International Journal of Mathematical Archive-7(2), 2016, 111-119
Available online through www.ijma.info ISSN 2229 – 5046

SIMPLE GRAPHOIDAL COVERING NUMBER OF SNAKE GRAPHS

J. SURESH SUSEELA[1], G. VENKAT NARAYANAN*[2]

[1]Department of Mathematics,
St. John's College, Tirunelveli 627002, India.

[2]Department of Mathematics,
St. Joseph's College of Engineering, Chennai 600 119, India.

(Received On: 20-01-16; Revised & Accepted On: 26-02-16)

ABSTRACT

A graphoidal cover of G is a collection ψ of (not necessarily open) paths in G, such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering of G and is denoted by $\eta(G)$. If every two paths in ψ have at most one common vertex, then it is called simple graphoidal cover of G. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by $\eta_s(G)$. Here we determine the simple graphoidal covering number of Snake graphs.

Keywords: *Simple Graphoidal Cover, Simple Graphoidal Covering Number, Triangular Snake graph, Quadrilateral Snake graph.*

Mathematics Subject Classification: *05C70.*

1. INTRODUCTION

By a graph G = (V, E) we mean a finite undirected graph without loop or multiple edges. The order and size of the G are denoted by p and q respectively. For theoretical terminology of graph we refer Harary [1]. All the graphs considered in this paper are assumed to be connected and non-trivial. If $P = (v_1, v_2, \ldots..v_n)$ be a path or cycle in a graph G, the vertices $v_2, v_3, \ldots\ldots v_{n-1}$ are called internal vertices of P and v_1, v_n are called external vertices of P. Two paths P and Q are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q. The concept of graphoidal cover was introduced by Dr. B.D. Acharya and Dr. E. Sampath Kumar [2]. The simple graphoidal cover was introduced by Dr. S. Arumugam and Dr. I. Shahul Hamid [3].

Definition 1.1 [1]: A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$

Definition 1.2 [3]: A simple graphoidal cover of a graph G is a graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by $\eta_s(G)$.

Corresponding Author: G. Venkat Narayanan[2], [2]Department of Mathematics,
St. Joseph's College of Engineering, Chennai 600 119, India.*

Definition 1.3 [2]: Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an interior vertex of ψ if it is an internal vertex of some path in ψ. Any vertex which is not an interior vertex of ψ is said to be an exterior vertex of ψ.

Theorem 1.4 [3]: For any simple graphoidal cover ψ of a (p, q) of graph G, let t_ψ denote the number of exterior vertices of ψ. Let t = $\min t_\psi$, where the minimum is taken over all simple graphoidal covers ψ of G. Then $\eta_s(G) = q - p + t$.

Theorem 1.5 [3]: For any graph G, $\eta_s(G) \geq q - p$. Moreover, the following are equivalent.

(i) $\eta_s(G) = q - p$.
(ii) There exists a simple graphoidal cover of G without exterior vertices.
(iii) There exists a set of P internally disjoint and edge disjoint induced paths without exterior vertices such that any two paths in P have at most one vertex in common.

Definition 1.6: A triangular snake is obtained from a path of $P = (u_1, u_2, \ldots, u_n)$ by joining u_i and u_{i+1} with a new vertex v_i, $1 \leq i \leq n-1$. (i.e.) every edge of P is replaced by a triangle C_3.

Definition 1.7: A double triangular snake consists of two triangular snake graphs that have a common path.

Definition 1.8: A triple triangular snake consists of three triangular snake graphs that have a common path.

Definition 1.9: An alternate triangular snake graph is obtained from a path of $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} (alternatively) to a new vertex v_i, $1 \leq i \leq n-1$.

Definition 1.10: An alternate double triangular snake graph consists of two alternate triangular snake graphs

Definition 1.11: The quadrilateral snake is obtained from a path of $P = (u_1, u_2, \ldots, u_n)$ by joining u_i and u_{i+1} with two new vertices v_i, w_i, $1 \leq i \leq n-1$. (i.e.) every edge of P is replaced by a cycle C_4.

Definition 1.12: The double quadrilateral snake consists of two quadrilateral snake graphs that have a common path.

Definition 1.13: The triple quadrilateral snake consists of three quadrilateral snake graphs that have a common path.

2. MAIN RESULTS

Theorem 2.1: Let G be a triangular snake graph then $\eta_s(G) = q - p + 1$.

