ON SOME PROPERTIES OF ROUGH APPROXIMATIONS
OF SUBGROUPS VIA AN EQUIVALENCE RELATION

Y. MADHAVI REDDY*1, E. KESHAVA REDDYAND2 AND P. VENKAT RAMAN3

1Plot No. 107, Sarada Apartments,
Street No. 6, Nallakunta, Hyderabad-500044, Talangana State, India.

2Professor, Headept of Mathematics, Jntua College of Engineering, Ananthapuramu.

3Honorary Professor & Head, Centre for Mathematics, JNIAS, Budhabhavan, Hyd., India.

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ABSTRACT

In 1982, Zdzislaw Pawlak introduced the theory of Rough sets to deal with the problems involving imperfect knowledge. This present research article studies some interesting properties of Rough approximations of subgroups via an equivalence relation. In this present work, a group structure is assigned to the universe set and a few results on the Rough approximations of subgroups of the universe set are established.

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INTRODUCTION

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by L. A. Zadeh in his classical paper [6] of 1965.

A polish applied mathematician and computer scientist Zdzislaw Pawlak introduced Rough set theory in his classical paper [3] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness. The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications.

In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering cosets of a subgroup. We investigate a few results on lower and upper approximations of subgroups.

1. PRELIMINARIES

In this section, some basic definitions that are necessary for further study of this work are presented.

In what follows \( \emptyset \) and \( U \) stand for the empty set and the universe set respectively.
1.1 Definition: A relation $R$ on a non-empty set $S$ is said to be an equivalence relation on $S$ if
(a) $xRy$ for all $x \in S$ (reflexivity)
(b) $xRy \iff yRx$ (symmetry)
(c) $xRy$ and $yRz \Rightarrow xRz$ (transitivity)

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation $R$ by the symbol $[x]$ and $\{ y \in S : yRx \}$.

1.2 Definition: Let $X \subseteq U$. Let $R$ be an equivalence relation on $U$. Then we define the following.
(a) The lower approximation of $X$ with respect to $R$ is the set of all objects, which can be for certain classified as $X$ using $R$. That is the set $R_l(X) = \{ x : R[x] \subseteq X \}$.
(b) The upper approximation of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as $X$ using $R$. That is the set $R_u(X) = \{ x : R[x] \cap X \neq \emptyset \}$.
(c) The boundary region of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not $X$ using $R$. That is the set $R_b(X) = R_u(X) - R_l(X)$.

It is clear that $R_b(X) \subseteq R_u(X) \subseteq R_l(X)$.

1.3 Definition: A set $X \subseteq U$ is said to be a Rough set with respect to an equivalence relation $R$ on $U$, if the boundary region $R_b(X) = R_u(X) - R_l(X)$ is non-empty.

1.4 Definition: A non-empty set of elements $G$ is said to form a group if in $G$ there is defined a binary operation, called the product and denoted by $\circ$, such that
(a) $a, b \in G \Rightarrow a \circ b \in G$ (Closure)
(b) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$ (Associative law)
(c) there exists an element $e \in G$ such that $a \circ e = e \circ a = a \quad \forall a \in G$. The element $e$ is called the identity element in $G$. (Existence of identity)
(d) for every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$. The element $a^{-1}$ is called the inverse element of $a$ in $G$. (Existence of inverse)

1.5 Definition: A group $G$ with respect to a binary operation $\circ$ is said to be an abelian group if $a \circ b = b \circ a$ for all $a, b$ in $G$.

1.6 Definition: A non-empty subset $H$ of a group $G$ with respect to a binary operation $\circ$ is said to be a subgroup of $G$ if $H$ itself forms a group with respect to $\circ$.

1.7 Remark: A non-empty subset $H$ of a group $G$ is a subgroup of $G$ if and only if
(i) $a, b \in H \Rightarrow a \circ b \in H$ and (ii) $a \in H \Rightarrow a^{-1} \in H$

1.8 Definition: A subgroup $H$ of a group $G$ is said to be a normal subgroup of $G$ if $x \in G$ and $h \in H$ then $xhx^{-1} \in H$.

