Let $G = (V, E)$ be a $(p, q)$ graph. An injection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ is called permutation labeling if the edge values are obtained by the number of permutations of the larger vertex label taken smaller vertex label at a time are all distinct. If a graph $G$ admits such labeling, it is called a permutation graph. In this paper we prove that cycle with one chord, cycle with twin chords, $P_2 + K_n$, book graph, tadpole and lotus inside a circle are permutation graphs.

**Keywords**: Permutation graphs, Join of two graphs.

**Subject classification number**: 05C78.

1 Introduction

Let $G$ be a simple, finite, connected and undirected graph with $p$ vertices and $q$ edges. We follow Harary[3] for the standard terminology and notations. Graph labeling is the assignment of real values or subsets of a set, subject to certain conditions, have been motivated due to its usefulness and applications in various fields. Permutation and combinations play an important role in combinatorial problems. In [4] Hegde et al. proved that $K_n$ is permutation graph if and only if $n \leq 5$. Further in [1] Baskar et al. proved that cycles, path, stars are permutation graphs. We present several definitions which are necessary for our present investigation.

Definition 1.1 A chord of a cycle is an edge joining two non-adjacent vertices of a cycle, where chord forms a triangle with two edges of the cycle.

Definition 1.2 Two chords of a cycle are said to be twin chords if they form a triangle with an edge of the cycle.

For positive integers $n$ and $p$ with $3 \leq p \leq n - 2$, $C_{n,p}$ is the graph consisting of a cycle $C_n$ with a pair of twin chords with which the edges of $C_n$ form cycles $C_p$, $C_3$ and $C_{n+1-p}$ without chords.

Definition 1.3 Let $G_1$ and $G_2$ be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. The join of $G_1$ and $G_2$ denoted by $G_1 + G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$, and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{uv/u \in V(G_1)$ and $v \in V(G_2)\}$.

Definition 1.4 Let $C_n$ be a cycle with $n$ vertices $\{u_1, u_2, \ldots, u_n\}$ and $K_{1,n}$ be the star graph with $n+1$ vertices $\{v, v_1, v_2, \ldots, v_n\}$, where $v$ is the apex vertex and $\{v_1, v_2, \ldots, v_n\}$
are the pendant vertices. The lotus inside a circle $C_n$, denoted by $LC_n$, is obtained by joining each $v_i$ to $u_i$ and $u_{i+1} \pmod{n}$, for $i = 1, 2, \ldots, n$.

**Definition 1.5** Let $G = (p, q)$ be a graph. An injection $f : V(G) \to \{1, 2, \ldots, p\}$ is called permutation labeling of $G$, if the induced edge function $g_f : E(G) \to N$ given by $g_f(uv) = f(u) \cdot P_{(v)},$ if $f(u) > f(v)$ and $f(v) \cdot P_{f(u)},$ if $f(u) > f(v)$ is injective, where $f(u) \cdot P_{f(u)}$ denotes the number of permutations of $f(u)$ things taken along $f(v)$ at a time. for all $u, v \in V(G)$.

## 2 Main Results

**Theorem 1** Cycle $C_n$ with one chord is permutation graph, for all $n \geq 4, n \in N$.

**Proof**: Let $G$ be the cycle with one chord. Let $\{v_1, v_2, \ldots, v_n\}$ be the successive vertices of $C_n$ and $\{e_1, e_2, \ldots, e_n\}$ be the edges of $C_n$, where $e_i = v_iv_{i+1}, 1 \leq i \leq n - 1$ and $e_n = v_nv_1$.

Let $e' = v_2v_n$ be the chord in cycle $C_n$. Here $|V(G)| = n$ and $|E(G)| = n + 1$.

We define injection $f : V(G) \to \{1, 2, \ldots, n\}$ by

$f(v_1) = 1, f(v_i) = 2i - 2; 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$,  
$= 2(n - i) + 3; \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$.

**Injectivity for edge labels:**

Here we note that $g_f(v_1v_2) = 2$, is the smallest edge label among all edge labels in graph $G$. For $2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$, $g_f$ is increasing for increasing value of $f(v_i)$ and so we get $g_f(v_iv_{i+1}) < g_f(v_{i+1}v_{i+2})$. Similarly for $\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$, $g_f$ is decreasing for decreasing value of $f(v_i)$ and so we get $g_f(v_iv_{i+1}) > g_f(v_{i+1}v_{i+2})$.

