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ADJOINT OPERATOR IN FUZZY 2-NORMED LINEAR SPACES

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ABSTRACT

In this paper, the definition of 2-adjoint operator on the fuzzy 2-normed linear spaces is introduced. It is shown that if $T : X \times X \to Y \times Y$ *be a strongly (weakly) fuzzy 2-bounded linear operator, then* $T^* : Y^* \times Y^* \to X^* \times X^*$ *(adjoint of* T *) is strongly (weakly) fuzzy 2-bounded linear operator and* $\|T(x,x')\|_\alpha^* = \|T^*(x,x')\|_\alpha^*$, *for each* $\alpha \in (0,1]$.

Keywords: *fuzzy 2-Adjoint operator, Fuzzy 2-Dual space.*

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [16] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler [6]. In 2009, Sundaram and Beaula [13] defined the concept of fuzzy 2-normed linear space and introduced fuzzy 2-linear operator.

Sinha, Lal and Mishra [12] introduced the concept of fuzzy 2-bounded linear operator on a fuzzy 2-normed linear space to another fuzzy 2-normed linear space and two types of (strong and weak) fuzzy 2-bounded linear operators were also defined. They also discussed the relation between strong fuzzy 2-bounded linear operator and weak fuzzy 2-bounded linear operator.

Ali Taghavi, Majid Mehdizadeh [14] introduced the definition of adjoint operator in fuzzy normed linear spaces and investigated some important general properties of adjoint fuzzy linear operators on fuzzy normed linear spaces.

In this paper we have introduced the concept of 2-adjoint linear operator on fuzzy 2-normed linear spaces. We have also investigated some properties of 2-adjoint linear operator of fuzzy 2-normed linear spaces.

2. PRELIMINARIES

In this section some definition and preliminaries results are given which will be used in this paper.

Definition 2.1 [10]: Let X be a linear space over a field F. A fuzzy subset N of $X \times X \times R$ (R is the set of real numbers) is called a fuzzy 2-norm on X if and only if

(N1). For all $t \in R$, with $t \le 0$, $N(x_1, x_2, t) = 0$

(N2). For all $t \in R$, with $t > 0$, $N(x_1, x_2, t) = 1$, if and only if x_1 and x_2 are linearly dependent.

(N3). $N(x_1, x_2, t)$ is invariant under any permutation of x_1, x_2.

(N4). For all $t \in R$, with $t > 0$

$$N(x_1, cx_2, t) = N\left(x_1, x_2, \frac{t}{|c|}\right) \text{ if } c \ne 0,\ c \in F.$$

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(N5). For all $s,t \in R$,

$$N(x_1, x_2 + x_2', s+t) \ge \min\{N(x_1, x_2, s), N(x_1, x_2', t)\}$$

(N6). $N(x_1, x_2, \bullet)$ is a non-decreasing function of R and

$$\lim_{t\to\infty} N(x_1, x_2, t) = 1.$$

Then (X, N) is called a fuzzy 2-normed linear space.

Later on we will be required the conditions

(N7). For all $\forall\, t > 0$, $N(x_1, x_2, t) > 0 \;\Rightarrow\; x_1, x_2$ are linearly dependent.

(N8). For x_1, x_2 linearly independent, $N(x_1, x_2, .)$ is continuous function of R and strictly increasing on $\{t : 0 < N(x_1, x_2, t) < 1\}$ of R.

Definition 2.2[11]: Let $T : A\times B \to C\times D$ be a fuzzy 2-linear operator, where A, B are subspaces of fuzzy 2-normed linear spaces (X, N_1) and C, D are subspaces of fuzzy 2-normed linear space (Y, N_2), then T is said to be fuzzy 2-continuous at $(x_0, x_0') \in A\times B$ for given $\varepsilon > 0$, $\alpha \in (0,1)$, $\exists\ \delta = \delta(\alpha, \varepsilon) > 0, \beta = \beta(\alpha, \varepsilon) \in (0,1)$, $\forall\ (x, x') \in A\times B$, $N_1\left[(x,x') - (x_0, x_0'), \delta\right] > \beta \Rightarrow N_2\left[T(x,x') - T(x_0, x_0'), \varepsilon\right] > \alpha$ if T is continuous at each point of $A\times B$ then it is fuzzy 2-continuous on $A\times B$.

