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ON SOME PROPERTIES OF ROUGH APPROXIMATIONS OF SUBGROUPSVIA AN EQUIVALENCE RELATION

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ABSTRACT

In 1982, Zdzislaw Pawlak introduced the theory of Rough sets to deal with the problems involving imperfect knowledge. This present research article studies some interesting properties of Rough approximations of subgroups via an equivalence relation. In this present work, a group structure is assigned to the universe set and a few results on the Rough approximations of subgroups of the universe set are established.

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INTRODUCTION

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of Artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of Fuzzy sets and Rough sets are powerful mathematical tools for modeling various types of uncertainties. Fuzzy set theory was introduced by *L. A. Zadeh* in his classical paper [6] of 1965.

A polish applied mathematician and computer scientist *Zdzislaw Pawlak* introduced Rough set theory in his classical paper [3] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with uncertainty or vagueness. The Rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application. Rough sets have been proposed for a very wide variety of applications.

In particular, the Rough set approach seems to be important for Artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct Rough sets by considering cosets of a subgroup. We investigate a few results on lower and upper approximations of subgroups.

1. PRELIMINARIES

In this section, some basic definitions that are necessary for further study of this work are presented.

In what follows ϕ and U stand for the empty set and the universe set respectively.

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1.1 Definition: A relation R on a non-empty set S is said to be an *equivalence relation* on S if
(a) xRx for all $x \in S$ (*reflexivity*)
(b) $xRy \Leftrightarrow yRx$ (*symmetry*)
(c) xRy and $yRz \Rightarrow xRz$ (*transitivity*)

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation R by the symbol $R[x]$ and $R[x] = \{y \in S : yRx\}$.

1.2 Definition: Let $X \subseteq U$. Let R be an equivalence relation on U. Then we define the following.
(a) The *lower approximation* of X with respect to R is the set of all objects, which can be for certain classified as X using R. That is the set
$R_*(X) = \{x : R[x] \subseteq X\}$.
(b) The *upper approximation* of X with respect to R is the set of all objects, which can be possibly classified as X using R. That is the set
$R^*(X) = \{x : R[x] \cap X \neq \phi\}$.
(c) The *boundary region* of X with respect to R is the set of all objects, which can be classified neither as X nor as not$-X$ using R. That is the set $\mathfrak{B}_R(X) = R^*(X) - R_*(X)$.
It is clear that $R_*(X) \subseteq X \subseteq R^*(X)$.

1.3 Definition: A set $X \subseteq U$ is said to be a*Roughset*with respect to an equivalence relation R on U, if the boundary region
$\mathfrak{B}_R(X) = R^*(X) - R_*(X)$ is non-empty.

1.4 Definition: A non-empty set of elements G is said to form a *group* if in G there is defined a binary operation, called the product and denoted by $\circ$, such that
(a) $a,b \in G \Rightarrow a \circ b \in G$ (*Closure*)
(b) $a \circ (b \circ c) = (a \circ b) \circ c$ for all a,b,c in G (*Associative law*)
(c) there exists an element $e \in G$ such that $a \circ e = e \circ a = a \quad \forall \ a \in G$. The element e is called the identity element in G. (*Existence of identity*)
(d) for every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$. The element a^{-1} is called the inverse element of a in G. (*Existence of inverse*)

1.5 Definition: A group G with respect to a binary operation $\circ$ is said to be an *abelian group* if $a \circ b = b \circ a$ for all a,b in G.

1.6 Definition: A non-empty subset H of a group G with respect to a binary operation $\circ$ is said to be a *subgroup* of G if H itself forms a group with respect to $\circ$.

1.7 Remark: A non-empty subset H of a group G is a subgroup of G if and only if
(i) $a,b \in H \Rightarrow a \circ b \in H$ and (ii) $a \in H \Rightarrow a^{-1} \in H$

1.8 Definition: A subgroup H of a group G is said to be a *normal subgroup* of G if $x \in G$ and $h \in H$ then $x h x^{-1} \in H$.

