FINITE DIMENSIONAL FUZZY ANTI 2- NORMED LINEAR SPACE

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ABSTRACT

In this paper we have generalized fuzzy anti 2-norm by introducing t-conorm in the earlier definition. The Riesz lemma and a few properties of finite dimensional fuzzy anti 2-normed linear space has been established with respect to t-conorm }.

Keywords: Fuzzy anti 2-norm, α -2-norm, Riesz lemma.

INTRODUCTION


In the present paper we have modified the definition of fuzzy anti 2-normed linear space. The Riesz lemma and important properties of finite dimensional fuzzy anti 2-normed linear space has been established with respect to t-conorm }.

PRELIMINARIES

Definition 2.1[10]: A binary operation } : [0,1] × [0,1] → [0,1] is a t-conorm if } satisfies the following condition:

(i) } is commutative and associative,
(ii) a } 0 = a, ∀a ∈ [0,1],
(iii) a } b ≤ c } d, whenever a ≤ c, b ≤ d and a, b, c, d ∈ [0,1].

Example: (i) a } b = a+b-ab (ii) a } b = max {a, b} (iii) a } b = {a+b, 1}

Definition 2.2[1]: Let X be a linear space over a real field F. A fuzzy subset } of X × X × R is called a fuzzy anti 2-norm on X if and only if it satisfies,

(Fa2-N1) for all t ∈ R with t ≤ 0, } (x₁, x₂, t) = 1

(Fa2-N2) for all t ∈ R with t > 0, } (x₁, x₂, t) = 0 if and only if x₁ and x₂ are linearly dependent.

(Fa2-N3) } (x₁, x₂, t) is invariant under any permutation.

(Fa2-N4) for all t ∈ R with t > 0, } (x₁, cx₂, t) = } (x₁, x₂, t/c) if c ≠ 0, c ∈ F

(Fa2-N5) for s,t ∈ R with t > 0 all } (x₁, x₂ + x₂', s + t) ≤ max } (x₁, x₂, s), } (x₁, x₂', t)

(Fa2-N6) } (x₁, x₂, t) is non-increasing function of t ∈ R and \( \lim_{t \to \infty} } (x₁, x₂, t) = 0. \)
Then \((X, N^*)\) is called a fuzzy anti 2-normed linear space. The following condition of fuzzy anti 2-norm \(N^*\) will be required later on,

\((\text{Fa2-N7})\) for \(t \in R\) with \(t > 0\), \(N^*(x_1, x_2, t) < 1\), \(\forall t > 0 \Rightarrow x_1 \) and \(x_2 \) are linearly dependent.

**Definition 2.3[1]:** Let \((X, N^*)\) be a fuzzy anti 2-normed linear space. A sequence \(\{x_n\} \) in \(X\) is said to be convergent to \(x \in X\) if \(\exists \) an integer \(n_0 \in N\) such that \(N^*(x_n - x, x, t) < r\), \(\forall n \geq n_0\).

**Definition 2.4[1]:** Let \((X, N^*)\) be a fuzzy anti 2-normed linear space. A sequence \(\{x_n\} \) in \(X\) is said to be cauchy sequence if \(\exists \) an integer \(n_0 \in N\) such that \(N^*(x_n - x_n, x, t) < r\), for all \(n \geq n_0\), \(p = 1, 2, 3, \ldots\)

**Definition 2.5[3]:** A subset \(A\) of a fuzzy anti 2-normed linear space \((X, N^*)\) is said to be bounded iff \(\exists t > 0\), \(N^*(Ax, x, t) < r\), \(\forall x, y \in A\).

**Definition 2.6[3]:** Let \((X, N^*)\) be a fuzzy anti 2-normed linear space. A subset \(B\) of \(X\) is said to be closed if any sequence \(\{x_n\} \) in \(B\) converges to \(x \in X\) that is \(\lim_{n \to \infty} N^*(x_n - x) = 0\), \(\forall t > 0 \Rightarrow x, y \in B\).

**Definition 2.7[1]:** A subset \(A\) of a fuzzy anti 2-normed linear space \((X, N^*)\) is said to be compact if any sequence \(\{x_n\} \) in \(A\) has a subsequence converging to an element of \(A\).

### 3. FUZZY ANTI 2-NORMED LINEAR SPACE

In this section we have modified the definition of fuzzy anti 2-norm with respect to a t-conorm \(\diamond\) and deduced some important results.

**Definition 3.1:** Let \(X\) be a linear space over a real field \(F\). A fuzzy subset \(N^*\) of \(X \times X \times R\) is called a fuzzy anti 2-norm on \(X\) if and only if it satisfies,

\((\text{Fa2-N1})\) for all \(t \in R\) with \(t \leq 0\), \(N^*(x_1, x_2, t) = 1\)

\((\text{Fa2-N2})\) for all \(t \in R\) with \(t > 0\), \(N^*(x_1, x_2, t) = 0\) if and only if \(x_1\) and \(x_2\) are linearly dependent.