Definition 2.3[11]: Let $T : A\times B \to C\times D$ be a fuzzy 2-linear operator, where A, B are subspaces of fuzzy 2-normed linear space (X, N_1) and C, D are subspaces of fuzzy 2-normed linear space (Y, N_2), then T is said to be strongly fuzzy 2-continuous at $(x_0, x_0') \in A\times B$ for given $\varepsilon > 0$, $\exists\ \delta > 0$ $\forall (x, x') \in A\times B$,

$$N_2\left[T(x,x') - T(x_0, x_0'), \varepsilon\right] \ge N_1\left[(x,x') - (x_0, x_0'), \delta\right].$$

Definition 2.4[11]: Let $T : A\times B \to C\times D$ be a fuzzy 2-linear operator, where A, B are subspaces of fuzzy 2-normed linear space (X, N_1) and C, D are subspaces of fuzzy 2-normed linear space (Y, N_2), then T is said to be weakly fuzzy 2-continuous at $(x_0, x_0') \in A\times B$ for given $\varepsilon > 0$, $\alpha \in (0,1)$, $\exists\ \delta = (\alpha, \varepsilon) > 0$ such that $\forall (x, x') \in A\times B$

$$N_1\left[(x,x') - (x_0, x_0'), \delta\right] \ge \alpha \Rightarrow N_2\left[T(x,x') - T(x_0, x_0'), \varepsilon\right] \ge \alpha.$$

Definition 2.5[12]: Let $T : A\times B \to C\times D$ be a fuzzy 2-linear operator, where A, B are subspaces of fuzzy 2-normed linear space (X, N_1) and C, D are subspaces of fuzzy 2-normed linear space (Y, N_2), then T is said to be strongly fuzzy 2- bounded on $A\times B$ if and only if, $\exists$ a positive real number M such that $\forall (x, x') \in A\times B$ and $\forall\, s \in R$,

$$N_2\left[T(x,x'), s\right] \ge N_1\left[(x,x'), \frac{s}{M}\right].$$

Definition 2.6[12]: Let $T : A\times B \to C\times D$ be a fuzzy 2-linear operator, where A, B are subspaces of (X, N_1) and C, D are subspaces of (Y, N_2), then T is said to be weakly fuzzy 2-bounded on $A\times B$ if for any $\alpha \in (0,1)$, $\exists\ M_\alpha > 0$ such that $\forall (x, x') \in A\times B, \forall\, t \in R$, $N_1\left((x,x'), \dfrac{t}{M_\alpha}\right) \ge \alpha \Rightarrow N_2(T(x,x'), t) \ge \alpha.$

Note: Let the linear space $Y = R$ or C. We define $N_2 : Y\times R \to [0,1]$ as

$$\begin{aligned} N_2(x,t) &= 1 \text{ if } t \le \|x\| \\ &= 0 \text{ if } t > \|x\| \end{aligned} \tag{A}$$

We find N_2 is a fuzzy norm on Y and thus (Y, N_2) is a fuzzy normed linear space.

A strongly fuzzy 2- bounded linear operator from $X\times X$ to Y where (X, N_1) is a fuzzy 2-normed linear space and Y is R or C with fuzzy norm defined by (A) is called strongly fuzzy 2- bounded linear functional. We will denote it by X^*.

Theorem 2.1[13]: Let (X, N_1) be a fuzzy 2-normed linear space. Assume that it satisfies (N7) i.e., $N(x_1, x_2, t) > 0$, For all $t > 0$ implies x_1 and x_2 are linearly dependent, define $\|x_1, x_2\|_\alpha = \inf\{t : N(x_1, x_2, t) \geq \alpha \in (0,1)\}$. Then $\{\|.,.\|_\alpha : \alpha \in [0,1)\}$ is an ascending family of 2-norms on X. These 2-norms are called α -2- norms on X corresponding to the fuzzy 2-norms.

