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OPERATOR INTERSECTION GRAPH OF A GROUP

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ABSTRACT

Let $(G, *)$ be a group with binary operation '$*$'. The Operator Intersection graph $\Gamma_{OI}(G)$ of G is a graph with $V(\Gamma_{OI}(G)) = G - e$ and two distinct vertices x and y are adjacent in $\Gamma_{OI}(G)$ if and only if $<x*y> \subseteq <x> \cap <y>$. In this paper, we want to explore how the group theoretical properties of G can effect on the graph theoretical properties of $\Gamma_{OI}(G)$. Some characterizations for fundamental properties of $\Gamma_{OI}(G)$ have also been obtained. Finally, we characterize certain classes of Operator Intersection Graph corresponding to finite abelian groups.

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1. INTRODUCTION

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph, for instance, see [1, 2]. In the present article, to any group G, we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting, let us introduce some necessary notation and definitions.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph $\Gamma = (V, E)$, V denote the set of all vertices and E denote the set of all edges in Γ. The degree $\deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $\deg_\Gamma(v)$ simply by $\deg(v)$. The order of Γ is defined $|V(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. A graph Γ is regular if the degrees of all vertices of Γ are the same. A vertex of degree 0 is known as an *isolated vertex* of Γ. A graph Ω is called a subgraph of Γ if $V(\Omega) \subseteq V(\Gamma)$, $E(\Omega) \subseteq E(\Gamma)$. Let $\Gamma = (V, E)$ be a graph and let $S \subseteq V$. A subgraph Ω of Γ is said to be an *induced subgraph* of Γ induced by S, if $V(\Omega) = S$ and each edge of Γ having its ends in S is also an edge of Ω. A simple graph Γ is said to be complete if every pair of distinct vertices of Γ are adjacent in Γ. A graph Γ is said to be connected if every pair of distinct vertices of Γ are connected by a path in Γ. The Union of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2,E_2)$ is a graph $\Gamma = (V,E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ The joint of two graphs $\Gamma_1 = (V_1,E_1)$ and $\Gamma_2 = (V_2,E_2)$ is a graph denoted by $\Gamma_1 + \Gamma_2 = (V,E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup$ {Edges joining every vertex of V_1 with every vertex of V_2}.

Let G be a group with identity e. The order of the group G is the number of elements in G and is denoted by $O(G)$. The order of an element a in a group G is the smallest positive integer k such that $a^k = e$. If no such integer exists, we say a has infinite order. The order of an element a is denoted $O(a)$. Let p be a prime number. A group G with $O(G) = p^k$ for some $k \in \mathbb{Z}^+$, is called a p-group.

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2. PREPARATION OF MANUSCRIPT

In this section, we observe certain basic properties of Operator Intersection graphs.

Definition 2.1: Let $(G, *)$ be a group with binary operation '$*$'. The Operator Intersection graph $\Gamma_{OI}(G)$ of G is a graph with $V(\Gamma_{OI}(G)) = G-e$, where e is an identity element of G and two distinct vertices x and y are adjacent in $\Gamma_{OI}(G)$ if and only if $< x * y > \subseteq < x > \cap < y >$.

Proposition 2.2: *Let $(G, *)$ be a group. For any non self inverse element $x \in G$, x and $x-1$ are adjacent in $\Gamma_{OI}(G)$.*

Proof: Let $(G, *)$ be a group with identity element e. Let $x \in G$ be non self inverse element. Since $x * x-1 = e$ and $< x > = < x-1 >$, $< e > \subseteq < x > \cap < x-1 >$. Hence, the result follows.

Theorem 2.3: Let $(G, *)$ be a group. Let $x \in G$. x is an isolated vertex in $\Gamma_{OI}(G)$ if and only if x is a self inverse element of G.

Proof: Let $(G, *)$ be a group. Let x be an isolated vertex in $\Gamma_{OI}(G)$. We have to prove that x is a self inverse element of G. Suppose not, x is non self inverse element of G. By Proposition 2.2, x is adjacent to $x-1$, which is a contradiction. Conversely assume that, x is a self inverse element of G. Clearly $< x > \cap < y > = \{e\}$ *or* $< x >$ for all $y \in G$. Also for all $y \in G$, $< x * y > \subseteq < x >$ or $\{e\}$. Hence x is an isolated vertex in $\Gamma_{OI}(G)$

Proposition 2.4: Let $(G, *)$ be a group. Any two distinct prime order elements are non adjacent in $\Gamma_{OI}(G)$.

Proof: Let $(G, *)$ be a group with identity element e. Let $x, y \in G$ be any two elements such that $O(x) = p$ and $O(y) = q$, where p, q are distinct prime. Clearly $< x > \cap < y > = \{e\}$. Therefore $< x * y > \not\subseteq < x > \cap < y >$. Hence the result follows.

