ERROR ESTIMATE OF THE FINITE FOURIER TRANSFORM
BY THE AVERAGED MODULUS OF SMOOTHNESS

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ABSTRACT

In this paper we will show how we can estimate the error for the Finite Fourier Transform by Averaged modulus of smoothness, for integrable 1-periodic functions defined on a closed interval and prove results.

1. INTRODUCTION

First, we will define a set of complex sequences of length $N+1$ by

$$S_N = \{c_0, c_1, \ldots, c_N \}$$

(1.1)

For every $N \in \mathbb{N}$, the finite Fourier transform (FFT) is the map from $S_N$ to itself defined by

$$F_N(c_j) = \frac{1}{N+1} \sum_{j=0}^{N} c_j e^{-2\pi i j k / N+1}$$

(1.2)

This sequence is periodic of period $N+1$, it can be extended to all $k \in \mathbb{Z}$, it is natural to think $\frac{k \pi n}{N+1}$ to $\frac{k \pi n}{2}$

The inverse of (FFT) is given as

$$F_N^{-1}(c_k) = \sum_{k_{min}}^{k_{max}} c_k e^{2\pi i j k / N+1}$$

(1.3)

A principal application of the (FFT) is to approximately compute samples of the Fourier transform of a function. If $f$ is a function defined on $[0, 1]$ then we define

$$\hat{f}_{N,k} = \frac{1}{N+1} \sum_{j=0}^{N} f(j / (N+1)) e^{-2\pi i j k / N+1}$$

(1.4)

The sum on the right-hand side of equation (1.4) is Riemann sum for the integral defining the $k^{th}$ Fourier coefficient

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx$$

(1.5)

1. Charles L. Epstein estimated the error of the (FFT) by modulus of continuity for continuous and continuous derivatives 1-periodic function on closed interval
2. Auslander and Grunbaum estimate the error of best approximation of (FFT) in terms of the sampling models and the frequency but are independent of the function
3. Ivanov obtained some results about approximation of measurable and bounded functions by Bernstein polynomials in $L_p [0, 1]$ space
4. Hirstov used a locally global norm for bounded functions and proved that the best one-sided approximations of $2\pi$-periodic bounded function with trigonometric polynomials of degree $n$ in the norm $L_p$

2. SOME USEFUL INFORMATION

Let $f$ be an integrable function on $[0, 1]$ then, we define the Averaged modulus of smoothness

$$\tau_k(f; \delta) = \left\| \omega_k(f; \delta) \right\|_{L_p}$$

(2.1)

$$= \left\| \int_a^b \left( \omega_k(f, x; \delta) \right)^p dx \right\|^{1/p}$$

where

$$\omega_k(f, x; \delta) = \sup \left\{ \left| \Delta_k^p f(t) \right| : t, t + kh \in \left[ x - \frac{k \delta}{2}, x + \frac{k \delta}{2} \right] \cap [a, b] \right\}$$

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And
\[ \Delta^k f(t) = \sum_{m=-n}^k (-1)^{m+k} \binom{k}{m} f(t + M h) \]
Such that
\[ \binom{k}{m} = \frac{k!}{m!(k-m)!} \]
Which called local moduli of smoothness.

An exponential polynomial of order \( M \) is a function of the form
\[ P(x) = \sum_{-M}^M \alpha_k e^{2\pi i k x} \quad \alpha \in \mathbb{C} \quad (2.2) \]
We denote the set of such functions by \( T_M \), then for each \( M \in \mathbb{N} \), there is a function \( p^* \in T_M \) that is a best approximation to \( f \) such that
\[ \|f - p^*\|_P = \min_{p \in T_M} \|f - p\|_P \quad (2.3) \]

**Proposition 2.1 [3]:** If \( f \) is an integrable on \([0, 1]\), then for each integer \( n \geq l \), we have
\[ \|f - p_n^*\|_{L^1} = \|f - p_n\|_{L^1} \]
where \( p_{n-1} \) is a polynomial of degree at most \( n-l \)