Proof: Let $\{u_1, u_2, u_3, u_4 \ldots u_n\}$ be the underlined path of G and $\{v_1, v_2, v_3, \ldots v_{n-1}\}$ be the vertices in G such that each v_i is adjacent to u_i, u_{i+1}. The collection of path of G given by

$P_i = (u_i, v_i, u_{i+1}, u_i)$, $1 \leq i \leq n-1$.

is a simple graphoidal cover of G in which u_1 is the only vertex which is not an internal. Therefore $\eta_s(G) \leq q - p + 1$. Now, let ψ be any simple graphoidal cover of G. Then ψ can contain at most (n–1) triangles and paths of length 1. Since every triangle can make two vertices internal, at most 2(n–1) vertices can be made internal in ψ. Therefore $t_\psi \geq 1$, since $p = 2n - 1$. Hence $\eta_s(G) \geq q - p + 1$. Thus $\eta_s(G) = q - p + 1$.

Theorem 2.2: Let G be a double triangular snake graph then $\eta_s(G) = q - p + n$

Proof: Let $\{u_1, u_2, u_3, u_4 \ldots u_n\}$ be the underlined path of G and $\{v_1, v_2, v_3, \ldots v_{n-1}, w_1, w_2, w_3, \ldots w_{n-1}\}$ be the vertices in G such that each v_i & w_i is adjacent to u_i, u_{i+1}. The collection of paths of G given by

$P_i = (u_i, v_i, u_{i+1}, u_i)$
$Q_i = (u_i, w_i)$
$R_i = (w_i, u_{i+1})$, $1 \leq i \leq n-1$.

is a simple graphoidal cover of G in which $u_1, w_1, w_2, \dots w_{n-1}$ are not an internal. Therefore $\eta_s(G) \le q-p+n$ Now, let ψ be any simple graphoidal cover of G. Now, let ψ be any simple graphoidal cover of G. Then ψ can contain at most (n–1) triangles and paths of length 1. Since every triangle can make two vertices internal, at most 2(n–1) vertices can be made internal in ψ. Therefore $t_\psi \ge n$, since p=3n–2. Hence $\eta_s(G) \ge q-p+n$ Thus $\eta_s(G) = q-p+n$

Theorem 2.3: Let G be a triple triangular snake graph, then $\eta_s(G) = q-p+2n-1$

Theorem 2.4: Let G be a quadrilateral snake graph, then $\eta_s(G) = q-p+1$.

Proof: Let $\{u_1, u_2, u_3, u_4 \dots u_n\}$ be the underlined path of G and $\{v_1, v_2, \dots, v_{n-1}, w_1, w_2, \dots, w_{n-1}\}$ be the vertices in G such that by joining u_i & u_{i+1} with two new vertices v_i *and* w_i by the edges (u_i, v_i), (v_i, w_i), (w_i, u_{i+1}). The collection of paths of G given by

$P_i = (u_i, v_i, w_i, u_{i+1}, u_i)$, $1 \le i \le n-1$.

is a simple graphoidal cover of G in which u_1 is not an internal. Therefore $\eta_s(G) \le q-p+1$. Now, let ψ be any simple graphoidal cover of G. Then ψ can contain at most (n–1) cycles of length 4 and paths of length 1. Since every cycles of length 4 can make three vertices internal, at most 3(n–1) vertices can be made internal in ψ. Therefore $t_\psi \ge 1$ since p=3n–2. Hence $\eta_s(G) \ge q-p+1$. Thus $\eta_s(G) = q-p+1$.

Theorem 2.5: Let G be a double quadrilateral snake graph then $\eta_s(G) = q-p+n$

Proof: Let $\{u_1, u_2, u_3, u_4 \dots u_n\}$ be the underlined path in G and $\{v_1, v_2, v_3, \dots, v_{n-1}, w_1, w_2, w_3, \dots, w_{n-1}, x_1, x_2, x_3, \dots x_{n-1}, y_1, y_2, y_3, \dots y_{n-1}\}$ be the vertices in G such that by joining u_i & u_{i+1} with four new vertice s v_i, w_i, x_i & y_i by the edges (u_i, v_i), (v_i, w_i), (w_i, u_{i+1}), (u_i, x_i), (x_i, y_i) & (y_i, u_{i+1}). The collection of paths of G given by

$P_i = (u_i, v_i, w_i, u_{i+1}, u_i)$

$Q_i = (u_i, x_i, y_i)$

$R_i = (y_i, u_{i+1})$, $1 \le i \le n-1$.

is a simple graphoidal cover of G in which u_1, $y_1, y_2, \dots y_{n-1}$ are not internal. Therefore $\eta_s(G) \le q-p+n$ Now, let ψ be any simple graphoidal cover of G. Then ψ can contain at most (n–1) cycle of length 4 and at most (n–1) paths of length 2. Since every cycle of length 4 can make three vertices internal and every path of length 2, can make one vertex internal. Therefore at most 4(n–1) vertices can be made internal in ψ. Therefore $t_\psi \ge n$ since p=5n–4. Hence $\eta_s(G) \ge q-p+n$ Thus $\eta_s(G) = q-p+n$