2. CONSTRUCTION OF ROUGH SETS

In the Literature of Rough set theory, information systems are considered. An information system is a pair $(U, \mathcal{A})$ where $\mathcal{A}$ is a set of attributes. Each attribute $a \in \mathcal{A}$ is a mapping $a : U \rightarrow V_a$ where $V_a$ is the
range set of the attribute \( a \in A \). Corresponding to each attribute \( a \in A \), an equivalence relation \( R_a \) is defined on \( U \) such that \( xR_y \Leftrightarrow a(x) = a(y) \). Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider an equivalence relation on a group to construct Rough sets and present a few results in this context.

In what follows \( G \) stands for a group and we take the universe set \( U \) to be \( G \). Let \( e \) be the identity element in \( G \).

2.1 Definition: Let \( H \) be a normal subgroup of a group \( G \) such that \( H \neq \{e\} \) and \( H \neq G \). We define a relation \( R \) on \( G \) as follows.

For \( x, y \in G \), \( xRy \Leftrightarrow x^{-1}y \in H \).

2.2 Proposition: The relation \( R \) on \( G \) is an equivalence relation on \( G \).

2.3 Remark: If \( x \in G \) then we denote the equivalence class of \( x \) under the equivalence relation \( R \) by the symbol \( [x] \) and we have \( [x] = \{ y \in G : yRx \} = xH \) thus the equivalence class of \( x \in G \) is the left coset \( xH \) of \( H \) in \( G \).

2.4 Proposition: For any subset \( A \) of \( G \), the following conditions are equivalent to one another.

\( (a) \quad R_e(A) = A \quad (b) \quad R^*(A) = A \quad (c) \quad R_e(A) = R^*(A) \)

Proof: For any subset \( A \) of \( G \), we always have \( R_e(A) \subseteq A \subseteq R^*(A) \).

Suppose \( R_e(A) = A \) and let \( x \in R^*(A) \). Then \( R[x] \cap A \neq \phi \)

\( \Rightarrow R[x] \cap R_e(A) \neq \phi \)

\( \Rightarrow \) there exists a point \( y \) in \( G \) such that \( y \in R[x] \cap R_e(A) \)

\( \Rightarrow yRx \) and \( y \in R_e(A) \)

\( \Rightarrow R[x] = R[y] \) and \( R[y] \subseteq A \)

\( \Rightarrow R[x] \subseteq A \)

\( \Rightarrow x \in A \)

This shows that \( R^*(A) \subseteq A \). Hence \( R^*(A) = A \).

This completes the proof of \( (a) \Rightarrow (b) \).

Now consider \( R^*(A) = A \). Let \( x \in A \). Then \( x \in R^*(A) \)

\( \Rightarrow R[x] \cap A \neq \phi \).

Let \( z \in R[x] \). Then \( xRz \Rightarrow R[z] = R[x] \)

\( \Rightarrow R[z] \cap A \neq \phi \) \( \Rightarrow z \in R^*(A) \) \( \Rightarrow z \in A \).

This proves that \( R[x] \subseteq A \) and hence \( x \in R_e(A) \).

Hence \( R_e(A) = A \). This completes the proof of \( (b) \Rightarrow (a) \).

Obviously (c) is equivalent to both (a) and (b).
3. ROUGH APPROXIMATIONS OF SUBGROUPS

In this section, we present a few properties of lower and upper rough approximations of subgroups of $G$.

3.1 Proposition: $\{e\}$ is a Rough set.

**Proof:** Since $R_e(\{e\}) \subseteq \{e\} \subseteq R^*(\{e\})$ either $R_e(\{e\}) = \emptyset$ or $R_e(\{e\}) = \{e\}$. Assume that $R_e(\{e\}) = \{e\}$. Then $e \in R_e(\{e\}) \Rightarrow H \subseteq \{e\} \Rightarrow H = \{e\}$.

This is a contradiction. Hence $R_e(\{e\}) = \emptyset$.

Let $x \in R^*(\{e\})$. Then $xH \cap \{e\} \neq \emptyset$.