**Proposition 2.2 [3]:** Let \( p_{n-1}(f) \) be the interpolation polynomial for \( f \) at the point \( h, 2h, \ldots, nh \)
i.e
\[ p_{n-1}(f; x) = \sum_{j=1}^n f(jh) l_{n-1,j-1}(\frac{x}{h} - 1) \quad (2.5) \]
where
\[ l_{n,j}(\frac{x}{h}) = l_{n-1,j-1}(\frac{x}{h} - 1) + (-1)^{n-j}(\frac{n}{j}) l_{n,0}(\frac{x}{h}) \]
Then
\[ f(x) - p_{n-1}(f; x) = \Delta_n f(0) l_{n,0}(\frac{x}{h}) + \varphi_n(f; x) - \sum_{j=0}^n \varphi_n(f; jh) l_{n,j}(\frac{x}{h}) + h^{-1} \int_0^1 \sum_{j=0}^n \varphi_n(f; jh + v) l_{n,j}(\frac{x-v}{h}) dv \quad (2.6) \]

**Lemma 2.1 [3]:** We have
\[ \gamma_n := \max \left\{ \sum_{j=1}^n \binom{n}{j}^{-1} |l_{n,j}(x)| : 0 \leq x \leq 1 \right\} = 1 \quad (2.7) \]

**Lemma 2.2:**
\[ \mu_{n,v} := \sum_{j=0}^n \binom{n}{j}^{-1} \max \{|l_{n,j}(x)| : v \leq x \leq v + 1 \} \quad (2.8) \]
Then\[ \mu_{n,v} \leq \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{v}}{v} \quad (2.9) \]
where\[ \sigma_v := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{v} \quad \sigma_0 := 0 \]
In particular, for \( v = 0 \), we have
\[ \mu_{n,0} = 1 + \sum_{j=1}^n \binom{n}{j}^{-1} \max \{|l_{n,j}(x)| : 0 \leq x \leq 1 \} \leq 2 \quad (2.10) \]

**Proposition 2.3 [3]:** For any function \( f \) which is integrable on \([0, 1]\) we have
\[ |f(x) - \sum_{j=1}^n f(jh) l_{n-1,j-1}(\frac{x}{h} - 1) - \frac{6 + 7\min(\sigma_v, \sigma_{v-1})}{(v)_2} t_n(f; h) | \leq \tau_f(h) \frac{1}{2^{2n+1}} \quad (2.11) \]
For \( x \in \left[ vh, (v+1)/h \right), h := 1/(n+1), v = 0, 1, \ldots, n \)
\[ \sigma_v := 1 + \frac{1}{2} + \cdots + \frac{1}{v} \quad \sigma_0 := 0 \]

**3. THE MAIN RESULTS**

In this section we state and prove our results for an integrable 1-periodic function

**Theorem 3.1:** If \( f(x) \) be an integrable, 1-periodic function on \([0, 1]\) then the best approximation \( p^* \) to \( f \) from \( T_M \) satisfies
\[ \|f - p^*\|_P \leq 6 \tau_f(h) \frac{1}{2^{2M}} \quad (3.1) \]
Proof: Since the interpolation polynomial

\[ P_{n-1}^*(f; x) = \sum_{j=1}^{n} f(jh)\left(\frac{x-jh}{h}\right) \]

In (2.12) are symmetric with respect to the middle of the interval \([0, 1]\), it is sufficient to prove it only for \(x \in [0, \frac{1}{2}]\) or for \(v = 0, ..., \lfloor(n-1)/2\rfloor\). For \(v = 0\), from lemma (2.1) and (2.2) using

\[ \|f(x) - P_{n-1}^*(f; x)\| = \left\| \int_0^1 f(x) - \sum_{j=0}^{n} \varphi_n(f; x) - \sum_{j=0}^{n} \varphi_n(f; jh) \left(\frac{x-jh}{h}\right) \right\| \]