Now we claim that $g_f(v_iv_{i+1}) \neq g_f(v_jv_{j+1}),$ for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n - 1$.

Suppose $g_f(v_iv_{i+1}) = g_f(v_jv_{j+1})$. We have $f(v_i) = r$, for some $r$, where $r$ is an even positive integer and $f(v_j) = t$, for some $t$, where $t$ is an odd positive integer. So $r + 2P_{r+1} = (r+2)! = (r+2)! = (r+2)!$, which is not possible as L.H.S of this equation is factorial of an even number and R.H.S is factorial of an odd number. Hence the claim is proved.

Further $g_f(v_{i+1}v_{i+2}) = n!$, which is the highest among all the edge labels in graph $G$. Also we have $g_f(v_nv_1) = 3$ and $g_f(v_2v_n) = 6$, which appear only once in $G$.

Hence the induced edge labeling $g_f : E(G) \to N$ is injective. So graph $G$ is permutation graph, for all $n \geq 4, n \in N$.

**Theorem 2** Cycle $C_n$ with twin chords is permutation graph, for all $n \geq 5, n \in N$.

**Proof**: Let $G$ be the cycle with twin chords. Let $\{v_1, v_2, \ldots, v_n\}$ be the successive vertices of cycle $C_n$ and let $e' = v_2v_n, e'' = v_3v_n$ be the two chords of $C_n$ , $|V(G)| = n$ and $|E(G)| = n + 2$.

We define the similar injection, as we have defined in Theorem 1.

We also apply the similar arguments for injectivity of edge labels in graph $G$.

Furthermore $g_f(v_3v_n) = 4!$, which is also distinct from all other edge labels. Hence the induced edge labeling $g_f : E(G) \to N$ is injective. So graph $G$ is permutation graph, for all $n \geq 5, n \in N$.

**Theorem 3** $P_2 + \overline{K_n}$ is permutation graph, for all $n \in N$.

**Proof**: Let $G = P_2 + \overline{K_n}$. Let $\{u_1, u_2\}$ be the vertices corresponding to $P_2$ and
Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices corresponding to $K_n$. Here $|V(G)| = n + 2$ and $|E(G)| = 2n + 1$. $E(G) = \{u_1u_2 \cup \{u_1v_i : 1 \leq i \leq n\} \cup \{uv_i, 1 \leq i \leq n\}$.

We define injection $f : V(G) \rightarrow \{1, 2, \ldots, n + 2\}$ as $f(u_1) = 1$, $f(u_2) = n + 2$, and $f(v_i) = i + 1, 1 \leq i \leq n$.

**Injectivity for edge labels:**

Here $g_f(u_1v_1) = i + 1$ are distinct for increasing value of $i$, $1 \leq i \leq n$. Also $g_f(u_2v_i) = i + 2$.

Claim 1: $g_f(u_1v_1) \neq g_f(u_2v_2)$, for all $i$, $1 \leq i \leq n$.

The highest value of $g_f(u_1v_1) = g_f(u_1v_n) = n + 1$, which is smaller than the smallest value of $g_f(u_2v_i) = g_f(u_2v_1) = n + 2$. So claim 1 is proved.

Claim 2: $g_f(u_1u_2) \neq g_f(u_1v_1)$, for all $i$, $1 \leq i \leq n$.

The highest value of $g_f(u_1v_1) = g_f(u_1v_n) = n + 1$, which is smaller than $g_f(u_1u_2) = n + 2$. So claim 2 is proved.

Claim 3: $g_f(u_1u_2) \neq g_f(u_2v_2)$, for all $i$, $1 \leq i \leq n$.

The smallest value of $g_f(u_2v_2) = g_f(u_2v_1) = (n + 1)(n + 2)$, for all $n$. But $g_f(u_1u_2) = (n + 1)(n + 2)$. So claim 3 is also proved.