$\Rightarrow e \in xH$

$\Rightarrow H^{-1} \subseteq H$

$\Rightarrow x \in H$

This shows that $R^*(\{e\}) \subseteq H$ (1)

Let $x \in H$. Then $xH = H$

$\Rightarrow xH \cap \{e\} = H \cap \{e\} = \{e\} \neq \emptyset$

$\Rightarrow x \in R^*(\{e\})$

This shows that $H \subseteq R^*(\{e\})$ (2)

From (1) and (2), $R^*(\{e\}) = H$.

Then $B_R(\{e\}) = H$.

Hence $\{e\}$ is a Rough set.

3.2 Remark: By the above Proposition-2.5, it follows that the lower approximation of a subgroup of $G$ is not necessarily a subgroup of $G$. Even though $\{e\}$ is a subgroup of $G$, its lower approximation is empty and hence $R_e(\{e\})$ is not a subgroup of $G$.

3.3 Proposition: $H$ is not a Rough set.

**Proof:** Clearly $R_e(H) \subseteq H \subseteq R^*(H)$.

Let $x \in H$. Then $xH = H \Rightarrow x \in R_e(H)$.

Then $H = R_e(H)$.

By proposition -2.4, $R_e(H) = H = R^*(H)$.

Hence $H$ is not a Rough set.

3.4 Proposition: Let $K$ be a subgroup of $G$. Then the following are equivalent.

(a) $H \subseteq K$

(b) $R_e(K) = K = R^*(K)$

(c) $R_e(K)$ is a subgroup of $G$

**Proof:** Let $K$ be a subgroup of $G$. Suppose that $H \subseteq K$.

Clearly $R_e(K) \subseteq K \subseteq R^*(K)$

Let $x \in K$. Since $H \subseteq K$, $xH \subseteq xK = K$

$\Rightarrow x \in R_e(K)$.
This shows that \( K \subseteq R_e(K) \implies K = R_e(K) \).

Hence \( R_e(K) = K = R_e^*(K) \).

This proves \((a) \implies (b)\).

Suppose that \( R_e(K) = K = R_e^*(K) \). Then \( R_e(K) \) is a subgroup of \( G \).

This proves \((b) \implies (c)\).

(b) Suppose that \( R_e(K) \) is a subgroup of \( G \)

\[ \implies e \in R_e(K) \]
\[ \implies H = eH \subseteq K \]

This proves \((c) \implies (a)\).

3.5 Proposition: If \( K \) is a subgroup of \( G \) then \( R^*_e(K) \) is a subgroup of \( G \).

Proof: Let \( K \) be a subgroup of \( G \).

Let \( x \in R^*_e(K) \).

\[ \implies xH \cap K \neq \emptyset \]
\[ \implies \text{there exists a point } y \text{ such that } y \in xH \cap K \]
\[ \implies y \in xH \text{ and } y \in K \]
\[ \implies x^{-1}y \in H \text{ and } y \in K \]
\[ \implies y^{-1}x \in H \text{ and } y^{-1} \in K \]

Since \( H \) is a normal subgroup of \( G \), we have
\( x \in G \) and \( y^{-1}x \in H \) imply that \( xy^{-1}xx^{-1} \in H \)
\[ \implies xy^{-1} \in H \]
\[ \implies y^{-1} \in x^{-1}H \]

Hence \( y^{-1} \in x^{-1}H \cap K \)
\[ \implies x^{-1}H \cap K \neq \emptyset \]
\[ \implies x^{-1} \in R^*_e(K) \]

Let \( x, y \in R^*_e(K) \).

\[ \implies xH \cap K \neq \emptyset \text{ and } yH \cap K \neq \emptyset \]
\[ \implies \text{there exist two elements } p \text{ and } q \text{ in } G \text{ such that } p \in xH \cap K \text{ and } q \in yH \cap K \]
\[ \implies p \in xH, q \in yH, p \in K \text{ and } q \in K. \]
\[ \implies pq \in xyH \text{ and } pq \in K \]
\[ \implies pq \in xyH \cap K \]
\[ \implies xyH \cap K \neq \emptyset \]
\[ \implies xy \in R^*_e(K) \]

Hence \( R^*_e(K) \) is a subgroup of \( G \).
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