ERROR ESTIMATE 0F THE FINITE FOURIER TRANSFORM
BY THE AVERAGED MODULUS OF SMOOTHNESS

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ABSTRACT

In this paper we will show how we can estimate the error for the Finite Fourier Transform by Averaged modulus of smoothness, for integrable 1-periodic functions defined on aclosed interval and prove results.

1. INTRODUCTION

First, we will define a set of complex sequences of lenght $N+1$ by

$$S_{N=} \{ c_0, c_1, \ldots\ldots\ldots c_n : c_j \in C \quad , \quad j = 0, 1, \ldots\ldots N \} \tag{1.1}$$

For every $N \in N$, the finite Fourier transform (FFT) is the map from S_N to itself defined by

$$F_{N(\langle c_j \rangle)_k} = \frac{1}{N+1} \sum_{j=0}^{N} c_j \, e^{\frac{-2\pi ijk}{N+1}} \tag{1.2}$$

This sequence is periodic of period $N+1$, it can be extended to all $K \in Z$, it is natural to think $k_{min = -\left[\frac{N+1}{2}\right]}$ to $k_{max = \left[\frac{N}{2}\right]}$

The inverse of (*FFT*) is given as

$$F_N^{-1}(\langle c_j \rangle) = \sum_{k=kmin}^{kmax} c_j \, e^{\frac{2\pi ijk}{N+1}} \tag{1.3}$$

Aprincipal application of the *(FFT)* is to approximately compute samples of the Fourier transform of a function If f is a function defined on [0, 1] then we define

$$\tilde{f}_{N,k} = \frac{1}{N+1} \sum_{j=1}^{N} f\left(\frac{j}{N+1}\right) e^{\frac{-2\pi ijk}{N+1}} \tag{1.4}$$

The sum on the right-hand side of equation (1.4) is Riemann sum for the integral defining the kth Fourier cofficient

$$\hat{f}(k) = \int_o^1 f(x) e^{-2\pi ikx} \, dx \tag{1.5}$$

1. Charles L.Epsten estimat the error of the (*FFT*) by modulus of continuity for continuous and continuous derivatives 1-periodic function on closed interval
2. Auslander and Grunbaum estimate the error of best approximation of (*FFT*) in terms of the sampling models and the frequency but are independent of the function
3. Ivanov obtained some results about approximation of measurable and bounded functions by bernstein polynomials in L_p [0, 1] space
4. Hirstov used a locally global norm for bounded functions and proved that the best one-sided approximations of 2π-periodic bounded function with trigonometric polynomials of degree *n* in the norm L_P

2. SOME USEFUL INFORMATION

Let *f* be an integrable function .on [0, 1] then, we define the Averaged modulus of smoothness

$$\tau_{k\,(f,\delta)_p} = \left\| \omega_{k(f,:;\delta)} \right\|_{L_p} \tag{2.1}$$

$$= \left[\int_a^b \left(\omega_k(f, x; \delta) \right)^p \right]^{\frac{1}{p}}$$

where

$$\omega_k \ (f, x; \delta) = sup \left\{ \left| \Delta_h^k f(t) \right| : t, t + kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2} \right] \cap [a, b] \right\}$$

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And

$$\Delta_h^k f(t) = \sum_{M=0}^{k} (-1)^{M+k} \binom{k}{m} f(t + Mh)$$

Such that

$$\binom{k}{m} = \frac{k!}{m!(k-m)!}$$

Which called local moduli of smoothness.

An exponential polynomial of order M is a function of the form

$$P(x) = \sum_{-M}^{M} \alpha_k \ e^{2\pi i k x} \qquad \alpha_k \in C \tag{2.2}$$

We denote the set of such functions by T_M, if f is an integrable,1-periodic function defined on [0,1] ,then for each $M \in N$, there is a function $p^* \in T_M$ that is a best approximation to f such that

$$\|f - p^*\|_{p = \min_{p \in T_M} \|f-p\|_p} \tag{2.3}$$

Proposition 2.1 [3]: If f is an integrable on $[0, 1]$, then for each integer $n \geq 1$, we have

$$\text{F(x)} = P_{n-1}(x) + \varphi_n(f; x) + \sum_{j=0}^{n} \frac{1}{h} \int_0^t \varphi_n \ (f; jh + v) l_{n,j}' \left(\frac{x-v}{h}\right) dv \tag{2.4}$$