\((\text{Fa2-N3})\) \(N^*(x_1, x_2, t)\) is invariant under any permutation of \(x_1\) and \(x_2\).

\((\text{Fa2-N4})\) for all \(t \in R\) with \(t > 0\)

\[N^*(x_1, cx_2, t) = N^*(x_1, x_2, t)\]

if \(c \neq 0, c \in F\)

\((\text{Fa2-N5})\) for \(s, t \in R\) with \(t > 0\) all \(N^*(x_1, x_2 + x_2, t) \leq N^*(x_1, x_2, t)\)

\((\text{Fa2-N6})\) \(N^*(x_1, x_2, t)\) is non-increasing function of \(t \in R\) and \(\lim_{t \to \infty} N^*(x_1, x_2, t) = 0\)

We further assume that for a fuzzy anti 2-normed linear space \((X, N^*)\),

\((\text{Fa2-N7})\) for all \(t \in R\) with \(t > 0\), \(N^*(x_1, x_2, t) < 1\), \(\forall t > 0 \Rightarrow x_1\) and \(x_2\) are linearly dependent.

\((\text{Fa2-N8})\) \(N^*(x_1, x_2, t)\) is a continuous function on \(R\) and strictly decreasing on the subset \(\{t : 0 < N^*(x_1, x_2, t) < 1\}\) of \(R\).

\((\text{Fa2-N9})\) \(a \diamond a = a\), \(\forall a \in [0, 1]\)

**Remark 3.1:** Let \(N^*\) be a fuzzy anti 2-norm on \(X\) then \(N^*(x_1, x_2, t)\) is non-increasing with respect to \(t\) for each \(x_1, x_2 \in X\).

**Proof:** Let \(t < s\). Then \(k = s - t > 0\), we have

\[N^*(x_1, x_2, t) = N^*(x_1, x_2, t) \diamond 0\] (by property of t-conorm)

\[= N^*(x_1, x_2, t) \diamond N^*(0, 0, k) \succeq N^*(x_1, x_2, t + k) = N^*(x_1, x_2, s)\]

Hence Proved
Example 3.1: Let \((X, \|\cdot\|)\) be 2-normed linear space and define \(a \triangleq b = a + b - ab\). Define \(N^* : X \times X \times R \to [0,1]\) by
\[
N^*(x_1, x_2, t) = \begin{cases} 
0, & \text{if } t > \|x_1, x_2\| \\
\frac{|x_1, x_2|}{t + |x_1, x_2|}, & \text{if } t \leq \|x_1, x_2\|, t > 0 \\
1, & \text{if } t \leq 0 
\end{cases}
\]
Then \(N^*\) is a fuzzy anti 2-norm on \(X\) with respect to the \(t\)-conorm \(\triangleleft\) and \((X, N^*)\) is a fuzzy anti 2-normed linear space with respect to the \(t\)-conorm \(\triangleleft\).

