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UNIQUENESS OF SOME GENERAL CLASS OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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ABSTRACT

We discuss some general class of differential polynomials of meromorphic functions and obtain two theorems which improve a result of Abhijit Banerjee [17].

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1. INTRODUCTION AND MAIN RESULTS

Let f and g be two nonconstant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. We say that $f$and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities and if we do not consider the multiplicities, then $f$and g are said to share the value a IM (ignoring multiplicities). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation S(r) denotes any quantity satisfying $S(r) = 0(T(r))$ as $r \to \infty$, outside a set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

Due to Nevanlinna [9], it is well known that if f and g share four distinct values CM, then f is a Mobius transformation of g.

Yang and Hua showed that similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem 1.1[12]: Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some (n+1)th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where c, c_1 and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.

Corresponding to entire functions, Xu and Qu proved the following result.

Theorem 1.2[10]: Let f and g be two nonconstant entire functions, $n \geq 12$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a IM, then either $f = dg$ for some (n+1)th root of unity d or $g(z) = c_1 e^{cz}$and $f(z) = c_2 e^{-cz}$ where $c, c_1$and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

To state the next result, we require the following definition.

Definition 1.3[4, 5]: Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} - \{\infty\}$, denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$ say that f, g share the value a with weight k.

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The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only it is an a-point of g with multiplicity $m(\leq k)$ and $z_0$is an a-point of f with multiplicity $m(> k)$ if and only if is an a-point of g with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that $f;\ g$ share the value a with weight k. Since $E_k(a;\ f) = E_k(a;\ g)$ implies $E_p(a;\ f) = E_p(a;\ g)$ for any integer $p(0 \leq p < k)$, clearly if f, g share (a, k), then f, g share (a, p) for any integer $p(0 \leq p < k)$, Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Theorem 1.4[5]: Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share $(a, 2)$, then either $f = dg$ for some (n+1)th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where $c, c_1$and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Theorem 1.5[17]: Let f and g be two nonconstant meromorphic functions such that $n > 22 - [5\Theta(\infty;\ f) + 5\Theta(\infty;\ g) + \min\{\Theta(\infty;\ f), \Theta(\infty;\ g)\}]$, where n is an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g^{'}$ share $(a, 0)$, then either $f = dg$ for some (n+1)th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$ where $c, c_1$and $c_2$are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Theorem 1.6[17]: Let f and g be two nonconstant meromorphic functions and $n > \max\{8, 12 - \{3\Theta(\infty, f) + 3\Theta(\infty, g)\}\}$ an integer. If for $a \in \mathbb{C} - \{0\}$, $f^n f'$ and $g^n g^{'}$ share $(a, 1)$, then either $f = dg$ for some (n+1)th root of unity d or $g(z) = c_1 e^{cz}$and $f(z) = c_2 e^{-cz}$ where $c, c_1$and $c_2$are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

Here in this paper, we partially extend Theorem 1.5 and Theorem 1.6 to a more general class of differential polynomials as.

Theorem 1.7: Let f and g be two nonconstant meromorphic functions such that $n > km + 3m + 4k + 14$, where n, m and k be positive integers. If for $a \in \mathbb{C} - \{0\}$, $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$ share $(a, 0)$, then either $f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv a^2$ or $f^n (f^m)^{(k)} \equiv g^n (g^m)^{(k)}$.

Theorem 1.8: Let$f$ and g be two nonconstant meromorphic functions such that $n > km + 3m + 3k + 11$, where n, m and k be positive integers. If for $a \in \mathbb{C} - \{0\}$, $f^n (f^m)^{(k)}$ and $g^n (g^m)^{(k)}$share $(a, 1)$, then either $f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv a^2$ or $f^n (f^m)^{(k)} \equiv g^n (g^m)^{(k)}$.

Definition 1.9[3]: For $a \in \mathbb{C} - \{\infty\}$, denote by $N(r, a;\ f| = 1)$ the counting function of simple a-points of f. For a positive integer m, denote by $N(r, a;\ f| \leq m)(N(r, a;\ f| \geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r, a;\ f| \leq m)(\overline{N}((r, a;\ f| \geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a;\ f| < m), (N(r, a;\ f| > m))\overline{N}(r, a;\ f| < m)$ and $\overline{N}(r, a;\ f| > m)$ are defined analogously.