Hence the induced edge labeling $g_f : E(G) \rightarrow N$ is injective. So graph $G$ is permutation graph, for all $n \in N$.

**Theorem 4** Tadpole $T_{m,n}$ is permutation graph, for all $m, n \in N$, $m \geq 3$.

**Proof :** Let $\{v_1, v_2, \ldots, v_n\}$ be the successive vertices corresponding to cycle $C_m$ and $\{v_{m+1}, v_{m+2}, \ldots, v_{n+m}\}$ be the successive vertices corresponding to $P_n$ in tadpole $T_{m,n}$.

Let $e = v_{n+m+1}$ be the bridge in tadpole $T_{m,n}$. Here $|V(G)| = m + n$ and $|E(G)| = m + n$.

We define injection $f : V(G) \rightarrow \{1, 2, \ldots, m + n\}$ by $f(v_1) = 1$.

$f(v_i) = 2i - 2$; $2 \leq i \leq \left[\frac{m}{2}\right] + 1$;

$= 2(m - i) + 3$, $\left[\frac{m}{2}\right] + 2 \leq i \leq m$,

$= i$, $m + 1 \leq i \leq m + n$.

**Injectivity for edge labels :**

For $2 \leq i \leq \left[\frac{m}{2}\right] + 1$, $g_f$ is increasing for increasing value of $f(v_i)$ and we get $g_f(v_{i+1}) < g_f(v_{i+1}v_i)$. Similarly for $\left[\frac{m}{2}\right] + 2 \leq i \leq m$, $g_f$ is decreasing for decreasing value of $f(v_i)$ and we get $g_f(v_{i+1}) > g_f(v_{i+1}v_i)$. Also $f(v_i)$ is increasing, for increasing values of $i$, $m + 1 \leq i \leq m + n$.

Claim : $g_f(v_{i+1}) \neq g_f(v_{j+1})$, $2 \leq i \leq \left[\frac{m}{2}\right]$, $\left[\frac{m}{2}\right] + 2 \leq j \leq m - 1$ and $m + 1 \leq t \leq m + n - 1$.

Assume if possible $g_f(v_{i+1}) = g_f(v_{j+1})$.

We have $f(v_i) = r$, $r$ being an even positive integer and $f(v_j) = t$, $t$ being an odd positive integer. So we get $t + 2P_r \Rightarrow t + 2P_r \Rightarrow \frac{(t+2)!}{2!} \Rightarrow (r+2)! \Rightarrow (t+2)! = (t+2)!$, which is not possible as L.H.S. is factorial of an even number and R.H.S. is factorial of an odd number. Hence $g_f(v_{i+1}) \neq g_f(v_{j+1})$.

Suppose $g_f(v_{i+1}) = g_f(v_{j+1})$.

Here we have the highest edge label of $g_f(v_{i+1})$ is $m - 1P_{m-1}$. We get the even and $mP_{m-1}$, when $m$ is odd. But the lowest edge label of $g_f(v_{i+1})$ is $(m + 2)!$, which is larger than the highest edge label of $g_f(v_{i+1})$. Hence $g_f(v_{j+1}) \neq g_f(v_{i+1})$.

Similarly by applying the above argument, we get $g_f(v_{i+1}) \neq g_f(v_{i+1})$.

Hence the claim is proved. Hence the induced edge labeling $g_f : E(G) \rightarrow N$ is injective. So graph $G$ is permutation graph, for all $m, n \in N$, $m \geq 3$. 

© 2016, IJMA. All Rights Reserved
Theorem 5 Book graph $B_n$ is permutation graph, for every positive integer $n$.

Proof: Let $B_n = K_{1,n} \times P_2$ be book graph. Let $\{u_{2i-1}, u_{2i}\}$ be the vertices of $i^{th}$ copy of $P_2$, $1 \leq i \leq n + 1$. Here $|V(B_n)| = 2n + 2$ and $|E(B_n)| = 3n + 1$. Here we have taken $\{u_1, u_2\}$ as the vertex set of central copy of $P_2$. Further $E(B_n) = \{u_1u_j, 3 \leq j \leq 2n + 1, j \text{ is odd}\} \cup \{u_2u_j, 4 \leq j \leq 2n + 2, j \text{ is even}\} \cup \{u_ju_{j+1}, 1 \leq j \leq 2n + 2, j \text{ is odd}\}$. We define injection $f : V(B_n) \to \{1, 2, \ldots, 2n + 2\}$, as $f(u_i) = i$, $1 \leq i \leq 2n + 2$.