Solution:
(i) \(\forall x_1, x_2 \in X \times X\) and \(\forall t \in R, t \leq 0\) we have \(N^*(x_1, x_2, t) = 1\).
(ii) \(\forall t \in R, t \leq 0\) if \(x_1, x_2\) are linearly dependent then \(\|x_1, x_2\| = 0\) so \(N^*(x_1, x_2, t) = 0\) with \(t > 0 \Rightarrow \|x_1, x_2\| < t\), \(\forall t > 0 \Rightarrow \|x_1, x_2\| = 0 \Rightarrow x_1, x_2\) are linearly dependent.
(iii) It is obvious that \(N^*(x_1, x_2, t)\) is invariant under any permutation.
(iv) If \(N^*(x_1, cx_2, t) = 0 \Leftrightarrow t > \|x_1, cx_2\| \Leftrightarrow t > |c| \|x_1, x_2\| \Leftrightarrow \frac{t}{|c|} > \|x_1, x_2\| \Rightarrow N^*(x_1, x_2, \frac{t}{|c|}) = 0\).
\[
N^*(x_1, x_2, t) = 1 \Leftrightarrow t \leq \|x_1, x_2\| \Leftrightarrow t \leq |t| \|x_1, x_2\| \Leftrightarrow \frac{t}{|t|} \leq \|x_1, x_2\| \Rightarrow N^*(x_1, x_2, \frac{t}{|t|}) = 1.
\]
(v) \(N^*(x_1, x_2, s) \triangleleft N^*(x_1, x_2, t) = N^*(x_1, x_2, s) + N^*(x_1, x_2, t) - N^*(x_1, x_2, s) N^*(x_1, x_2, t)\).
If \(s > \|x_1, x_2\|\) and \(t > \|x_1, x_2\|\) so \(s + t > \|x_1, x_2\| + \|x_1, x_2\|\) then \(N^*(x_1, x_2 + x_2', s + t) = 0\) and \(N^*(x_1, x_2, s) \triangleleft N^*(x_1, x_2, t) = 0 + 0 - 0 = 0\).
So \(N^*(x_1, x_2 + x_2', s + t) = N^*(x_1, x_2, s) N^*(x_1, x_2, t)\).
If \(s > \|x_1, x_2\|\) and \(t \leq \|x_1, x_2\|\) then \(N^*(x_1, x_2, s) \triangleleft N^*(x_1, x_2', t) = 0 + 1 = 1\).
If \(s \leq \|x_1, x_2\|\) and \(t \leq \|x_1, x_2\|\) then \(N^*(x_1, x_2, s) \triangleleft N^*(x_1, x_2', t) = 1 + 0 = 1\).
Then in all the above three cases,
\(N^*(x_1, x_2, s) \triangleleft N^*(x_1, x_2', t) = 1 \geq N^*(x_1, x_2 + x_2', s + t)\).
Thus \(N^*(x_1, x_2 + x_2', s + t) \leq N^*(x_1, x_2, s) N^*(x_1, x_2, t)\).
(vi) From the definition if \(t > \|x_1, x_2\|\), then \(\lim_{t \to \infty} N^*(x_1, x_2, t) = 0\). Thus \((X, N^*)\) is a fuzzy anti 2-normed linear space with respect to the \(t\)-conorm \(\triangleleft\).

Example 3.2: Let \((X, \|\cdot\|)\) be 2-normed linear space and define \(a \triangleq b = \min\{a+b, 1\}\). Define \(N^* : X \times X \times R \to [0,1]\) by
\[
N^*(x_1, x_2, t) = \begin{cases} 
0, & \text{if } t > \|x_1, x_2\| \\
\frac{|x_1, x_2|}{t + |x_1, x_2|}, & \text{if } t \leq \|x_1, x_2\|, t > 0 \\
1, & \text{if } t \leq 0 
\end{cases}
\]
Then \(N^*\) is a fuzzy anti 2-norm on \(X\) with respect to the \(t\)-conorm \(\triangleleft\) and \((X, N^*)\) is a fuzzy anti 2-normed linear space with respect to the \(t\)-conorm \(\triangleleft\).

Solution:
(i) From the definition we have \(N^*(x_1, x_2, t) = 1\) if \(\forall t \in R, t \leq 0\).
(ii) If \( t > 0 \) and \( t \leq \|x_1, x_2\| \) the \( N^*(x_1, x_2, t) = \frac{|x_1, x_2|}{t + \|x_1, x_2\|} \) if \( x_1, x_2 \) are linearly dependent so \( x_1, x_2 = 0 \) therefore \( N^*(x_1, x_2, t) = 0 \).

Conversely, \( N^*(x_1, x_2, t) = 0 \) then \( t > \|x_1, x_2\| \) \( \forall t \Rightarrow \|x_1, x_2\| = 0 \) , so \( x_1, x_2 \) are linearly dependent.

(iii) It is obvious that \( N^*(x_1, x_2, t) \) is invariant under any permutation of \( x_1 \) and \( x_2 \).

(iv) If \( N^*(x_1, c x_2, t) = 0 \) \( \Leftrightarrow \|x_1, c x_2\| \) \( \Leftrightarrow \left\| \frac{t}{\|x_1, c x_2\|} \right\| \leq \left\| \frac{t}{\|x_1, x_2\|} \right\| \) \( \Leftrightarrow N^*(x_1, x_2, t) = 0 \).

If \( N^*(x_1, c x_2, t) = \frac{\|x_1, c x_2\|}{t + \|x_1, c x_2\|} \) \( \Leftrightarrow \|x_1, c x_2\| \leq \|x_1, x_2\| \) \( \Leftrightarrow N^*(x_1, x_2, t) = \frac{\|x_1, x_2\|}{t + \|x_1, x_2\|} = \frac{\|x_1, c x_2\|}{t + \|x_1, c x_2\|} \)

(v) \( N^*(x_1, x_2, s) * N^*(x_1, x_2', t) = \min \left\{ N^*(x_1, x_2, s) + N^*(x_1, x_2', t), 1 \right\} \) If \( \|x_1, x_2\| \geq s \) and \( \|x_1, x_2'\| \geq t \) then

\[
N^*(x_1, x_2, s) + N^*(x_1, x_2', t) = \frac{\|x_1, x_2\|}{s + \|x_1, x_2\|} + \frac{\|x_1, x_2'\|}{s + \|x_1, x_2'\|} \geq 1
\]

Since \( s + \|x_1, x_2\| > st \).

