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SOME NEW PERMUTATION GRAPHS

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Abstract

Let $G = (V, E)$ be a (p, q) graph. An injection $f : V(G) \to \{1, 2, \dots, p\}$ is called permutation labeling if the edge values are obtained by the number of permutations of the larger vertex label taken smaller vertex label at a time are all distinct. If a graph G admits such labeling, it is called a permutation graph. In this paper we prove that cycle with one chord, cycle with twin chords, $P_2 + \overline{K_n}$, book graph, tadpole and lotus inside a circle are permutation graphs.

Key words : Permutation graphs, Join of two graphs.
Subject classification number: 05C78.

1 Introduction

Let G be a simple, finite, connected and undirected graph with p vertices and q edges. We follow Harary[3] for the standard terminology and notations. Graph labeling is the assignment of real values or subsets of a set, subject to certain conditions, have been motivated due to its usefulness and applications in various fields. Permutation and combinations play an important role in combinatorial problems. In [4] Hegde et al. proved that K_n is permutation graph if and only if $n \leq 5$. Further in [1] Baskar et al. proved that cycles, path, stars are permutaion graphs. We present several definitions which are necessary for our present investigation.

Definition 1.1 A *chord* of a cycle is an edge joining two non-adjacent vertices of a cycle, where chord forms a triangle with two edges of the cycle.

Definition 1.2 Two chords of a cycle are said to be twin chords if they form a triangle with an edge of the cycle.

For positive integers n and p with $3 \leq p \leq n - 2$, $C_{n,p}$ is the graph consisting of a cycle C_n with a pair of twin chords with which the edges of C_n form cycles C_p, C_3 and C_{n+1-p} without chords.

Definition 1.3 Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \phi$. The join of G_1 and G_2 denoted by $G_1 + G_2$ is the graph with vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$, and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{uv/u \in V(G_1)$ and $v \in V(G_2)\}$.

Definition 1.4 Let C_n be a cycle with n vertices $\{u_1, u_2, \dots, u_n\}$ and $K_{1,n}$ be the star graph with $n+1$ vertices $\{v, v_1, v_2, \dots, v_n\}$, where v is the apex vertex and $\{v_1, v_2, \dots, v_n\}$

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are the pendant vertices. The lotus inside a circle C_n, denoted by LC_n, is obtained by joining each v_i to u_i and $u_{i+1}(mod n)$, for $i = 1, 2, \ldots, n$.

Definition 1.5 Let $G = (p, q)$ be a graph. An injection $f : V(G) \rightarrow \{1, 2, \ldots, p\}$ is called *permutation labeling* of G, if the induced edge function $g_f : E(G) \rightarrow N$ given by $g_f(uv) = ^{f(u)} P_{f(v)}$, if $f(u) > f(v)$ and $^{f(v)}P_{f(u)}$, if $f(v) > f(u)$ is injective, where $^{f(u)}P_{f(v)}$ denotes the number of permutations of $f(u)$ things taken along $f(v)$ at a time. for all $u, v \in V(G)$.