Theorem 2.6: Let G be triple quadrilateral snake graph then $\eta_s(G) = q-p+2n-1$

Theorem 2.7: Let G be an alternate triangle snake graph then

$$\eta_S(G) = \begin{cases} q-p+\left(\dfrac{n}{2}\right), & \textit{triangular paths starts at } u_1 \textit{ and } n \equiv 0 \pmod 4 \textit{ or } n \equiv 2 \pmod 4 \\ q-p+\left(\dfrac{n+1}{2}\right), & \textit{triangular paths starts at } u_1 \textit{ or } u_2 \textit{ and } n \equiv 1 \pmod 4 \textit{ or } n \equiv 3 \pmod 4 \\ q-p+\left(\dfrac{n+2}{2}\right), & \textit{triangular paths starts at } u_2 \textit{ and } n \equiv 0 \pmod 4 \textit{ or } n \equiv 2 \pmod 4 \end{cases}$$

Proof: Let $\{u_1, u_2, u_3, u_4 \dots u_n\}$ be the underlined path and $\{v_1, v_2, v_3, \dots, v_{n-1}\}$ be the vertices in G such that each v_i is adjacent to $u_{i,} u_{i+1}$.

Case-(i): alternate triangular snake starts at u_1

Subcase-(i): $n \equiv 0 \pmod 4$ **or** $n \equiv 2 \pmod 4$

The collection of paths of G given by
$P_0 = (u_1, v_1, u_2, u_1)$,
$P_i = (u_{2i}, u_{2i+1})$,
$Q_i = (u_{2i+1}, v_{i+1}, u_{2i+2}, u_{2i+1})$, $1 \le i \le \left(\frac{n-2}{2}\right)$

is a simple graphoidal cover of G in which $u_1, u_3, u_5, \ldots\ldots, u_{n-1}$ are not internal. Therefore $\eta_s(G) \le q - p + \left(\frac{n}{2}\right)$.

Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most n vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n}{2}\right)$, since $p = \left(\frac{3n}{2}\right)$.

Hence $\eta_s(G) \ge q - p + \left(\frac{n}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n}{2}\right)$.

Subcase-(ii): $n \equiv 1 (\mathrm{mod}\, 4)$ **or** $n \equiv 3 (\mathrm{mod}\, 4)$

The collection of paths of G is given by
$P_i = (u_{2i-1}, v_i, u_{2i}, u_{2i-1})$,
$Q_i = (u_{2i}, u_{2i+1})$, $1 \le i \le \left(\frac{n-1}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_3, u_5, \ldots\ldots u_n$ are not internal. Therefore $\eta_s(G) \le q - p + \left(\frac{n+1}{2}\right)$. Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most (n–1) vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n+1}{2}\right)$, since $p = \left(\frac{3n-1}{2}\right)$.

Hence $\eta_s(G) \ge q - p + \left(\frac{n+1}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+1}{2}\right)$.

Case-(ii): alternate triangular snake starts at u_2

Subcase-(i): $n \equiv 0 (\mathrm{mod}\, 4)$ **or** $n \equiv 2 (\mathrm{mod}\, 4)$

The collection of paths of G given by
$P = (u_1, u_2)$
$P_i = (u_{2i}, v_i, u_{2i+1}, u_{2i})$,
$Q_i = (u_{2i+1}, u_{2i+2})$, $1 \le i \le \left(\frac{n-2}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, u_8, u_{10} \ldots\ldots, u_n$ are not internal. Therefore $\eta_s(G) \le q - p + \left(\frac{n+2}{2}\right)$. Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-2}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most (n–2) vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n+2}{2}\right)$, since $p = \left(\frac{3n-2}{2}\right)$. Hence $\eta_s(G) \ge q - p + \left(\frac{n+2}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+2}{2}\right)$.

Subcase-(ii): $n \equiv 1 (\mathrm{mod}\, 4)$ **or** $n \equiv 3 (\mathrm{mod}\, 4)$
The collection of paths of G given by
$P_i = (u_{2i-1}, u_{2i})$,
$Q_i = (u_{2i}, v_i, u_{2i+1}, u_{2i})$, $1 \le i \le \left(\frac{n-1}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, u_8, u_{10}........, u_{n-1}$ are not internal. Therefore $\eta_s(G) \le q - p + \left(\frac{n+1}{2}\right)$. Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most (n–1) vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n+1}{2}\right)$ since $p = \left(\frac{3n-1}{2}\right)$. Hence $\eta_s(G) \ge q - p + \left(\frac{n+1}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+1}{2}\right)$.

Theorem 2.8: Let G be an alternate double triangle snake graph, then $\eta_S(G) = q - p + n$.