Definition 2.7[4]: Let (X, N_1) be a fuzzy 2- normed linear space satisfying (N7), (N8). Let $T \in X^*$ and $\{\|.,.\|_\alpha^1 : \alpha \in (0,1)\}$ be the family of α -2 norm of N_1

Define $\|T\|_\alpha^* = \underset{(x_1,x_2)\in X}{\vee} \dfrac{|T(x_1, x_2)|}{\|x_1, x_2\|_\alpha^1}, \forall \alpha \in (0,1)$

Then $\{\|\ \|_\alpha^* : \alpha \in (0,1)\}$ is an ascending family of norm on X^*. Again we define

$$N^*(T, s) = \sup\{\alpha \in (0,1) : \|T\|_*^\alpha \leq s\} \text{ for } (T, s) \neq (0,0)$$
$$= 0 \text{ for } (T, s) = (0,0)$$

Then N^* is a fuzzy norm on X^* so (X^*, N^*) is a fuzzy normed linear space. We call X^* the strong fuzzy dual space of X^*.

Notation 2.1: Let (X, N_1) and (Y, N_2) are fuzzy 2-normed linear spaces. Denote $B(X,Y) =$ set of all strongly fuzzy 2-bounded linear operators defined from $X \times X$ to $Y \times Y$.

Notation 2.2: Denote $B'(X,Y) =$ set of all weakly fuzzy 2- bounded linear operators defined from $X \times X$ to $Y \times Y$.

Definition 2.8[13]: A fuzzy 2-linear functional F is a real valued function $A \times B$ where A, B are subspaces of fuzzy 2-normed linear spaces (X, N) such that

(1) $F(x + x', y + y') = F(x, y) + F(x, y') + F(x', y) + F(x', y')$

(2) $F(\alpha x, \beta y) = \alpha\ \beta\ F(x, y), \quad \alpha, \beta \in [0,1]$.

F is said to be bounded with respect to α -2-norm if there exists a constant $k \in [0,1]$ such that $|F(x, y)| \leq k\|x, y\|_\alpha$, for every $(x, y) \in A \times B$. If F is bounded then norm of F is defined as
$\|F\| = \text{glb}\{k : |(x, y)| \leq k\|x, y\|_\alpha \text{ for every } (x, y) \in A \times B\}$.

Theorem 2.2 [3]: Let (X, N) be a fuzzy 2-normed linear space and Z be a subspace of X. Let F be a fuzzy 2-bounded linear functional on M then F can be extended to a fuzzy 2-linear functional F_0 defined on the whole space (X, N) such that $\|F_0\| = 1$.

THE MAIN RESULTS

In this section we have defined fuzzy 2-adjoint linear operator and some important results related to it.

Definition 3.1: Let (X, N_1) and (Y, N_2) are two fuzzy 2-normed linear spaces and $T \in B(X,Y)(B'(X,Y))$. Operator $T^* : Y^* \times Y^* \to X^* \times X^*$ is defined by
$(T^*(F_1, F_2))(x, x') = (\ (F_1, F_2)T(x, x')), \ \ \forall\ (F_1, F_2) \in Y^* \times Y^*, (x, x') \in X \times X = X^2$.
Then T^* is called fuzzy 2-adjoint operator of T.

Theorem 3.1: Let (X, N_1) be a fuzzy 2-normed linear space satisfying (N7) and (N8). Then for each $\alpha \in (0,1)$ then there exists a strongly fuzzy 2- bounded linear functional $F \in X^*$ such that $\|F\|_\alpha^* = 1$ and $F(x_0, x_0') = \|x_0, x_0'\|_\alpha^1$ for every α belong to (0, 1).

Proof: Since (X, N_1) be a fuzzy 2-normed linear space satisfying (N7) then $\left(X, \|.,.\|_\alpha^1\right)$ is a 2- normed linear space for each $\alpha \in (0,1)$. By 2- Hahn Banach theorem over 2- normed linear space $\left(X, \|.,.\|_\alpha^1\right), \exists\ F \in X^*$ such that $\|F\|_\alpha = 1$ and $F(x_0, x_0') = \|x_0, x_0'\|_\alpha^1$.