2 Main Results

Theorem 1 Cycle C_n with one chord is permutation graph, for all $n \geq 4, n \in N$.
Proof : Let G be the cycle with one chord. Let $\{v_1, v_2, \ldots, v_n\}$ be the successive vertices of C_n and $\{e_1, e_2, \ldots, e_n\}$ be the edges of C_n, where $e_i = v_i v_{i+1}$, $1 \leq i \leq n - 1$ and $e_n = v_n v_1$.
Let $e' = v_2 v_n$ be the chord in cycle C_n. Here $| V(G) |= n$ and $| E(G) |= n + 1$.
We define injection $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ by
$f(v_1) = 1$,
$f(v_i) = 2i - 2; 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$,
$\quad = 2(n - i) + 3; \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$.
Injectivity for edge labels:
Here we note that $g_f(v_1 v_2) = 2$, is the smallest edge label among all edge labels in graph G. For $2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1$, g_f is increasing for increasing value of $f(v_i)$ and so we get $g_f(v_i v_{i+1}) < g_f(v_{i+1} v_{i+2})$. Similarly for $\lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n$, g_f is decreasing for decreasing value of $f(v_i)$ and so we get $g_f(v_i v_{i+1}) > g_f(v_{i+1} v_{i+2})$.
Now we claim that $g_f(v_i v_{i+1}) \neq g_f(v_j v_{j+1})$, for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n - 1$. Suppose $g_f(v_i v_{i+1}) = g_f(v_j v_{j+1})$. We have $f(v_i) = r$, for some r, where r is an even positive integer and $f(v_j) = t$, for some t, where t is an odd positive integer. So $^{r+2}P_r = ^{t+2} P_t \Longrightarrow \frac{(r+2)!}{2!} = \frac{(t+2)!}{2!} \Longrightarrow (r + 2)! = (t + 2)!$, which is not possible as L.H.S. of this equation is factorial of an even number and R.H.S. is factorial of an odd number. Hence the claim is proved.
Further $g_f(v_{\lfloor \frac{n}{2} \rfloor + 1} v_{\lfloor \frac{n}{2} \rfloor + 2}) = n!$, which is the highest among all the edge labels in graph G. Also we have $g_f(v_1 v_n) = 3$ and $g_f(v_2 v_n) = 6$, which appear only once in G.
Hence the induced edge labeling $g_f : E(G) \rightarrow N$ is injective. So graph G is permutation graph, for all $n \geq 4, n \in N$.
Theorem 2 Cycle C_n with twin chords is permutation graph, for all $n \geq 5, n \in N$.
Proof : Let G be the cycle with twin chords. Let $\{v_1, v_2, \ldots, v_n\}$ be the successive vertices of cycle C_n and let $e' = v_2 v_n$, $e'' = v_3 v_n$ be the two chords of C_n, $| V(G) |= n$ and $| E(G) |= n + 2$.
We define the similar injection, as we have defined in Theorem 1.
We also apply the similar arguments for injectivity of edge labels in graph G.
Furthermore $g_f(v_3 v_n) = 4!$, which is also distinct from all other edge labels. Hence the induced edge labeling $g_f : E(G) \rightarrow N$ is injective. So graph G is permutation graph, for all $n \geq 5, n \in N$.
Theorem 3 $P_2 + \overline{K_n}$ is permutation graph, for all $n \in N$.
Proof : Let $G = P_2 + \overline{K_n}$. Let $\{u_1, u_2\}$ be the vertices corresponding to P_2 and