Using

\[ \varphi_n(f; x) = \varphi_n(f; vh + t) = \frac{(-1)^{n-v}}{h} \int_0^1 \Delta_h^n f(x - vy) dy \]

To obtain

\[ \|\varphi_n(f; x)\|_p \leq \frac{1}{\ell} \tau_n(f; h) \]

By using (2.10) we have

\[ \|f(x) - P_{n-1}^*(f; x)\|_p \leq 6 \tau_n(f; h) \]

Corollary 3.1: If \(f\) is an integrable, 1-periodic function on \([0, 1]\) with \(l\) integrable 1-periodic derivatives, then the approximation \(P^*\) to \(f\) from \(T_M\) satisfies

\[ \|f - p^*\|_p \leq 6^{l+1} \frac{\ell!}{(2\pi)^l} \]

Now, we can state and prove our results for integrable 1-periodic functions.

Theorem 3.2: For \(M \in N\) and \(f\), an integrable 1-periodic function defined on \([0, 1]\), we have estimates

\[ |\hat{f}_{2M,k} - \hat{f}(k)| \leq 2\|p^* - f\|_p \]

for \(|k| \leq M\) (3.3)

Proof: Since \(p^* \in T_M\), a simple calculation shows that for \(|k| \leq M\) we have

\[ |\hat{f}_{2M,k} - \hat{p}_{2M,k} + \hat{p}_{2M,k} - \hat{f}(k)| \]

Then

\[ |\hat{f}_{2M,k} - \hat{p}_{2M,k}| = \left| \left[ \int_0^1 \left( \frac{1}{2M+1} \sum_{j=0}^{2M} f\left( \frac{j}{2M+1} \right) - p^* \left( \frac{j}{2M+1} \right) e^{-2nixk} \right) dx \right] \right|_p^1 \]

In the second line we use the triangle inequality.

\[ \left| \left[ \int_0^1 \left( f(x) - p^*(x) \right) e^{-2nixk} \right] dx \right|_p^1 \leq \left[ \int_0^1 \left( f(x) - p^*(x) \right) e^{-2nixk} \right] dx \]  

Then

\[ \|f(k) - \hat{p}_{2M,k}\|_p \leq \|f - p^*\|_p \]

Corollary 3.2: Suppose that \(f\) is an integrable 1-periodic function with \(l \geq 0\) an integrable 1-periodic derivative; then

\[ |\hat{f}_{2M,k} - \hat{f}(k)| \leq 12 \frac{\ell!}{(2\pi)^l} \]

For \(|k| \leq M\)

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Proof: From theorem (3.2)

\[ |\tilde{f}_{2M,k} - \hat{f}(k)| \leq 2\|p^* - f\|_P \]

Then from corollary (3.2)

\[ \|f - p^*\|_P \leq 6^{l+1} \frac{\tau_f}{(2\pi M)^P} \]

Then

\[ |\tilde{f}_{2M,k} - \hat{f}(k)| \leq 2 \cdot 6^{l+1} \frac{\tau_f}{(2\pi M)^P} \]

\[ \leq 12 \frac{6^{l+1}}{(2\pi M)^P} \]

Corollary 3.3: suppose that \( f \) is an integrable 1-periodic function then

\[ |\tilde{f}_{2M,k}| \leq |\hat{f}(k)| + 6\tau_f \left( \frac{1}{2\pi M} \right) \]

Proof: by theorem (3.2)

\[ |\tilde{f}_{2M,k} - \hat{f}(k)| \leq 2\|f - p^*\|_P \]

\[ |\tilde{f}_{2M,k} - \hat{f}(k)| \leq |\tilde{f}_{2M,k} - \tilde{p}_{2M,k} + \tilde{p}_{2M,k} - \hat{f}(k)| \]

\[ \leq |\tilde{f}_{2M,k} - \tilde{p}_{2M,k}| + |\tilde{p}_{2M,k} - \hat{f}(k)| \]

\[ |\tilde{f}_{2M,k} - \tilde{p}_{2M,k}| \leq \|f - p^*\|_P \]