Theorem 3.2: Let (X, N_1) and (Y, N_2) be fuzzy 2-normed linear spaces. If $T \in B(X, Y)$ then $T^* \in B(Y^*, X^*)$ and $\forall\, \alpha \in (0,1]$, $\|T\|_\alpha^* = \|T^*\|_\alpha^*$.

Proof: For each $(F_1, F_2) \in Y^* \times Y^*$ we get $T^*(F_1, F_2) = (F_1, F_2)T$ and so $T^*(F_1, F_2)$ is fuzzy 2-continuos linear operator. So for each $(F_1, F_2) \in Y^* \times Y^*$ we have $T^*(F_1, F_2) \in X^* \times X^*$ clearly, since T is strongly fuzzy 2-bounded on $X \times X$. Hence it is uniformly fuzzy 2- bounded linear operator so $\exists\ M\ (> 0) \in R$ such that

$\|T(x, x')\|_\alpha^2 \le M\|x, x'\|_\alpha^1, \ \ \forall\ (x, x')\ (\ne 0) \in X^2,$

As $\|T\|_\alpha^* = \bigvee_{(x,x') \in X^2, (x,x') \ne 0} \dfrac{\|T(x, x')\|_\alpha^2}{\|(x, x')\|_\alpha^1} (\le M),$

$$\text{So } \|T(x, x')\|_\alpha^2 \le \|T\|_\alpha^* \|x, x'\|_\alpha^1 \tag{1}$$

Let $(x, x') \in X \times X$ and $(F_1, F_2) \in Y^* \times Y^*$, therefore from (1), we have

$\left\|\left(T^*(F_1, F_2)\right)(x, x')\right\|_\alpha^* = \|(F_1, F_2)(T(x, x'))\|_\alpha^* \le |(F_1, F_2)|\ \|T(x, x')\|_\alpha^2 \le\ |(F_1, F_2)|\ \|T\|_\alpha^*\ \|(x, x')\|_\alpha^1, \ \forall\ \alpha \in (0,1]$

So T^* is 2-continuos and

$$\|T^*\|_\alpha^* \le \|T\|_\alpha^* \le M, \quad \forall\ \alpha \in (0,1] \tag{2}$$

So T^* is strongly fuzzy 2- bounded and so $T^* \in B(Y^*, X^*)$.

For $\varepsilon > 0$, there exists (x, x') such that $\|(x, x')\|_\alpha = 1$ and $\|T(x, x')\|_\alpha^2 \ge \|T\|_\alpha^* - \varepsilon$, by theorem (3.1) there exists $(F_1, F_2) \in Y^* \times Y^*$ such that $\|(F_1, F_2)\|_\alpha^* = 1$ and $(F_1, F_2)(T(x, x')) = \|T(x, x')\|_\alpha^2$.

So $\left|\left(T^*(F_1, F_2)\right)(x, x')\right| = |(F_1, F_2)(T(x, x'))| = \|T(x, x')\|_\alpha^2 \ge \|T\|_\alpha^* - \varepsilon, \forall\, \alpha \in (0,1]$.

Thus $\|T^*\|_\alpha^* \ge \|T^*(F_1, F_2)\|_\alpha^* > \|T\|_\alpha^* - \varepsilon, \forall\, \alpha \in (0,1]$,

$$\text{So } \|T^*\|_\alpha^* \ge \|T\|_\alpha^*, \forall\, \alpha \in (0,1], \tag{3}$$

(2) and (3) implies that

$\|T^*\|_\alpha^* = \|T\|_\alpha^*, \forall\, \alpha \in (0,1]$

Theorem 3.3: Let (X, N_1) be a fuzzy 2-normed linear space. If $T \in B'(X, Y)$ then $T^* \in B'(Y^*, X^*)$ and $\|T(x, x')\|_\alpha^* = \|T^*(x, x')\|_\alpha^* \ \ \forall\ \alpha(0,1]$.

Proof: The proof is similar to theorem (3.2) and therefore, it is omitted.

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