$\{v_1, v_2, \ldots, v_n\}$ be the vertices corresponding to $\overline{K_n}$. Here $|\ V(G)\ | = n+2$ and $|\ E(G)\ | = 2n+1$. $E(G) = \{u_1u_2\} \cup \{u_1v_i; 1 \le i \le n\} \cup \{u_2v_i, 1 \le i \le n\}$.
We define injection $f : V(G) \to \{1, 2, \ldots, n+2\}$ as $f(u_1) = 1$, $f(u_2) = n+2$, and $f(v_i) = i+1, 1 \le i \le n$.
Injectivity for edge labels:
Here $g_f(u_1v_i) = i+1$ are distinct for increasing value of i, $1 \le i \le n$. Also $g_f(u_2v_i) = {}^{n+2}P_{f(v_i)}$ are distinct for increasing values of i, $1 \le i \le n$.
Claim 1 : $g_f(u_1v_i) \ne g_f(u_2v_i)$, for all i, $1 \le i \le n$.
The highest value of $g_f(u_1v_i) = g_f(u_1v_n) = n+1$, which is smaller than the smallest value of $g_f(u_2v_i) = g_f(u_2v_1) = (n+1)(n+2)$. So claim 1 is proved.
Claim 2 : $g_f(u_1u_2) \ne g_f(u_1v_i)$, for all i, $1 \le i \le n$.
The highest value of $g_f(u_1v_i) = g_f(u_1v_n) = n+1$, which is smaller than $g_f(u_1u_2) = n+2$. So claim 2 is proved.
Claim 3 : $g_f(u_1u_2) \ne g_f(u_2v_i)$, for all i,$1 \le i \le n$.
The smallest value of $g_f(u_2v_i) = g_f(u_2v_1) = (n+1)(n+2)$, for all n. But $g_f(u_1u_2) = (n+1) < (n+1)(n+2)$. So claim 3 is also proved.
Hence the induced edge labeling $g_f : E(G) \to N$ is injective. So graph G is permutation graph, for all $n \in N$.
Theorem 4 Tadpole $T_{m,n}$ is permutation graph, for all $m, n \in N$, $m \ge 3$.
Proof : Let $\{v_1, v_2, \ldots, v_m\}$ be the successive vertices corresponding to cycle C_m and $\{v_{m+1}, v_{m+2}, \ldots v_{m+n}\}$ be the successive vertices corresponding to P_n in tadpole $T_{m,n}$. Let $e = v_mv_{m+1}$ be the bridge in tadpole $T_{m,n}$. Here $|\ V(G)\ | = m+n$ and $|\ E(G)\ | = m+n$.
We define injection $f : V(G) \to \{1, 2, \ldots, m+n\}$ by
$f(v_1) = 1$,
$f(v_i) = 2i-2$; $2 \le i \le \lfloor \frac{m}{2} \rfloor + 1$;
$\quad = 2(m-i)+3$; $\lfloor \frac{m}{2} \rfloor + 2 \le i \le m$,
$\quad = i$; $m+1 \le i \le m+n$.
Injectivity for edge labels :
For $2 \le i \le \lfloor \frac{m}{2} \rfloor + 1$, g_f is increasing for increasing value of $f(v_i)$ and so we get $g_f(v_iv_{i+1}) < g_f(v_{i+1}v_{i+2})$. Similarly for $\lfloor \frac{m}{2} \rfloor + 2 \le i \le m$, g_f is decreasing for decreasing value of $f(v_i)$ and so we get $g_f(v_iv_{i+1}) > g_f(v_{i+1}v_{i+2})$. Also $f(v_i)$ is increasing, for increasing values of i, $m+1 \le i \le m+n$.
Claim : $g_f(v_iv_{i+1}) \ne g_f(v_jv_{j+1}) \ne g_f(v_tv_{t+1}), 2 \le i \le \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 2 \le j \le m-1$ and $m+1 \le t \le m+n-1$.
Assume if possible $g_f(v_iv_{i+1}) = g_f(v_jv_{j+1})$.
We have $f(v_i) = r$, r being an even positive integer and $f(v_j) = t$, t being an odd positive integer. So we get ${}^{r+2}P_r = {}^{t+2}P_t \implies \frac{(r+2)!}{2!} = \frac{(t+2)!}{2!} \implies (r+2)! = (t+2)!$, which is not possible as L.H.S. is factorial of an even number and R.H.S. is factorial of an odd number. Hence $g_f(v_iv_{i+1}) \ne g_f(v_jv_{j+1})$.
Suppose $g_f(v_jv_{j+1}) = g_f(v_tv_{t+1})$.
Here we have the highest edge label of $g_f(v_jv_{j+1})$ is ${}^{m-1}P_{m-3}$, when m is even and ${}^{m}P_{m-2}$, when m is odd. But the lowest edge label of $g_f(v_tv_{t+1})$ is $(m+2)!$, which is larger than the highest edge label of $g_f(v_jv_{j+1})$. Hence $g_f(v_jv_{j+1}) \ne g_f(v_tv_{t+1})$.
Similarly by applying the above argument, we get $g_f(v_tv_{t+1}) \ne g_f(v_iv_{i+1})$.
Hence the claim is proved. Hence the induced edge labeling $g_f : E(G) \to N$ is injective. So graph G is permutation graph, for all $m, n \in N$, $m \ge 3$.