Definition 1.10[5]: Denote by $N_2(r;\ a;\ f)$ the sum $\overline{N}(r, a;\ f) + \overline{N}(r, a;\ f| \geq 2)$.

Definition 1.11[1, 15, 16]: Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. Denote by $\overline{N_L}(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, denote by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q = 1$, and denote by $N_E^{2)}(r, 1; f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way, one can define $\overline{N_L}(r, 1; g), N_E^{1)}(r, 1; g), N_E^{2)}(r, 1; g)$.

Definition 1.12(cf. [1]): Let k be a positive integer. Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, and a 1-point of g with multiplicity q. Denote by $\overline{N}_{f>k}(r, 1; g)$ the reduced counting function of those 1-points of f and g such that $p > q = k. \overline{N}_{g>k}(r, 1; f)$ is defined analogously.

Definition 1.13([4, 5]): Let f, g share a value IM. Denote by $\overline{N_*}(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly
$\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N_*}(r, a; f, g) = \overline{N}_L(r, a;\ f) + \overline{N}_L(r, a;\ g)$.

Definition 1.14([6]): Let $a, b \in \mathbb{C} \cup \{\infty\}$. Denote by $N(r, a; f|g = b)$ the counting function of those a-points of f, counted according to multiplicity, which are the b-points of g.

Definition 1.15([6]): Let $a, b \in \mathbb{C} \cup \{\infty\}$. Denote by $N(r, a; f|g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. LEMMAS

In this section, we present some lemmas which will be needed in the sequel. Let f, g, F, G be four nonconstant meromorphic functions. Henceforth, We will denote by h and H the following two functions:

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right), H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

Lemma 2.1[15]: Let f, g be two nonconstant meromorphic functions such that they share $(1, 0)$ and $h \neq 0$. Then

$$T(r,f) \le N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g) + 2\overline{N}\left(r,\frac{1}{f}\right) + 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g) + S(r,f) + S(r,g)$$

Lemma 2.2[17]: Let f, g be two nonconstant meromorphic functions such that they share (1,1) and $h \neq 0$. Then

$$T(r,f) \le N_2\left(r,\frac{1}{f}\right) + N_2(r,f) + N_2\left(r,\frac{1}{g}\right) + N_2(r,g) + \frac{1}{2}\overline{N}\left(r,\frac{1}{f}\right) + \frac{1}{2}\overline{N}(r,f) + S(r,f) + S(r,g)$$

Lemma 2.3[18]: Let f and g be two meromorphic functions. If f and g share 1 CM, then one of the following must occur:

(i) $T(r,f) + T(r,g) \le 2\left\{N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + N_2(r,f) + N_2(r,g)\right\} + S(r,f) + S(r,g)$

(ii) either $f \equiv g$ or $fg \equiv 1$.

Lemma 2.4[15]: Let f be a nonconstant meromorphic function. Then

$$N\left(r,\frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + S(r,f).$$

3. PROOF OF THEOREM 1.7

Let $F = \frac{f^n(f^m)^{(k)}}{a}$ and $G = \frac{g^n(g^m)^{(k)}}{a}$. Since $f^n(f^m)^{(k)}$ and $g^n(g^m)^{(k)}$ share $(a, 0)$. If possible, we suppose that $H \not\equiv 0$. Then by Lemma 2.1, we obtain

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + 2\overline{N}\left(r,\frac{1}{F}\right) + 2\overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,F) + S(r,G)$$

We see that

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) + N_2(r,F) &\le N_2\left(r,\frac{1}{f^n(f^m)^{(k)}}\right) + N_2\left(r,f^n(f^m)^{(k)}\right) + S(r,f) \\ &\le N_2\left(r,\frac{1}{f^n}\right) + N_2\left(r,\frac{1}{(f^m)^{(k)}}\right) + 2\overline{N}\left(r,f^n(f^m)^{(k)}\right) + S(r,f) \\ &\le 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{(f^m)^{(k)}}\right) + 2\overline{N}(r,f) + S(r,f) \\ &\le 2\overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right) + (k+2)\overline{N}(r,f) + S(r,f) = (k+m+4)T(r,f) + S(r,f). \end{aligned}$$

Therefore,

$$N_2\left(r,\frac{1}{F}\right) + N_2(r,F) \le (k+m+4)T(r,f) + S(r,f) \tag{3.1}$$