where P_{n-1} is a polynomial of degree at most *n-1*

Proposition 2.2 [3]: Let $P_{n-1}^*(f)$ be the interpolation polynomial for f at the point h, 2h,.........,nh
i.e

$$P_{n-1}^*(f; x) = \sum_{j=1}^{n} f(ih) l_{n-1,j-1} \left(\frac{x}{h} - 1\right) \tag{2.5}$$

where

$$l_{n,j} \left(\frac{x}{h}\right) = l_{n-1,j-1} \left(\frac{x}{h} - 1\right) + (-1)^{n-j} \binom{n}{j} l_{n,0} \left(\frac{x}{h}\right)$$

Then

$$f(x) - P_{n-1}^*(f; x) = \Delta_h^n f(0) l_{n,0} \left(\frac{x}{h}\right) + \varphi_n(f; x) - \sum_{j=0}^{n} \varphi_n(f; jh) l_{n,j} \left(\frac{x}{h}\right) + h^{-1} \int_o^t \sum_{j=o}^{n} \varphi_n(f; jh + v) \ l_{n,j}' \left(\frac{x-v}{h}\right) dv \tag{2.6}$$

Lemma 2.1 [3]: We have

$$y_n := max \left\{ \sum_{j=1}^{n} \binom{n}{j}^{-1} |l_{n,j}(x)| : 0 \leq x \leq 1 \right\} = 1 \tag{2.7}$$

Lemma 2.2:

$$\mu_{n,v} := \sum_{j=o}^{n} \binom{n}{j}^{-1} max\{|l_{n,j}(x)| : v \leq x \leq v + 1\} \tag{2.8}$$

Then

$$\mu_{n,v} \leq \frac{1 + \sigma_v + \sigma_{v+1}}{\binom{n}{v}} \tag{2.9}$$

$V = 0, 1, 2,........\frac{n-1}{2}$
where

$$\sigma_v := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{v} \qquad \sigma_0 := 0$$

In particular, for v = 0, we have

$$\mu_{n,o} = 1 + \sum_{j=1}^{n} \binom{n}{j}^{-1} max\{|l_{n,j}(x)| : o \leq x \leq 1\} \leq 2 \tag{2.10}$$

Proposition 2.3 [3]: For any function f which is integrable on [0, 1] we have

$$\left| f(x) - \sum_{j=1}^{n} f(jh) l_{n-1,j-1} \left(\frac{x}{h} - 1\right) \right| \leq \frac{6 + 7min(\sigma_v, \sigma_{n-v})}{\binom{n}{v}} \tau_n(f; h) \tag{2.11}$$

For x $\in [vh, (v + 1)/h$, h:=1/(n+1), v = 0, 1,.....,n $\sigma_v := 1 + \frac{1}{2} + \cdots + \frac{1}{v}$
$$\sigma_0 := 0$$

3. THE MAIN RESELTS

In this section we state and prove our reselts for an integrabal 1-periodic function

Theorem 3.1: If $f(x)$ be an integrable, 1-periodic function on [0,1] then the best approximation p^* to f from T_M satisfies

$$\|f - p^*\|_p \leq 6 \ \tau_{f \left(\frac{1}{2\pi M}\right)} \tag{3.1}$$

Proof: Since the interpolation polynomial
$$P_{n-1}^*(f;x) = \sum_{j=1}^n f(jh)l_{n-1,j-1}\left(\frac{x}{h}-1\right)$$

In (2.12) are symmetric with respect to the middle of the interval [0, 1], it is sufficient to prove it only for x $\in \left[0,\frac{1}{2}\right]$ or for v = 0,......, [(n-1)/2]. For v = 0, from lemma (2.1) and (2.2) using

$$\| f(x) - p_{n-1}^*(f,x)\| = \left\|\Delta_h^n f(0)l_{n,0}\left(\frac{x}{h}\right) + \varphi_n(f;x) - \sum_{j=o}^n \varphi_n(f;jh)\left(\frac{x}{h}\right) + h^{-1}\int_0^t \sum_{j=0}^n \varphi_n(f;jh+v)l_{n,j}'\left(\frac{x-v}{h}\right)dv\right\|$$