Theorem 5 Book graph B_n is permutation graph, for every positive integer n.
Proof : Let $B_n = K_{1,n} \times P_2$ be book graph. Let $\{u_{2i-1}, u_{2i}\}$ be the vertices of i^{th} copy of P_2, $1 \leq i \leq n+1$. Here $\mid V(B_n) \mid = 2n+2$ and $\mid E(B_n) \mid = 3n+1$.
Here we have taken $\{u_1, u_2\}$ as the vertex set of central copy of P_2. Further $E(B_n) = \{u_1u_j, 3 \leq j \leq 2n+1, j \text{ is odd}\} \bigcup \{u_2u_j, 4 \leq j \leq 2n+2, j \text{ is even}\} \bigcup \{u_ju_{j+1}, 1 \leq j \leq 2n+2, j \text{ is odd}\}$. We define injection $f : V(B_n) \to \{1, 2, \ldots, 2n+2\}$, as $f(u_i) = i$, $1 \leq i \leq 2n+2$.
Injectivity for edge labels :
As $g_f(u_1u_j)$ ($3 \leq j \leq 2n+1$, j is odd), is increasing, for increasing values of j, we get $g_f(u_1u_j) < g_f(u_1u_{j+1})$. Similarly $g_f(u_2u_j) < g_f(u_2u_{j+1})$, for $4 \leq j \leq 2n+2$, j is even. Moreover $g_f(u_1u_2) = 2$ is the smallest edge label in graph B_n.
Claim: $g_f(u_1u_j)$ ($3 \leq j \leq 2n+1$, j is odd) $\neq g_f(u_2u_j)$ ($1 \leq j \leq 2n+2$, j is even) $\neq g_f(u_tu_{t+1})$ ($1 \leq t \leq 2n+2$, t is odd).
Here $g_f(u_1u_j) = {}^jP_1 = j$, is always an odd number and $g_f(u_2u_j) = {}^jP_2 = j(j-1)$, is always an even number. So clearly $g_f(u_1u_j) \neq g_f(u_2u_j)$.
Now it is enough to prove $g_f(u_2u_j) \neq g_f(u_tu_{t+1})$
Suppose $g_f(u_2u_j) = g_f(u_tu_{t+1})$, for some j and t. So ${}^jP_2 = (t+1)! \Longrightarrow j(j-1) = (t+1)!$. Now as j is even let $j = 2p$, for some $p \in N$, $p > 1$ and for odd t, $t = 2p-1$, for some $p \in N$, $p > 1$.
So $2p(2p-1) = (2p-1+1)!$
$\Rightarrow 2p(2p-1) = (2p)!$
$\Rightarrow 2p-2 = 0$ or $2p-2 = 1$
$\Rightarrow p = 1$ or $p = \frac{3}{2}$, which is not possible as $p \in N$ and $p > 1$.
Hence the claim is proved. So the induced edge labeling $g_f : E(G) \to N$ is injective. So graph B_n is permutation graph, for every positive integer n.
Theorem 6 Lotus inside a circle is permutation graph.
Proof : Let LC_n be a lotus inside a circle. Let $\{u_1, u_2, \ldots, u_n\}$ be the successive vertices corresponding to cycle C_n and $\{v, v_1, v_2, \ldots, v_n\}$ be the successive vertices corresponding to star $K_{1,n}$,where v is the central vertex of $K_{1,n}$. Here $\mid V(LC_n) \mid = 2n+1$ and $\mid E(LC_n) \mid = 4n$. Also $E(LC_n) = \{u_iu_{i+1}, 1 \leq i \leq n\} \bigcup \{v_iu_i, 1 \leq i \leq n\} \bigcup \{v_iu_{i+1}, 1 \leq i \leq n \bigcup \{vv_i, 1 \leq i \leq n\}$.
We define injection $f : V(LC_n) \to \{1, 2, \ldots, 2n+1\}$ by $f(v) = 2n+1$ and $f(u_i) = 2i, 1 \leq i \leq n$, $f(v_i) = 2i-1, 1 \leq i \leq n$.
Injectivity for edge labels:
Claim 1: For all u_i, $1 \leq i \leq n,{}^rP_{r-2} \neq^{r+2} P_r$, where $f(u_i) = r$= an even positive integer.
Suppose claim 1 is not true, so ${}^rP_{r-2} =^{r+2} P_r \Rightarrow \frac{r!}{2!} = \frac{(r+2)!}{2!} \Rightarrow 1 = (r+2)(r+1) \Rightarrow r = \frac{-3 \pm \sqrt{5}}{2}$, which contradicts the choice of r. Hence Claim 1 is proved.
Claim 2: For $\{v_iu_i, 1 \leq i \leq n\}$ and $\{v_iu_{i+1}, 1 \leq i \leq n\}$, ${}^rP_{r-1} \neq^{r+2} P_{r-1}$, where $f(u_i) = r$ = an even positive integer.
Suppose claim 2 is not true, so ${}^rP_{r-2} =^{r+2} P_{r-1} \Rightarrow \frac{r!}{1!} = \frac{(r+2)!}{3!} \Rightarrow 6 = (r+2)(r+1)$ $\Rightarrow r = 1$ or $r = -4$, which contradicts the choice of r. Hence claim 2 proved.
Claim 3: For $\{vv_i, 1 \leq i \leq n\}$, ${}^{2n+1}P_r \neq^{2n+1} P_{r-2}$, for $1 \leq r \leq n$.
Suppose Claim 3 does not hold. So we get ${}^{2n+1}P_r =^{2n+1} P_{r-2} \Rightarrow \frac{(2n+1)!}{(2n+1-r)!} = \frac{(2n+1)!}{(2n+1-r+2)!} \Rightarrow (2n-r+3)! = (2n-r+1)!$
Suppose $2n-r = t$, we get $t^2+5t+5 = 0 \Rightarrow t = \frac{-5 \pm \sqrt{5}}{2}$ and hence $t < 0 \Rightarrow 2n-r < 0$