Injectivity for edge labels:
As $g_f(u_1u_j) \ (3 \leq j \leq 2n + 1, j \text{ is odd})$, is increasing, for increasing values of $j$, we get $g_f(u_1u_j) < g_f(u_1u_{j+1})$. Similarly $g_f(u_2u_j) < g_f(u_2u_{j+1})$, for $4 \leq j \leq 2n + 2, j \text{ is even}$. Moreover $g_f(u_1u_2) = 2$ is the smallest edge label in graph $B_n$.

Claim: $g_f(u_1u_j) \ (3 \leq j \leq 2n + 1, j \text{ is odd}) \neq g_f(u_2u_j) \ (1 \leq j \leq 2n + 2, j \text{ is even}) \neq g_f(u_1u_{j+1}) \ (1 \leq t \leq 2n + 2, t \text{ is odd})$. Here $g_f(u_1u_j) = jP_1 = j$, is always an odd number and $g_f(u_2u_j) = jP_2 = j(j - 1)$, is always an even number. So clearly $g_f(u_1u_j) \neq g_f(u_2u_j)$. Now it is enough to prove $g_f(u_2u_j) \neq g_f(u_1u_{j+1})$.

Suppose $g_f(u_2u_j) = g_f(u_1u_{j+1})$, for some $j$ and $t$. So $jP_2 = (t + 1)! \Rightarrow j(j - 1) = (t + 1)$. Now as $j$ is even let $j = 2p$, for some $p \in N, p > 1$ and for odd $t, t = 2p - 1$, for some $p \in N, p > 1$.

So $2p(2p - 1) = (2p - 1)!$
$\Rightarrow (2p)(2p - 1) = (2p)$
$\Rightarrow 2p - 2 = 0 \text{ or } 2p - 2 = 1$
$\Rightarrow p = 1 \text{ or } p = \frac{3}{2}$, which is not possible as $p \in N$ and $p > 1$.

Hence the claim is proved. So the induced edge labeling $g_f : E(G) \to N$ is injective. So graph $B_n$ is permutation graph, for every positive integer $n$.

Theorem 6 Lotus inside a circle is permutation graph.

Proof: Let $LC_n$ be a lotus inside a circle. Let $\{u_1, u_2, \ldots, u_n\}$ be the successive vertices corresponding to cycle $C_n$ and $\{v_1, v_2, v_3, \ldots, v_n\}$ be the successive vertices corresponding to star $K_{1,n}$, where $v$ is the central vertex of $K_{1,n}$. Here $|V(LC_n)| = 2n + 1$ and $|E(LC_n)| = 4n$. Also $E(LC_n) = \{u_1u_{i+1}, 1 \leq i \leq n\} \cup \{v_iu_1, 1 \leq i \leq n\}$.

We define injection $f : V(LC_n) \to \{1, 2, \ldots, 2n + 1\}$ by $f(v) = 2n + 1$ and $f(u_i) = 2i, 1 \leq i \leq n, f(v_i) = 2i - 1, 1 \leq i \leq n$.

Injectivity for edge labels:
Claim 1: For all $u_i, 1 \leq i \leq n$, $rP_{r-2} \neq r+2 P_r$, where $f(u_i) = r$ is an even positive integer.

Suppose claim 1 is not true, so $rP_{r-2} = r+2 P_r \Rightarrow \frac{r!}{(r-2)!} = \frac{(r+2)!}{2r}$ \Rightarrow $r = \frac{3 + \sqrt{5}}{2}$, which contradicts the choice of $r$. Hence Claim 1 is proved.

Claim 2: For $\{v_iu_j, 1 \leq i \leq n\}$ and $\{v_{i+1}u_1, 1 \leq i \leq n\}$, $rP_{r-1} \neq r+2 P_{r-1}$, where $f(u_i) = r = an even positive integer.