In this case \( N^*(x_1, x_2, s) * N^*(x_1, x_2', t) = 1 \geq N^*(x_1, x_2 + x_2', s + t) \).

If \( \|x_1, x_2\| \geq s \) and \( \|x_1, x_2'\| < t \) then either \( \|x_1, x_2 + x_2'\| \geq s + t \) or \( \|x_1, x_2 + x_2'\| < s + t \).

Now, \( N^*(x_1, x_2, s) + N^*(x_1, x_2', t) = \frac{\|x_1, x_2\|}{s + \|x_1, x_2\|} + \frac{\|x_1, x_2'\|}{s + \|x_1, x_2\|} < 1 \).

Hence \( N^*(x_1, x_2, s) * N^*(x_1, x_2', t) = \frac{\|x_1, x_2\|}{s + \|x_1, x_2\|} \).

If \( \|x_1, x_2 + x_2'\| \geq s + t \) then consider

\[
N^*(x_1, x_2 + x_2', s + t) - N^*(x_1, x_2, s) * N^*(x_1, x_2', t) = \frac{\|x_1, x_2 + x_2'\|}{s + t + \|x_1, x_2 + x_2'\|} - \frac{\|x_1, x_2\|}{s + \|x_1, x_2\|} \leq \frac{\|x_1, x_2\|}{s + t + \|x_1, x_2\|} - \frac{\|x_1, x_2\|}{s + \|x_1, x_2\|} = \frac{s \|x_1, x_2\| - t \|x_1, x_2\|}{s + t + \|x_1, x_2\|} (s + \|x_1, x_2\|) \leq 0, \text{ Since } \|x_1, x_2\| \leq t, 0, \text{ Since } s \leq \|x_1, x_2\| \text{ so } st < \|x_1, x_2\| .
\]

So, \( N^*(x_1, x_2 + x_2', s + t) < N^*(x_1, x_2, s) * N^*(x_1, x_2', t) \).

If \( \|x_1, x_2 + x_2'\| < s + t \) then

\[
N^*(x_1, x_2 + x_2', s + t) = \frac{\|x_1, x_2\|}{s + \|x_1, x_2\|} = N^*(x_1, x_2, s) * N^*(x_1, x_2', t)
\]
If \[ \|x_1, x_2\| < s \text{ and } \|x_1, x_2\| \geq t \] then in the similar way we can show that
\[ N^*(x_1, x_2 + x'_2, s + t) \leq N^*(x_1, x_2, s) \odot N^*(x_1, x'_2, t). \]

If \[ \|x_1, x_2\| < s \text{ and } \|x_1, x'_2\| \geq t \] then \[ N^*(x_1, x_2, s) + N^*(x_1, x'_2, t) = 0 + 0 < 1. \]
Therefore, \[ N^*(x_1, x_2, s) \odot N^*(x_1, x'_2, t) = 0. \]

Also \[ \|x_1, x_2 + x'_2\| \leq \|x_1, x_2\| + \|x_1, x'_2\| < s + t \] and \[ N^*(x_1, x_2 + x'_2, s + t) = 0. \]

So \[ N^*(x_1, x_2 + x'_2, s + t) = N^*(x_1, x_2, s) \odot N^*(x_1, x'_2, t). \]

So \[ N^*(x_1, x_2 + x'_2, s + t) \leq N^*(x_1, x_2, s) \odot N^*(x_1, x'_2, t) \]

(vi) If \[ t > \|x_1, x_2\| \] then from the definition \[ \lim_{t \to \infty} N^*(x_1, x_2, t) = 0. \]
If \( x_1, x_2 \) are not independent and \( t \leq \|x_1, x_2\| \) then
\[ \lim_{t \to \infty} N^*(x_1, x_2, t) = 0. \]

If \( x_1, x_2 \) are linearly dependent and \( t \leq \|x_1, x_2\| \) then \( \lim_{t \to \infty} N^*(x_1, x_2, t) = 0. \)

Hence \( \lim_{t \to \infty} N^*(x_1, x_2, t) = 0. \forall x_1, x_2 \in X \times X. \)

Thus \( N^* \) is a fuzzy anti 2-norm on \( X \) with respect to the \( t \)-conorm \( \diamond \) and \( (X, N^*) \) is a fuzzy anti 2-normed linear space with respect to the \( t \)-conorm \( \diamond \).