$\Rightarrow r > 2n$, which is contradiction as $1 \leq r \leq n$. So claim 3 is proved.
Claim 4: $g_f(u_{i+1}u_i) \neq g_f(v_iu_i) \neq g_f(v_iu_{i+1}) \neq g_f(vv_i)$, for $1 \leq i \leq n$
We note that $g_f(u_{i+1}u_i) =^r P_{r-2}$, where $f(u_{i+1} = r$ = an even positive integer, $r > 2$, $1 \leq i \leq n$, $r \leq n$. $g_f(v_iu_i) =^q P_{q-1}$, where $f(u_i) = q$ = an even positive integer, $1 \leq i \leq n$, $1 \leq q \leq n$. $g_f(v_iu_{i+1}) =^t P_{t-3}$, where $f(u_{i+1}) = t$ = an even positive integer,$t > 2$, $1 \leq i \leq n$, $t \leq n$. $g_f(vv_i) =^{2n+1} P_{2i-1}$, $1 \leq i \leq n$.
Suppose $g_f(u_{i+1}u_i) = g_f(v_iu_i) \Rightarrow {}^rP_{r-2} =^q P_{q-1} \Rightarrow \frac{r!}{2!} = \frac{q!}{1!} \Rightarrow \frac{q!}{r!} = \frac{1}{2}$ and for if $g_f(v_iu_i) = g_f(v_iu_{i+1}) \Rightarrow {}^qP_{q-1} =^t P_{t-3} \Rightarrow q! = \frac{t!}{3!} \Rightarrow \frac{q!}{r!} = \frac{1}{6}$,
But the ratio of factorial of two consecutive even integers(greater than 2) is atmost $\frac{1}{12}$ and so $g_f(u_{i+1}u_i) \neq g_f(v_iu_i) \neq g_f(v_iu_{i+1})$. Similarly we get$g_f(v_iu_{i+1}) \neq g_f(vv_i)$. Hence the claim is proved.
Hence the induced edge labeling $g_f : E(LC_n) \to N$ is injective. So Graph LC_n is permutation graph, for all n.

3 Figures and Examples

Illustration 1 Permutation labeling of cycle C_8 with one chord and cycle C_8 with twin chords are given in following *Figure 1* .