Theorem 2.5: Let G be any group. $\Gamma_{OI}(G)$ is complete if and only if G is a cyclic group of prime order.

Proof: Let G be a cyclic group of prime order p. Clearly, every element of G other than identity is a generator of G. Let $x \in G - e$. By Proposition 2.2, x and x^{-1} are adjacent. Let $y \in G - e$ be an element other than x^{-1}. Clearly $< x * y > = < x > \cap < y >$. Therefore x and y are adjacent in $\Gamma_{OI}(G)$. Hence $\Gamma_{OI}(G)$ is complete.

Conversely assume that $\Gamma_{OI}(G)$ is complete. Let p and q be two distinct prime such that $p|O(G)$ and $q|O(G)$. By Cauchy's Theorem, G has two elements x, y *such* that $O(x) = p$ and $O(y) = q$. Clearly $< x > \cap < y > = \{e\}$. Therefore x and y are non adjacent in $\Gamma_{OI}(G)$. Therefore $O(G) = p^n$ for some prime p.

Claim: $O(G) = p$

Suppose $O(G) = p^n$, $n > 1$.

Case (i): G has distinct subgroups of order p.

Let $x, y \in G$ such that $O(x) = O(y) = p$ and $< x > \neq < y >$. Clearly $< x > \cap < y > = \{e\}$. Therefore x and y are non adjacent in $\Gamma_{OI}(G)$,which is a contradiction.

Case (ii): G has unique subgroup of order p.

Let H be the subgroup of order p. Therefore $H = < x >$ for some $x \in G$.

Let y be an element of G such that $O(y) = p^k$ for some $1 < k \leq n$. Clearly $< y > \cap < x > = < x > = H$ and $y \notin H$.

Suppose $x * y \in H$. Since $x \in H$ and H is a subgroup of G, $x-1 * (x * y) \in H$. i.e., $y \in H$, which is a contradiction.

From both cases we conclude that $O(G) = p$ for some prime p.

Proposition 2.6: Let $(G, *)$ be a cyclic group. Any two generators of G are adjacent in $\Gamma_{OI}(G)$.

Proof: Let $(G, *)$ be a cyclic group. Let $x, y \in G$ be any two generators of G.

Clearly $< x > \cap < y > = G$. Therefore $< x * y > \subseteq < x > \cap < y >$. Hence, the result follows.

Remark 2.7: The converse of the Proposition 2.6 is not true.

Consider the group $\mathbb{Z}_6$. *In* $\Gamma_{OI}(\mathbb{Z}_6)$, 2 and 4 are adjacent but they are not generators of G.

Theorem 2.8: For any finite group G, $\Gamma_{OI}(G)$ is a tree if and only if G is isomorphic to one of the groups $\mathbb{Z}_2$ *or* $\mathbb{Z}_3$.

Proof: Clearly $\Gamma_{OI}(\mathbb{Z}_2) = K_1$ and $\Gamma_{OI}(\mathbb{Z}_3) = k_2$ and hence $\Gamma_{OI}(G)$ is a tree when G is either $\mathbb{Z}_2$ *or* $\mathbb{Z}_3$. Conversely, assume that $\Gamma_{OI}(G)$ is a tree. Suppose $p|O(G)$ for some prime number $p \geq 5$. Then G has an element of order p and hence $K_{p-1}(p \geq 5)$ is a sub graph of $\Gamma_{OI}(G)$ and so $\Gamma_{OI}(G)$ is not a tree. Suppose $2|O(G)$, then G has self inverse element. Therefore by Theorem 2.3, $\Gamma_{OI}(G)$ has an isolated vertex, a contradiction to the assumption. Therefore $3|O(G)$. Suppose G contains an element a such that $O(a) \geq 9$. Then the subgraph induced by $< a >$ contains $K_6 \subseteq \Gamma_{OI}(G)$, again a contradiction to the assumption. Therefore every element of G has an order three. In this case G has more than 3 elements, then $\Gamma_{OI}(G)$ is disconnected. Hence G has exactly 2 elements of order 3 or 1 element of order 2. From this either $G \cong \mathbb{Z}_2$ *or* $G \cong \mathbb{Z}_3$.

Theorem 2.9: Let G be a group of order pq, where $p < q$ and p, q are two distinct primes. Then $\Gamma_{OI}(G) \cong (K_{p-1} \cup K_{q-1}) \cup K_{\emptyset(pq)}$, if G is cyclic, where $\emptyset$ is the Euler function and $\Gamma_{OI}(G) \cong qK_{p-1} \cup K_{q-1}$, *if* G is non cyclic.