Then from theorem (3.1)

\[ \|f - p^*\|_P \leq 6 \tau_f \left( \frac{1}{2\pi M} \right) \]

Then

\[ |\tilde{f}_{2M,k}| \leq \|f - p^*\|_P + |\hat{f}(k)| \]

\[ \leq 6 \tau_f \left( \frac{1}{2\pi M} \right) + |\hat{f}(k)| \]

Theorem (3.3) there is a universal constant \( C \) such that, if \( f \) is an integrable 1-periodic function, then

\[ |\tilde{f}_{2M(x)} - \tilde{f}_{2M(x)}| \leq C \log M \|f - p^*\|_P \]

Where \( p^* \in T_M \) is a best approximation.

Proof: Let Dirichlet kernel

\[ f_{2M(x)} = \int_0^1 D_M (x - y) f(y) dy \]

Then

\[ \tilde{f}_{2M(x)} = \frac{1}{2M+1} \sum_{j=0}^{2M} D_M (x - \frac{j}{2M+1}) f \left( \frac{j}{2M+1} \right) \]

We observe that

\[ |\tilde{f}_{2M(x)} - f_{2M(x)}| \leq |\tilde{f}_{2M(x)} - p^*(x)| + |p^*(x) - f_{2M(x)}| \]

\[ \leq \left[ \int_0^1 \left( \frac{1}{2M+1} \sum_{j=0}^{2M} D_M (x - \frac{j}{2M+1}) \right) \left| f \left( \frac{j}{2M+1} \right) - p^* \left( \frac{j}{2M+1} \right) \right| dx \right]^\frac{1}{2} \]

\[ \leq \left[ \int_0^1 \left( \frac{1}{2M+1} \sum_{j=0}^{2M} D_M (x - \frac{j}{2M+1}) \right) \left| f \left( \frac{j}{2M+1} \right) - p^* \left( \frac{j}{2M+1} \right) \right| dx \right]^\frac{1}{2} \]

\[ \leq \|f - p^*\|_P \cdot \left[ \int_0^1 \sum_{j=0}^{2M} |D_M (x - \frac{j}{2M+1})| dx \right] \]

Both the sum and the integral in the last line are bounded by a constant times \( \log M \)

Corollary 3.4: if \( f \) is an integrable 1-periodic function whose averaged modulus satisfies

\[ \tau_f(\delta) = o\left( (\log \delta)^{-1} \right) \]

Then the (FFT) partial sum \( \{\tilde{f}_{2M}\} \) converges to \( f \) on \([0, 1]\) if \( f \) has \( l \) integrable periodic derivatives and \( \tau_f^l \) satisfies the above estimate, then for each \( 1 \leq j \leq l \) the sequence \( \{\tilde{f}_{2M(j)}\} \) converges to \( \tilde{f}_{2M(j)} \)

Proof: let \( p^* \in T_M \) be a best approximation to \( f \), we will use the triangle inequality to conclude that

\[ |\tilde{f}_{2M(x)} - f_{2M(x)}| \leq |\tilde{f}_{2M(x)} - p^*(x)| + |p^*(x) - f_{2M(x)}| \]
Applying theorem (3.1), we see that

\[ \|f_{2M(x)} - f_{M(x)}\| \leq (C \log M + 6) \tau_f \left( \frac{1}{2M\pi} \right) \]  

(3.8)

The estimate in (3.7) implies the right-hand side of equation (3.8) tends to zero as \( M \) tends to infinity. The last inequality tells us that \( \|f_{2M(x)} - f_{M(x)}\| \) tends to zero as \( M \) tends to infinity.

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REFERENCES


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