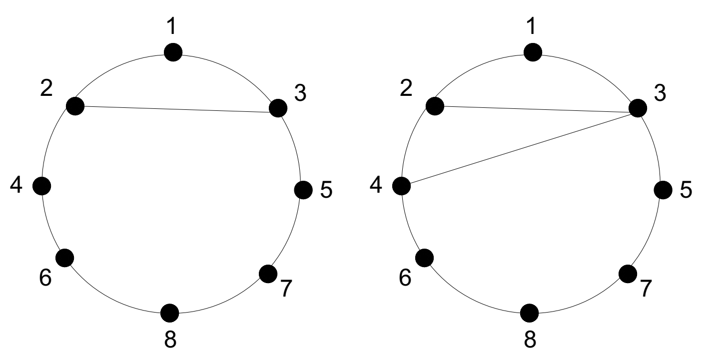

Figure 1 : permutation labeling of cycle C_8 with one chord and cycle C_8 with twin chords.

Illustration 2 Permutation labeling of $P_2 + \overline{K_5}$ is given in following *Figure 2* .

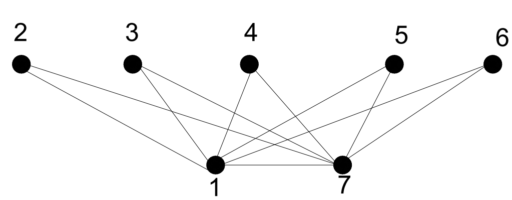

Figure 2 : permutation labeling of $P_2 + \overline{K_5}$

Illustration 3 Permutation labeling of $T_{4,3}$ is given in following *Figure 3*.

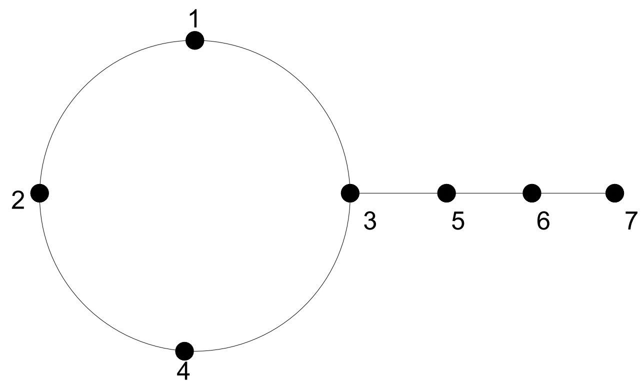

Figure 3 : permutation labeling of $T_{4,3}$

Illustration 4 Permutation labeling of B_4 is given in following *Figure 4*.

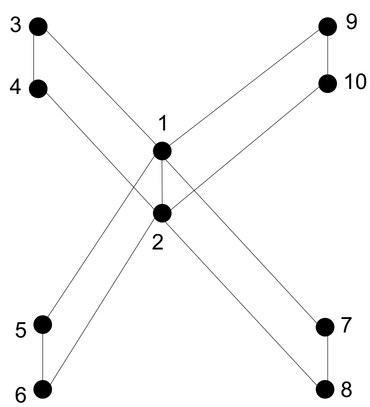

Figure 4 : permutation labeling of B_4

Illustration 5 Permutation labeling of LC_4 is given in following *Figure 5*.

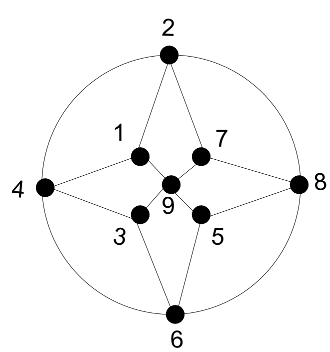

Figure 5 : permutation labeling of LC_4

4 Conclusion and Open Problems

Due to the present investigation, six new results related to permutation graphs are found. Here We also put open problems related to permutation labeling.
Problem 1 : To prove or disprove star of some graph is permutation.
Problem 2 : To characterize graphs G and H such that join of G and H is permutation.

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