On the similar lines we can write (3.1) for the function G as

$$N_2\left(r,\frac{1}{G}\right) + N_2(r,G) \le (k+m+4)T(r,g) + S(r,g) \tag{3.2}$$

Since

$$nT(r,f) = T(r,f^n) = T\left(r,\frac{f^n(f^m)^{(k)}a}{a(f^m)^{(k)}}\right)$$

$$\leq T(r,F) + T\left(r, \frac{1}{(f^m)^{(k)}}\right) + T(r,a) + S(r,f)$$
$$= T(r,F) + (km+m)T\left(r, \frac{1}{f}\right) + S(r,f) \tag{3.3}$$

Therefore
$$(n-km-m)T(r,f) \leq T(r,F) + S(r,f) \tag{3.4}$$

Similarly
$$(n-km-m)T(r,g) \leq T(r,G) + S(r,g) \tag{3.5}$$

$$(n-km-m)\{T(r,f)+T(r,g)\} \leq \{T(r,F)+T(r,G)\} + S(r,f) + S(r,g) \tag{3.6}$$

Suppose that
$$T(r,F)+T(r,G) \leq 2\left\{N_2\left(r,\frac{1}{F}\right) + N_2(r,F)\right\} + 2\{N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\} + 3\left\{\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{G}\right)\right\}$$
$$+ 3\{\overline{N}(r,F) + \overline{N}(r,G)\} + S(r,f) + S(r,g).$$

Then from (3.1), (3.2), (3.6) and (3.7) we have
$$(n-km-m)\{T(r,f)+T(r,g)\}$$
$$\leq 2\{N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\} + 3\left\{N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G)\right\}$$
$$+ S(r,f) + S(r,g)$$
$$\leq 2(k+m+4)\{T(r,f)+T(r,g)\}$$
$$+ 3\left\{N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{g}\right) + (k+1)\left\{\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)\right\} + k\{\overline{N}(r,f) + \overline{N}(r,g)\} + 2\overline{N}(r,f)\right.$$
$$\left. + 2\overline{N}(r,g)\right\} + S(r,f) + S(r,g)$$
$$\leq \{4k+2m+14\}\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g)$$

which implies that
$$\{n-km-3m-4k-14\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$$
a contradiction since $n > km+3m+4k+14$, where $m > k-1$.

Then by Lemma 2.3, it follows that either
$F.G \equiv 1$ or $F \equiv G$ that is either $f^n(f^m)^{(k)}g^n(g^m)^{(k)} \equiv a^2$ or $f^n(f^m)^{(k)} = g^n(g^m)^{(k)}$.

4. PROOF OF THEOREM 1.8

Let $F = \frac{f^n(f^m)^{(k)}}{a}$ and $G = \frac{g^n(g^m)^{(k)}}{a}$. Since $f^n(f^m)^{(k)}$ and $g^n(g^m)^{(k)}$ share $(a,1)$. Suppose that $H \not\equiv 0$. Then by Lemma 2.2, we obtain
$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G) + \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G)$$

Suppose that
$$T(r,F)+T(r,G) \leq 2\{N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\} + \frac{1}{2}\overline{\{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right)$$
$$+\overline{N}(r,G)\} + S(r,F) + S(r,G) \tag{4.1}$$

Then from (3.1),(3.2),(3.6) and (4.1) we have
$$(n-km-m)\{T(r,f)+T(r,g)\}$$
$$\leq 2\{N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\} + \frac{1}{2}\overline{\{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}(r,G)\}$$
$$+ S(r,F) + S(r,G)$$
$$\leq 2(k+m+4)\{T(r,f)+T(r,g)\}$$
$$+ \frac{1}{2}\left\{N\left(r,\frac{1}{f}\right) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + (k+1)\overline{N}\left(r,\frac{1}{g}\right) + k\overline{N}(r,g) + 2\overline{N}(r,f)\right.$$
$$\left. + 2\overline{N}(r,g)\right\} + S(r,f) + S(r,g) \leq (3k+2m+11)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g)$$

which implies that
$$\{n-km-3m-3k-11\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$$
a contradiction since $n > km+3m+3k+11$, where $m > k-1$.

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Then by Lemma 2.3, it follows that either
$F.G \equiv 1$ or $F \equiv G$ that is either $f^n(f^m)^{(k)}g^n(g^m)^{(k)} \equiv a^2$ or $f^n(f^m)^{(k)} = g^n(g^m)^{(k)}$.

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