Example 3.3: Let \( (X, \|\|) \) be 2-normed linear space and define \( a \odot b = \min\{a + b, 1\} \). Define \( N^*: X \times X \times R \to [0,1] \) by \( \|x_1, x_2\| \ = \ \frac{\|x_1, x_2\|}{2t - \|x_1, x_2\|} \) if \( t > \|x_1, x_2\| \)
\[ N^*(x_1, x_2, t) = \begin{cases} \frac{\|x_1, x_2\|}{2t - \|x_1, x_2\|}, & \text{if } t > \|x_1, x_2\| \\ 1, & \text{if } t \leq \|x_1, x_2\| \end{cases} \]

Then \( N^* \) satisfies all the condition of fuzzy anti 2-norm with respect to t-conorm \( \diamond \). So \( N^* \) is a fuzzy anti 2-norm on \( X \) with respect to the \( t \)-conorm \( \diamond \) and \( (X, N^*) \) is a fuzzy anti 2-normed linear space with respect to the \( t \)-conorm \( \diamond \).

Theorem 3.1: Let \( (X, N^*) \) be a fuzzy anti 2-normed linear space with respect to a \( t \)-conorm \( \diamond \) satisfying (Fa2-N7) and (Fa2-N9). Then for any \( \alpha \in (0,1) \) the function \( N^*: X \times X \times R \to [0,\infty) \) defined as
\[ \|x_1, x_2\| = \alpha \left\{ t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha \right\} \alpha \in (0,1). \]
is a 2-norm on \( X \). Then \( \|\|_\alpha : \alpha \in (0,1) \) is an ascending family of 2-norm on a linear space \( X \).

Proof:
(i) For \( x_1, x_2 \) for \( t \leq 0 \), so \( N^*(x_1, x_2, t) \leq 1 - \alpha \) is not possible.
So \( \left\{ t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha \right\} \geq 0 \), \( \alpha \in (0,1) \Rightarrow \|x_1, x_2\| \geq 0, \alpha \in (0,1) \).

(ii) It is obvious \( \left\{ t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha \right\} = 0 \Rightarrow \forall t > 0, N^*(x_1, x_2, t) < 1 \)
So by (Fa2-N7) \( x_1 \) and \( x_2 \) are linearly dependent.

Conversely, \( x_1 \) and \( x_2 \) are linearly dependent
\[ \Rightarrow \left\{ t > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha \right\} = 0, \forall \alpha \in (0,1) \Rightarrow \|x_1, x_2\| = 0. \]
(iii) If \( c = 0 \) it is obvious. If \( c \neq 0 \) then

\[
\left\| x_1, cx_2 \right\| = \land \left\{ s > 0 : N^*(x_1, cx_2, s) \leq 1 - \alpha \right\} \\
= \land \left\{ s > 0 : N^* \left( x_1, x_2, \frac{s}{c} \right) \leq 1 - \alpha \right\} \\
= \land \left\{ \forall x \in X, N^*(x_1, x_2, s) \leq 1 - \alpha \right\} \\
= \land \left\{ \forall x \in X, N^*(x_1, x_2, s) \leq 1 - \alpha \right\}
\]

(iv) \( \left\| x_1, x_2 \right\| + \left\| x_1, x_2 \right\| = \land \left\{ s > 0 : N^*(x_1, x_2, t) \leq 1 - \alpha \right\} \lor \left\{ s > 0 : N^*(x_1, x_2, s) \leq 1 - \alpha \right\} \forall \alpha \in (0, 1)
\]

So \( \land \left\{ s > 0 : N^*(x_1, x_2, t) \lor N^*(x_1, x_2, s) \leq 1 - \alpha \right\} \)

Hence \( \left\| \cdot, \cdot \right\| : \alpha \in (0, 1) \) is a 2-norm on \( X \).

Theorem 3.2: Let \( (X, N^*) \) be a fuzzy anti 2-normed linear space satisfying (Fa2-N7) and (Fa2-N8) Also, if \( \left\| \cdot, \cdot \right\| : \alpha \in (0, 1) \) be an ascending family of norms of \( X \), defined by \( \left\| x_0, x_0' \right\| = \land \left\{ s > 0 : N^*(x_0, x_0', s) \leq 1 - \alpha \right\} \alpha \in (0, 1) \) Then for \( x_0, x_0' \) (linearly independent) \( X \) \( \alpha \in (0, 1) \) and \( s > 0 \) in \( R \),

\[
\left\| x_0, x_0' \right\| = s \Leftrightarrow N^*(x_0, x_0', s) = 1 - \alpha.
\]

Proof: let \( \left\| x_0, x_0' \right\| = s \) then \( s > 0 \). Then \( \exists \) a sequence \( \left\{ s_n \right\}_{n \to \infty} \) such that \( s_n \to s \) as \( n \to \infty \) and

\[
N^*(x_0, x_0', s_n) \leq 1 - \alpha, \quad \forall n \in N.
\]