Using

$$\varphi_n(f;x) = \varphi_n(f;vh+t)$$
$$= \frac{(-1)^{n-v}}{h\binom{n}{v}}\int_0^h \Delta_y^n f(x-vy)dy$$

$$\|\varphi_n(f;x)\|_P \le \frac{1}{\binom{n}{v}}\tau_n(f;h)$$

To obtain

$$\|f(x) - P_{n-1}^*(f;x)\|_P \le \left\|\tau_n(f;h)\left[\max_{0\le t\le1}|l_{n,0}(t)| + 1 + \max_{0\le t\le1}\sum_{j=0}^n \frac{1}{\binom{n}{j}}|l_{n,j}(t)| + \sum_{j=0}^n \frac{1}{\binom{n}{j}}\int_0^t |l_{n,j}'(v)|dv\right]\right\|_P$$

By using (2.10) we have
$$\|f(x) - P_{n-1}^*(f;x)\|_P \le 6\tau_n(f;h)$$

Corollary 3.1: If f is an integrable, 1-periodic function on [0,1] with l integrable 1-periodic dervatives ,then the approximation P^* to f from T_M satisfies
$$\|f - p^*\|_P \le 6^{l+1}\frac{\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l} \tag{3.2}$$

Now, we can state and prove our results for integrable 1-periodic function.

Theorem 3.2: for $M \in N$ and f, an integrable 1-periodic function defined on [0, 1], we have estimates
$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \le 2\|p^* - f\|_P \qquad for \ |k| \le M \tag{3.3}$$
where p* $\in T_M$

Proof: Since $p^* \in T_M$, a simple calculation shows that for $|k| \le M$ we have
$$\left|\tilde{f}_{2M,k} - \tilde{p}_{2M,k}^* + \tilde{p}_{2M,k}^* - \hat{f}_{(k)}\right|$$
Then

$$\left|\tilde{f}_{2M,k} - \tilde{p}_{2M,k}^*\right| = \left[\left|\int_0^1 \left(\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right) - p^*\left(\frac{j}{2M+1}\right)\right]e^{\frac{-2\pi ijk}{2M+1}}\right)^p dx\right|\right]^{\frac{1}{p}}$$

$$\le \left[\int_0^1 \left(\left|\frac{1}{2M+1}\sum_{j=0}^{2M}\left[f\left(\frac{j}{2M+1}\right) - p^*\left(\frac{j}{2M+1}\right)\right]e^{\frac{-2\pi ijk}{2M+1}}\right|\right)^P dx\right]^{\frac{1}{p}}$$

$$\left|\tilde{f}_{2M,k} - \tilde{p}_{2M,k}^*\right| \le \|f - p^*\|_P$$

In the second line we use the triangle inquality.

$$\left[\left|\int_0^1 \left(\int_0^1 [f(x) - p^*(x)]e^{-2\pi ikx}\right)^P dx\right|\right]^{\frac{1}{P}} \le \left[\int_0^1 \left|\left(\int_0^1 [f(x) - p^*(x)]e^{-2\pi ikx}\right)^P\right|dx\right]^{\frac{1}{P}}$$

$$\left|\hat{f}(k) - \tilde{p}_{2M,k}\right| \le \|f - p^*\|_P$$

Then

$$\left|\tilde{f}_{2M,k} - \tilde{p}_{2M,k}^* + \hat{p}_{2M,k}^* - \hat{f}_{(k)}\right| \le \left|\tilde{f}_{2M,k} - \tilde{p}_{2M,k}^*\right| + \left|\hat{f}_{(k)} - \hat{p}_{2M,k}^*\right|$$

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \le \|f - p^*\|_P + \|f - p^*\|_P$$
$$\le 2\|p^* - f\|_P$$

Corollary 3.2: suppose that f is an integrabal 1-periodic function with $l \ge 0$ an integrabal 1-periodic derivatives; then
$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \le 12\frac{6^l \tau_f^l\left(\frac{1}{2M\pi}\right)}{(2\pi M)^l} \tag{3.4}$$
For $|k| \le M$

Proof: From theorem (3.2)

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|p^* - f\|_P$$

Then from corollary (3.2)

$$\|f - p^*\|_P \leq 6^{l+1} \frac{\tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Then