2. CONSTRUCTION OF ROUGH SETS

In the Literature of Rough set theory, information systems are considered. An *information system* is a pair $(U, \mathcal{A})$ where $\mathcal{A}$ is a set of attributes. Each attribute $a \in \mathcal{A}$ is a mapping $a : U \rightarrow V_a$ where V_a is the range set of the attribute $a \in \mathcal{A}$. Corresponding to each attribute $a \in \mathcal{A}$, an equivalence relation R_a is defined on U such that $xR_a y \Leftrightarrow a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct Rough sets. We consider an equivalence relation on a group to construct Rough sets and present a few results in this context.

In what follows G stands for a group and we take the universe set U to be G. Let e be the identity element in G.

2.1 Definition: Let H be a normal subgroup of a group G such that $H \neq \{e\}$ and $H \neq G$. We define a relation R on G as follows.
For $x, y \in G$, $xRy \Leftrightarrow x^{-1}y \in H$.

2.2 Proposition: The relation R on G is an equivalence relation on G.

2.3 Remark: If $x \in G$ then we denote the equivalence class of x under the equivalence relation R by the symbol $R[x]$ and we have $R[x] = \{y \in G : yRx\} = xH$ thus the equivalence class of $x \in G$ is the left coset xH of H in G.

2.4 Proposition: For any subset A of G, the following conditions are equivalent to one another.
(a) $R_*(A) = A$ (b) $R^*(A) = A$ (c) $R_*(A) = R^*(A)$

Proof: For any subset A of G, we always have $R_*(A) \subseteq A \subseteq R^*(A)$.
Suppose $R_*(A) = A$ and let $x \in R^*(A)$. Then $R[x] \cap A \neq \phi$
$\Rightarrow R[x] \cap R_*(A) \neq \phi$
$\Rightarrow$ there exists a point y in G such that $y \in R[x] \cap R_*(A)$
$\Rightarrow yRx$ and $y \in R_*(A)$
$\Rightarrow R[x] = R[y]$ and $R[y] \subseteq A$
$\Rightarrow R[x] \subseteq A$
$\Rightarrow x \in A$

This shows that $R^*(A) \subseteq A$. Hence $R^*(A) = A$.

This completes the proof of (a)$\Rightarrow$(b).

Now consider $R^*(A) = A$. Let $x \in A$. Then $x \in R^*(A)$
$\Rightarrow R[x] \cap A \neq \phi$.

Let $z \in R[x]$. Then $xRz \Rightarrow R[z] = R[x]$
$\Rightarrow R[z] \cap A \neq \phi \Rightarrow z \in R^*(A) \Rightarrow z \in A$.

This proves that $R[x] \subseteq A$ and hence $x \in R_*(A)$.

Hence $R_*(A) = A$. This completes the proof of (b)$\Rightarrow$ (a).

Obviously (c) is equivalent to both (a) and (b).

3. ROUGH APPROXIMATIONS OF SUBGROUPS

In this section, we present a few properties of lower and upper rough approximations of subgroups of G .

3.1 Proposition: $\{e\}$ is a Rough set.

Proof: Since $R_*(\{e\}) \subseteq \{e\} \subseteq R^*(\{e\})$ either $R_*(\{e\}) = \phi$ or $R_*(\{e\}) = \{e\}$ .Assume that $R_*(\{e\}) = \{e\}$. Then $e \in R_*(\{e\}) \Rightarrow H \subseteq \{e\} \Rightarrow H = \{e\}$.
This is a contradiction. Hence $R_*(\{e\}) = \phi$.
Let $x \in R^*(\{e\})$. Then $xH \cap \{e\} \neq \phi$.
$\Rightarrow e \in xH$
$\Rightarrow H^{-1} \in H$
$\Rightarrow x \in H$
This shows that $R^*(\{e\}) \subseteq H$ (1)

Let $x \in H$. Then $xH = H$
$\Rightarrow xH \cap \{e\} = H \cap \{e\} = \{e\} \neq \phi$
$\Rightarrow x \in R^*(\{e\})$
This shows that $H \subseteq R^*(\{e\})$(2)

From (1) and (2), $R^*(\{e\}) = H$.
Then $\mathfrak{B}_R(\{e\}) = H$.
Hence $\{e\}$ is a Rough set.