Therefore \( \lim_{n \to \infty} N^*(x_0, x_0', s_n) \leq 1 - \alpha \Rightarrow N^*(x_0, x_0', \lim_{n \to \infty} s_n) \leq 1 - \alpha \Rightarrow N^*(x_0, x_0', s) \leq 1 - \alpha \)

Let \( \alpha \in (0, 1) \), \( x_0, x_0' \) (linearly dependent) \( X \) and \( s = \left\| x_0, x_0' \right\| = \land \left\{ s > 0 : N^*(x_0, x_0', s) \leq 1 - \alpha \right\} \)

Therefore \( N^*(x_0, x_0', s) \leq 1 - \alpha \)

(1)

If possible let \( N^*(x_0, x_0', s) < 1 - \alpha \) then by continuity of \( N^*(x_0, x_0', s) \) at \( s \), there exist \( s' < s \) such that \( N^*(x_0, x_0', s') < 1 - \alpha \), which is impossible since

\[
s = \land \left\{ s > 0 : N^*(x_0, x_0', s) \leq 1 - \alpha \right\}.
\]

Thus \( N^*(x_0, x_0', s) \geq 1 - \alpha \)

(2)

From (1) and (2) it follows that \( N^*(x_0, x_0', s) = 1 - \alpha \). Thus

\[
\left\| x_0, x_0' \right\| = s \Rightarrow N^*(x_0, x_0', s) = 1 - \alpha
\]

(3)
Next if \( N^*(x_0, x'_0, s) = 1 - \alpha, \alpha \in (0,1) \) then
\[
\|x_0, x'_0\|_\alpha = \bigwedge \{ t : N^*(x_0, x'_0, s) \leq 1 - \alpha \} = s.
\] (4)

Hence from (3) and (4) we have for \( \alpha \in (0,1) \), \( x_0, x'_0 \) (linearly independent) \( \in X \) and for \( s > 0, \|x_0, x'_0\| = s \iff N^*(x_0, x'_0, s) = 1 - \alpha. \) Hence proved.

**Theorem 3.3:** Let \( (X, N^*) \) be a fuzzy anti 2-normed linear space with respect to a t- conorm \( \diamond \) satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9).

Let \( \|x_1, x_2\|_\alpha = \bigwedge \{ t : N^*(x_1, x_2, t) \leq 1 - \alpha \}, \alpha \in (0,1). \) Also, let \( N^*_1 : X \times X \times R \to [0,1] \) be defined by
\[
N^*_1(x_1, x_2, t) = \begin{cases} 
1 - \alpha : \|x_1, x_2\|_\alpha \leq t \\
1, \text{ otherwise}
\end{cases}
\]
Then \( N^*_1 = N^* \).

**Proof:** Let \((x_0, x'_0, t_0) \in X \times X \times R \) and \( \alpha \in (0,1) \). To prove this we consider the following cases:

**Case (i):** For any \((x_0, x'_0) \in X \times X \) and \( t \leq 0, N^*(x_0, x'_0, t_0) = N^*_1(x_0, x'_0, t_0) = 1. \)

**Case (ii):** Let \( x_0, x'_0 \) (linearly dependent), \( t_0 > 0. \)
Then \( N^*(x_0, x'_0, t_0) = 0 \) also \( \|x_1, x_2\|_\alpha = 0 \) so \( N^*_1(x_0, x'_0, t_0) = 0 \)

**Case (iii):** Let \( x_0, x'_0 \) (linearly independent), \( t_0 > 0 \) such that \( N^*(x_0, x'_0, t_0) = 1. \) By theorem (3.2), we have \( N^*(x_0, x'_0, t_0) = 1 - \alpha. \) Since \( N^*(x_0, x'_0, t_0) = 1 > 1 - \alpha \) it follows that
\[
N^*(x_0, x'_0, t_0) = 1 - \alpha < N^*(x_0, x'_0, t_0) \quad \text{and since} \quad N^*(x_0, x'_0, t_0) \quad \text{is strictly non-increasing.}
\]
So \( t_0 < \|x_1, x_2\|_\alpha, \forall \alpha \in (0,1). \) So \( N^*_1(x_0, x'_0, t_0) = \bigwedge \{ 1 - \alpha : \|x_1, x_2\|_\alpha \leq t_0 \} = 1. \)
Thus \( N^*(x_0, x'_0, t_0) = N^*_1(x_0, x'_0, t_0) = 1. \)

**Case (iv):** Let \( x_0, x'_0 \) (linearly independent), \( t_0 > 0 \) such that \( N^*(x_0, x'_0, t_0) = 0. \)
As \( \|x_0, x'_0\|_\alpha = \bigwedge \{ t : N^*(x_0, x'_0, t) \leq 1 - \alpha \}, \alpha \in (0,1) \).
As \( N^*(x_0, x'_0, \|x_0, x'_0\|_\alpha) = 1 - \alpha \) as \( N^* \) is decreasing.