Suppose claim 2 is not true, so $rP_{r-2} = r+2 P_{r-1} \Rightarrow \frac{r!}{(r-2)!} = \frac{(r+2)!}{2r} \Rightarrow 6 = (r + 2)(r + 1) \Rightarrow r = 1$ or $r = -4$, which contradicts the choice of $r$. Hence claim 2 proved.

Claim 3: For $\{v_iu_1, 1 \leq i \leq n\}$, $2n+1 P_r \neq 2n+1 P_{r-2}$, for $1 \leq r \leq n$.

Suppose Claim 3 does not hold. So we get $2n+1 P_r = 2n+1 P_{r-2} \Rightarrow (2n+1)! = \frac{(2n+1)!}{(2n+1-r-2)!} = (2n-r+3)! = (2n-r+1)!$
Suppose $2n - r = t$, we get $t^2 + 5t + 5 = 0 \Rightarrow t = \frac{-5 \pm \sqrt{5}}{2}$ and hence $t < 0 \Rightarrow 2n - r < 0$
$r > 2n$, which is contradiction as $1 \leq r \leq n$. So claim 3 is proved.

Claim 4: $g_f(u_{i+1}u_i) \neq g_f(v_iu_i) \neq g_f(v_iu_{i+1})$, for $1 \leq i \leq n$.

We note that $g_f(u_{i+1}u_i) = r P_{r-2}$, where $f(u_{i+1}) = r$ is an even positive integer, $r > 2$, $1 \leq i \leq n$, $r \leq n$. $g_f(v_iu_i) = q P_{q-1}$, where $f(u_i) = q$ is an even positive integer, $1 \leq i \leq n$, $1 \leq q \leq n$. $g_f(v_iu_{i+1}) = t P_{t-3}$, where $f(u_{i+1}) = t$ is an even positive integer, $t > 2$, $1 \leq i \leq n$, $t \leq n$. $g_f(v_iu_i) = 2n+1 P_{2n-1}$, $1 \leq i \leq n$.

Suppose $g_f(u_{i+1}u_i) = g_f(v_iu_i) \Rightarrow r P_{r-2} = q P_{q-1} \Rightarrow \frac{r}{r-2} = \frac{q}{q-1} \Rightarrow \frac{q}{q-1} = \frac{r}{r-2} \Rightarrow q! = (q-1)! \Rightarrow q = 2$. And for if $g_f(v_iu_i) = g_f(v_iu_{i+1}) \Rightarrow q P_{q-1} = t P_{t-3} \Rightarrow q! = t! \Rightarrow q! = t! \Rightarrow q = t$. But the ratio of factorial of two consecutive even integers (greater than 2) is almost $\frac{1}{2}$ and so $g_f(u_{i+1}u_i) \neq g_f(v_iu_i) \neq g_f(v_iu_{i+1})$. Similarly we get $g_f(v_iu_{i+1}) \neq g_f(v_iu_i)$. Hence the induced edge labeling $g_f : E(LC_n) \rightarrow N$ is injective. So Graph $LC_n$ is permutation graph, for all $n$.

3 Figures and Examples

Illustration 1 Permutation labeling of cycle $C_8$ with one chord and cycle $C_8$ with twin chords are given in following Figure 1.

![Figure 1: permutation labeling of cycle $C_8$ with one chord and cycle $C_8$ with twin chords.](image)

Illustration 2 Permutation labeling of $P_2 + K_5$ is given in following Figure 2.

![Figure 2: permutation labeling of $P_2 + K_5$.](image)

Illustration 3 Permutation labeling of $T_{1,3}$ is given in following Figure 3.
Illustration 4  Permutation labeling of $B_4$ is given in following Figure 4.

Illustration 5  Permutation labeling of $LC_4$ is given in following Figure 5.
4 Conclusion and Open Problems

Due to the present investigation, six new results related to permutation graphs are found. Here we also put open problems related to permutation labeling.

Problem 1: To prove or disprove star of some graph is permutation.

Problem 2: To characterize graphs $G$ and $H$ such that join of $G$ and $H$ is permutation.

References


Source of support: Nil, Conflict of interest: None Declared

[Copyright 2016. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]