2 t + c_5)y} = \frac{du}{-w_{111} - [2c_2 t + c_3 + c_1 y + c_2 y^2]u}$$

$$w_{111} = c_1 = c_2 = c_4 = c_6 = 0$$

$$\therefore\ \frac{dt}{2c_5 t} = \frac{dy}{c_5 y} = \frac{du}{-c_3 u} \tag{3.49}$$

Let $\frac{dt}{2c_5 t} = \frac{dy}{c_5 y}$

$$\therefore\ \eta = \frac{y}{\sqrt{t}} \tag{3.50}$$

Let $\frac{dt}{2c_5 t} = \frac{du}{-c_3 u}$

$$\therefore\ u = t^{\alpha} f(\eta) \tag{3.51}$$

Now $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2}$ is transformed into the below equations

SOLUTIONS OF THE DIFFUSION EQUATIONS

Constant	Similarity Variables	Transformed Equation
$c_1 = c_2 = c_4 = c_6 = 0$	$\eta = \frac{y}{\sqrt{t}}\ ,\ f(\eta) = \frac{u}{t^{\alpha}}$	$f'' = \alpha f - \frac{1}{2}\eta f'$
$c_1 = c_2 = c_4 = c_5 = 0$	$\eta = y\ ,\ f(\eta) = \frac{u}{\exp(\beta t)}$	$f'' - \beta f = 0$
$c_1 = c_2 = c_5 = 0$	$\eta = y - \frac{c_4}{c_6}\, t\ ,\ f(\eta) = u\ \exp(c_3/c_6)\, t$	$f'' + \frac{c_4}{c_6} f' + c_3/c_6\, f = 0$
$c_1 = c_3 = c_4 = c_5 = c_6 = 0$	$\eta = \frac{y}{t}\ ,\ f(\eta) = u\sqrt{y}\exp\left[\left(\frac{y^2}{4}\right)/t\right]$	$f'' - \eta f' + \frac{3}{4} f = 0$
$c_1 = c_2 = c_3 = c_4 = c_6 = 0$	$\eta = \frac{y}{\sqrt{t}},\ f(\eta) = u + [a(2t + y^2)]/4c_5$	$f'' + \frac{1}{2}\,\eta f' + = 0$

4. CONCLUSION

The carried out analysis in this paper, shows the effectiveness of the method in obtaining invariant solution of the diffusion equation. The characteristic function method shows clear simplifications, systematic procedure as well as elegance in the derivation based on the methods of invariance analysis as it is illustrated in the present work.

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