It follows that, \( \|x_0, x'_0\|_\alpha < t_0, \forall \alpha \in (0,1), \) by (Fa2-N6). Therefore,
\[
\|x_1, x_2\|_\alpha < t_0 \implies N^*_1(x_0, x'_0, t_0) = \bigwedge \{ 1 - \alpha : \|x_0, x'_0\|_\alpha \leq t_0 \} = 0,
\]
Thus \( N^*(x_0, x'_0, t_0) = N^*_1(x_0, x'_0, t_0) = 0. \)

**Case (v):** Let \( x_0, x'_0 \) (linearly independent), \( t_0 > 0, \) s.t., \( 0 < N^*(x_0, x'_0, t_0) < 1. \)
Let, \( N^*(x_0, x'_0, t_0) = 1 - \beta, \) as \( \|x_0, x'_0\|_\beta = \bigwedge \{ t : N^*(x_0, x'_0, t) \leq 1 - \beta \} \)
as \( N^* \) is non-increasing function of \( t, \) we have \( \|x_0, x'_0\|_\beta \leq t_0 \)
So \( N_t^*(x_0, x'_0, t_0) \leq 1 - \beta \). Therefore,

\[
N_t^*(x_0, x'_0, t_0) \leq N^*(x_0, x'_0, t_0)
\]

(1)

As \( N_t^*(x_0, x'_0, t_0) = 1 - \beta \Leftrightarrow \|x_1, x_2\|_\beta = t_0 \).

If \( \beta < \alpha < 1 \) and let \( \|x_0, x'_0\|_{\beta} = t_1 \) then \( N_t^*(x_0, x'_0, t_1) = 1 - \alpha < 1 - \beta = N_t^*(x_0, x'_0, t_0) \)

As \( N_t^*(x_0, x'_0, \cdot) \) is monotonically decreasing so \( t_0 < t_1 \) since \( \|x_0, x'_0\|_{\beta} = t_1 > t_0 \).

So \( N_t^*(x_0, x'_0, t_0) > 1 - \beta = N_t^*(x_0, x'_0, t_0) \)

(2)

So from (1) and (2) we have \( N_t^*(x_0, x'_0, t_0) = N^*(x_0, x'_0, t_0) \).

Hence proved.

Lemma 3.4: Let \( (X, N^*) \) be a fuzzy anti 2-normed linear space with respect to t- conorm \( \diamond \) satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9), every sequence is convergent if and only if it is convergent with respect to its corresponding \( \alpha \)-2-norms, \( \alpha \in (0,1) \).

Proof: Let \( (X, N^*) \) be a fuzzy anti 2-normed linear space satisfying (Fa2-N7), (Fa2-N8), (Fa2-N9) and \( \{x_n\} \) be a sequence in \( X \) such that \( x_n \to x \) as \( n \to \infty \).

\[
\lim_{n \to \infty} N^*(x_n - x, x_0, t) = 0, \quad \forall t > 0.
\]

Let \( 0 < \alpha < 1 \). So, \( \lim_{n \to \infty} N^*(x_n - x, x_0, t) = 0 < 1 - \alpha \Rightarrow \exists n_0(t), \) such that

\[
N^*(x_n - x, x_0, t) < 1 - \alpha \quad \forall n \geq n_0(t, \alpha).
\]

Now, \( \|x_n - x, x_0\|_\alpha = \bigwedge \{ t \geq 0 : N^*(x_n - x, x_0, t) \leq 1 - \alpha \} \)

\[
\Rightarrow \|x_n - x, x_0\|_\alpha \leq t, \quad \forall n \geq n_0(t, \alpha)
\]

Since \( t > 0 \) is arbitrary, \( \|x_n - x, x_0\|_\alpha \to 0 \) as \( n \to \infty \), \( \forall \alpha \in (0,1) \).

Conversely, suppose that \( \|x_n - x, x_0\|_\alpha \to 0 \) as \( n \to \infty \), \( \forall \alpha \in (0,1) \).

Then for \( \alpha \in (0,1), \varepsilon > 0, \exists n_0(\alpha, \varepsilon) \) such that \( \|x_n - x, x_0\|_\alpha < \varepsilon, \forall n \geq n_0(\alpha, \varepsilon), \alpha \in (0,1) \).