Proof:

Case (i): Let G be a cyclic group. In this case G has an unique p - Sylow subgroup namely H_p and an unique q -Sylow subgroup namely H_q. Clearly $\Gamma_{OI}(H_p) \cong K_{p-1}$ *and* $\Gamma_{OI}(H_q) \cong K_{q-1}$. Note that all the elements in $G - (H_p \cup H_q)$ are generators of G and so $|G - (H_p \cup H_q)| = \emptyset(pq)$. Since any two generators of G are adjacent, the graph induced by generators is $K_{\emptyset(pq)}$. Let $x \in G$ be generator and $y \in H_p$. Suppose $x * y \in H_p$, then $x * y * y-1 \in H_p$, i.e., $x \in H_p$, which is a contradiction. Suppose $x * y \in H_q$, then $<x* y> \not\subseteq <x> \cap < y>$. Therefore generator of G is not adjacent to non generator elements. Hence, $\Gamma_{OI}(G) \cong (K_{p-1} \cup K_{q-1}) \cup K_{\emptyset(pq)}$.

Case (ii): Let G be a non-cyclic group. In this case G has q p –Sylow subgroups and a unique q -Sylow subgroup. Hence $\Gamma_{OI}(G) \cong qK_{p-1} \cup K_{q-1}$.

Proposition 2.10: Let G be a finite group of order n with no self inverse element and q be number of edges in $\Gamma_{OI}(G)$. Then $q \geq \frac{n-1}{2}$. Moreover, this bound is sharp.

Proof: By Proposition 2.2, x and $x-1$ are adjacent for all $x \in G-e$. Hence $q \geq \frac{n-1}{2}$. Moreover, for the group $\mathbb{Z}_3$, $\Gamma_{OI}(\mathbb{Z}_3) \cong K_{1,1}$ and for this graph the bound is sharp.

We now characterize the groups G for which the associated graph $\Gamma_{OI}(G)$ attains this bound.

Theorem 2.11: Let G be a group of order n and no self inverse element. Let q be number of edges in $\Gamma_{OI}(G)$. $q = \frac{n-1}{2}$ if and only if $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$.

Proof: Assume that $\Gamma_{OI}(G)$ is a graph with $\frac{n-1}{2}$ edges. In view of Proposition 2.2, we get $\Gamma_{OI}(G)$ is an union of K_2. Suppose $p \geq 5$ be a prime number such that $p|O(G)$, then G has an element of order p and so K_{p-1} is a subgraph of $\Gamma_{OI}(G)$, which is a contradiction. Since G has no self inverse element, $O(G)$ must be 3^n. Suppose G has an element of order 3^k for some $k \geq 2$, then $\Gamma_{OI}(G)$ contains K_6 as a sub graph, which is a contradiction. Therefore every element of G has order 3. Hence $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$. Conversely assume that $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \ldots \times \mathbb{Z}_3$. Let $x, y \in G$ such that $< x > \cap < y > = \{e\}$. Clearly x and y are non adjacent. Therefore the adjacent vertices of x is $x-1$ only. Hence $\Gamma_{OI}(G)$ is union of K_2.

Theorem 2.12: Let G be a cyclic group of order p^2. Then $\Gamma_{OI}(G) \cong K_{\emptyset(p^2)} \cup K_{p-1}$, where p is a prime and $\varphi(n)$ is an Euler function.

Proof: Let G be a cyclic group of order p^2. Let A be the set of elements of order p and B be the set of elements of order p^2. Since G is a cyclic group, the elements in B are generators of G. Clearly $|A|=p-1$ *and* $|B|= \Phi(p^2)$. Therefore by Proposition 2.6, the graph induced by the set B is $K_{\emptyset(p^2)}$ and by Theorem 2.5, the graph induced by the set A is K_{p-1}. Let $x \in A$ and $y \in B$. Suppose $x * y \in A$. Since $A \cup \{e\}$ is a subgroup of order p, $x^{-1} * x * y \in A$. Therefore $y \in A$, which is a contradiction. Suppose $x * y \in B$. $< x > \cap < y > = < x >$. Therefore $< x * y > \not\subseteq < x > \cap < y >$. So x and y are non adjacent. Hence the proof follows.

Theorem 2.13: Let $G \cong \mathbb{Z}_p\times\mathbb{Z}_p\times\ldots\times\mathbb{Z}_p$ and $O(G) = p^n$. Then $\Gamma_{OI}(G)\cong K_{p-1} \cup K_{p-1} \cup \ldots \cup K_{p-1}$, where p is a prime.

Proof: Let $G \cong \mathbb{Z}_p \times\mathbb{Z}_p \times \ldots \times\mathbb{Z}_p$. Clearly G can be divided into $\frac{p^n-1}{p-1}$ distinct subgroup of order p. By Theorem 2.5, the graph induced by the subgroup of order p is K_{p-1}. Let x, y be two distinct elements from two distinct subgroups. Note that $< x > \cap < y >= \{e\}$. Therefore x, y are non adjacent in $\Gamma_{OI}(G)$. Hence the proof follows.

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