3.2 Remark: By the above Proposition-2.5, it follows that the lower approximation of a subgroup of G is not necessarily a subgroup of G . Even though $(\{e\})$ is a subgroup of G , its lower approximation is empty and hence $R_*(\{e\})$ is not a subgroup of G .

3.3 Proposition: H is not a Rough set.

Proof: Clearly $R_*(H) \subseteq H \subseteq R^*(H)$.

Let $x \in H$. Then $xH = H \Rightarrow x \in R_*(H)$.
Then $H = R_*(H)$.
By proposition -2.4, $R_*(H) = H = R^*(H)$.
Hence H is not a Rough set.

3.4 Proposition: Let K be a subgroup of G . Then the following are equivalent.
(a) $H \subseteq K$ (b) $R_*(K) = K = R^*(K)$ (c) $R_*(K)$ is a subgroup of G

Proof: Let K be a subgroup of G . Suppose that $H \subseteq K$.
Clearly $R_*(K) \subseteq K \subseteq R^*(K)$

Let $x \in K$. Since $H \subseteq K$, $xH \subseteq xK = K$
$\Rightarrow x \in R_*(K)$.

This shows that $K \subseteq R_*(K) \;\Rightarrow\; K = R_*(K)$.

Hence $R_*(K) = K = R^*(K)$.

This proves $(a) \Rightarrow (b)$.

Suppose that $R_*(K) = K = R^*(K)$. Then $R_*(K)$ is a subgroup of G.
This proves $(b) \Rightarrow (c)$.

(b) Suppose that $R_*(K)$ is a subgroup of G
$\Rightarrow \; e \in R_*(K)$
$\Rightarrow \; H = eH \subseteq K$
This proves $(c) \Rightarrow (a)$.

3.5 Proposition: If K is a subgroup of G then $R^*(K)$ is asubgroup of G.

Proof: Let K be a subgroup of G.
Let $x \in R^*(K)$.
$\Rightarrow \; xH \bigcap K \neq \phi$
$\Rightarrow$ there exists a point y such that $y \in xH \bigcap K$
$\Rightarrow \; y \in xH$ and $y \in K$
$\Rightarrow \; x^{-1}y \in H$ and $y \in K$
$\Rightarrow \; y^{-1}x \in H$ and $y^{-1} \in K$

Since H is a normal subgroup of G, we have
$x \in G$ and $y^{-1}x \in H$ imply that $xy^{-1}xx^{-1} \in H$
$\Rightarrow \; xy^{-1} \in H$
$\Rightarrow \; y^{-1} \in x^{-1}H$

Hence $y^{-1} \in x^{-1}H \bigcap K$
$\Rightarrow \; x^{-1}H \bigcap K \neq \phi$
$\Rightarrow \; x^{-1} \in R^*(K)$

Let $x, y \in R^*(K)$
$\Rightarrow \; xH \bigcap K \neq \phi$ and $yH \bigcap K \neq \phi$
$\Rightarrow$ there exist two elements p and q in G such that $p \in xH \bigcap K$ and $q \in yH \bigcap K$
$\Rightarrow \; p \in xH, q \in yH, p \in K \;\; and \;\; q \in K$.
$\Rightarrow \; pq \in xyH$ and $pq \in K$
$\Rightarrow \; pq \in xyH \bigcap K$
$\Rightarrow \; xyH \bigcap K \neq \phi$
$\Rightarrow \; xy \in R^*(K)$

Hence $R^*(K)$ is asubgroup of G.

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