Now,

\[
N^*(x_n - x, x_0, \varepsilon) = \bigwedge \{ 1 - \alpha : \|x_n - x, x_0\|_\alpha \leq \varepsilon \}
\]

\[
\Rightarrow N^*(x_n - x, x_0, \varepsilon) \leq 1 - \alpha, \quad \forall n \geq n_0(\alpha, \varepsilon), \alpha \in (0,1).
\]

\[
\Rightarrow \lim_{n \to \infty} N^*(x_n - x, x_0, \varepsilon) = 0, \quad \forall t > 0.
\]

Thus \( x_n \) converges to \( x \).

Hence Proved.

Corollary 3.5: Let \( (X, N^*) \) be a fuzzy anti 2-normed linear space with respect to t- conorm \( \diamond \) satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9). \( W \subseteq X \) is closed in \( (X, N^*) \) if and only if it is closed with respect to its corresponding \( \alpha \)-2-norms, \( \alpha \in (0,1) \).

Theorem 3.6 (Riesz lemma): Let \( W \) be a closed and proper subspace of a fuzzy anti 2-normed linear space \( (X, N^*) \) with respect to t- conorm \( \diamond \) satisfying (Fa2-N7) (Fa2-N8) and (Fa2-N9). Then for each \( \varepsilon > 0 \) there exist \( y_1, y_2 \in (X - W)^2 \) such that \( N^*(y_1, y_2, \varepsilon) \leq 1 - \alpha \) and \( N^*(y_1 - w, y_2 - w, \varepsilon) \leq 1 - \alpha \) for all \( w(y,1) \leq 1 - \alpha \) and \( w \in W \).
Proof: As \[ \left\| x_1, x_2 \right\|_\alpha = \bigwedge \{ \alpha : N^\alpha(x_1, x_2, t) \leq 1 - \alpha, \alpha \in (0, 1) \} \] is an ascending family of fuzzy \(\alpha\)-2-norm on a linear space \(X\). Now by Riesz lemma for 2-normed linear space, it follows that for any \(\varepsilon > 0\) there exist \(y_1, y_2 \in (X - W)^t\) such that \[ \left\| y_1, y_2 \right\|_\alpha = 1 \] and \[ \left\| y_1 - w, y_2 - w \right\|_\alpha > 1 - \varepsilon \] \(\forall w \in W\).

Now, from theorem (3.3), for all \(\alpha \leq 1 - \alpha\) we have

\[ N^\alpha(y_1, y_2, t) = \bigwedge \{ 1 - \alpha : \left\| y_1, y_2 \right\|_\alpha \leq t \} \]
\[ \Rightarrow N^\alpha(y_1, y_2, 1) = \bigwedge \{ 1 - \alpha : \left\| y_1, y_2 \right\|_\alpha \leq 1 \} \]
\[ \Rightarrow N^\alpha(y_1, y_2, 1) \leq 1 - \alpha \]

Again,

\[ N^\alpha(y_1 - w, y_2 - w, t) = \bigwedge \{ 1 - \alpha : \left\| y_1 - w, y_2 - w \right\|_\alpha \leq t \} \]
\[ \Rightarrow N^\alpha(y_1 - w, y_2 - w, \varepsilon) = \bigwedge \{ 1 - \alpha : \left\| y_1 - w, y_2 - w \right\|_\alpha \leq \varepsilon \} \]
\[ \Rightarrow N^\alpha(y_1 - w, y_2 - w, w) \leq 1 - \alpha. \]

Hence proved.

Theorem 3.7: Let \((X, N^\alpha)\) be a fuzzy anti 2-normed linear space with respect to a \(t\)-conorm \(\diamond\) satisfying (Fa2-N7), (Fa2-N8) and (Fa2-N9). If the set \(\{ x_1, x_2 : N^\alpha(x_1, x_2, 1) \leq 1 - \alpha, \alpha \in (0, 1) \}\) is compact then \(X\) is a space of finite dimension.

Proof: It can be easily verified that \(\{ x_1, x_2 : N^\alpha(x_1, x_2, 1) \leq 1 - \alpha, \alpha \in (0, 1) \}\) and \(\left\{ x_1, x_2 : \left\| x_1, x_2 \right\|_\alpha \leq 1, \alpha \in (0, 1) \right\}\) by applying Riesz lemma 3.6, it can be proved that if for some \(\alpha \in (0, 1)\) the set \(\{ x_1, x_2 : \left\| x_1, x_2 \right\|_\alpha \leq 1, \alpha \in (0, 1) \}\) is compact then \(X\) is of finite dimensional. By lemma (3.4), it follows that , for some \(\alpha \in (0, 1)\), \(\{ x_1, x_2 : N^\alpha(x_1, x_2, 1) \leq 1 - \alpha \}\) is compact then \(X\) is a space of finite dimensional.

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