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2.6^{l+1} \frac{\tau_f^l\left(\frac{1}{2M\pi}\right)}{(2M\pi)^l}$$

$$\leq 12 \frac{6^l \tau_f^l\left(\frac{1}{2\pi M}\right)}{(2\pi M)^l}$$

Corollary 3.3: suppose that *f* is an integrable 1-periodic function then

$$\left|\tilde{f}_{2M,k}\right| \leq \left|\hat{f}(k)\right| + 6\tau_f\left(\frac{1}{2\pi M}\right) \tag{3.5}$$

Proof: by theorem (3.2)

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq 2\|f - p^*\|_P$$

$$\left|\tilde{f}_{2M,k} - \hat{f}(k)\right| \leq \left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k} + \tilde{p}^*_{2M,k} - \hat{f}(k)\right|$$
$$\leq \left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right| + \left|\tilde{p}^*_{2M,k} - \hat{f}(k)\right|$$

$$\left|\tilde{f}_{2M,k} - \tilde{p}^*_{2M,k}\right| \leq \|f - p^*\|_P$$

Then from theorem (3.1)

$$\|f - p^*\|_p \leq 6\,\tau_f\left(\frac{1}{2\pi M}\right)$$

Then

$$\left|\tilde{f}_{2M,k}\right| \leq \|f - p^*\|_P + \left|\hat{f}(k)\right|$$
$$\leq 6\,\tau_f\left(\frac{1}{2\pi M}\right) + \left|\hat{f}(k)\right|$$

Theorem (3.3) there is a universal constant *C* such that, if *f* is an integrabal 1-periodic function, then

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq C \log M \|f - p^*\|_P$$

Where $p^* \in T_M$ is a best approximation.

Proof: Let Dirichlet kernel

$$f_{2M}(x) = \int_0^1 D_M(x - y)f(y)dy$$

Then

$$\tilde{f}_{2M}(x) = \frac{1}{2M+1}\sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right) f\left(\frac{j}{2M+1}\right)$$

We observe that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq \left|\tilde{f}_{2M(x)} - p^*(x)\right| + \left|p^*(x) - f_{2M(x)}\right|$$

$$\leq \left[\left|\int_0^1 \left(\frac{1}{2M+1}\sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right)\left[f\left(\frac{j}{2M+1}\right) - p^*\left(\frac{j}{2M+1}\right)\right] + \int_0^1 D_M(x-y)[p^*(y) - f(y)]dy\right)^P dx\right|\right]^{\frac{1}{P}}$$

$$\leq \left[\int_0^1 \left|\left(\frac{1}{2M+1}\sum_{j=0}^{2M} D_M\left(x - \frac{j}{2M+1}\right)\left[f\left(\frac{j}{2M+1}\right) - p^*\left(\frac{j}{2M+1}\right)\right] + \int_0^1 D_M(x-y)[p^*(y) - f(y)]dy\right)^P\right| dx\right]^{\frac{1}{P}}$$

$$\leq \|f - p^*\|_p \cdot \left[\frac{1}{2M+1}\sum_{j=0}^{2M}\left|D_M\left(x - \frac{j}{2M+1}\right)\right| + \int_0^1 |D_M(x-y)|\,dy\right]$$

Both the sum and the integral in the last line are bounded by a constant times log M

Corollary 3.4: if *f* is an integrabal 1-periodic function whose averaged modulus satisfies

$$\tau_f(\delta) = o(|\log \delta|^{-1}) \tag{3.7}$$

Then the (FFT) partial sum $(\tilde{f}_{2M})$ converges to f on [0, 1] if f has *l* integrabal periodic derivatives and τ_f^l satisfies the above estimate, then for each $1 \leq j \leq l$, the sequence $\langle \partial_x^i \tilde{f}_{2M}\rangle$ converges to $\partial_x^j f$

Proof: let p*$\in T_M$ be a best approximation to f, we will use the triangle inquality to conclude that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \leq \left|\tilde{f}_{2M(x)} - p^*(x)\right| + \left|p^*(x) - f_{2M(x)}\right|$$

Applying theorem (3.1), we see that

$$\left|\tilde{f}_{2M(x)} - f_{2M(x)}\right| \le (C\log M + 6)\tau_f\left(\frac{1}{2M\pi}\right) \tag{3.8}$$

The estimate in (3.7) implies the right-hand side of equation (3.8) tends to zero as M tends to infinity. The last inquality tell us that $\left\|\tilde{f}_{2M(x)} - f_{2M(x)}\right\|$ tends to zero as M tends to infinity

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