Proof: Let $\{u_1, u_2, u_3, u_4....u_n\}$ be the underlined path in G and $\{v_1, v_2, v_3,...., v_{n-1}, w_1, w_2, w_3,.....w_{n-1}\}$ be the vertices in G such that each v_i & w_i is adjacent to u_i , u_{i+1} . Here we have two cases.

Case-(i): Alternate double triangular snake starts at u_1

Subcase-(i): $n \equiv 0 \pmod 4$ **or** $n \equiv 2 \pmod 4$

The collection of paths of G given by

$P = (u_1, v_1, u_2, u_1)$,

$Q = (u_1, w_1)$,

$R = (w_1, u_2)$,

$P_i = (u_{2i}, u_{2i+1})$,

$Q_i = (u_{2i+1}, v_{i+1}, u_{2i+2}, u_{2i+1})$,

$R_i = (u_{2i+1}, w_{i+1})$,

$S_i = (w_{i+1}, u_{2i+2}), \quad 1 \le i \le \left(\frac{n-2}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_3, u_5,......, u_{n-1}, w_1, w_2,........, w_{\left(\frac{n}{2}\right)}$ are not internal. Therefore $\eta_s(G) \le q - p + n$ Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most n vertices can be made internal. Therefore $t_\psi \ge n$, since $p = 2n$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Subcase-(ii): $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$

The collection of paths of G given by

$P_i = (u_{2i-1}, v_i, u_{2i}, u_{2i-1})$,

$Q_i = (u_{2i-1}, w_i)$

$R_i = (w_i, u_{2i})$,

$S_i = (u_{2i}, u_{2i+1}), \quad 1 \le i \le \left(\frac{n-1}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_3, u_5,......, u_n, w_1, w_2,........., w_{\left(\frac{n-1}{2}\right)}$ are not internal. Therefore $\eta_s(G) \le q - p + n$ Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most (n–1) vertices can be made internal. Therefore $t_\psi \ge n$, since $p = 2n - 1$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Case-(ii): Alternate double triangular snake starts at u_2

Subcase-(i): $n \equiv 0(\bmod 4)$ or $n \equiv 2(\bmod 4)$

The collection of paths of G given by
$P = (u_1, u_2)$
$P_i = (u_{2i}, v_i, u_{2i+1}, u_{2i})$,
$Q_i = (u_{2i}, w_i)$,
$R_i = (w_i, u_{2i+1})$,
$S_i = (u_{2i+1}, u_{2i+2}),\ 1 \le i \le \left(\frac{n-2}{2}\right)$.
is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, ..., u_n, w_1, w_2, ..., w_{\left(\frac{n-2}{2}\right)}$ are not internal. Therefore $\eta_s(G) \le q - p + n$ Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-2}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most (n–2) vertices can be made internal.

Therefore $t_\psi \ge n$, since $p = 2n - 2$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Subcase-(ii): $n \equiv 1(\bmod 4)$ or $n \equiv 3(\bmod 4)$

The collection of paths of G is given by
$P_i = (u_{2i-1}, u_{2i})$
$Q_i = (u_{2i}, v_i, u_{2i+1}, u_{2i})$,
$R_i = (u_{2i}, w_i)$,
$S_i = (w_i, u_{2i+1}),\ 1 \le i \le \left(\frac{n-1}{2}\right)$.
is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_{6,} ......, u_{n-1}, w_1, w_2, ..., w_{\left(\frac{n-1}{2}\right)}$ are not internal.

Therefore $\eta_s(G) \le q - p + n$ Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ triangles and each triangle can make two vertices internal in ψ, at most (n–1) vertices can be made internal.
Therefore $t_\psi \ge n$, since $p = 2n - 1$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Theorem 2.9: Let G be an alternate quadrilateral snake graph, then

$$\eta_s(G) = \begin{cases} q - p + \left(\frac{n}{2}\right), \text{ triangular paths starts at } u_1 \text{ and } n \equiv 0(\bmod 4) \text{ or } n \equiv 2(\bmod 4) \\ q - p + \left(\frac{n+1}{2}\right), \text{triangular paths starts at } u_1 \text{ or } u_2 \text{ and } n \equiv 1(\bmod 4) \text{ or } n \equiv 3(\bmod 4) \\ q - p + \left(\frac{n+2}{2}\right), \text{ triangular paths starts at } u_2 \text{ and } n \equiv 0(\bmod 4) \text{ or } n \equiv 2(\bmod 4) \end{cases}$$

Proof: Let $\{u_1, u_2, u_3, u_4 .... u_n\}$ be the underlined path in G and $\{v_1, v_2, v_3, ..., v_{n-1}, w_1, w_2, w_3, ... w_{n-1}\}$ be the vertices in G such that joining u_i & u_{i+1} with two new vertices v_i, w_i by the edges (u_i, v_i), (v_i, w_i), (w_i, u_{i+1}). Here we have two cases.

Case-(i): Alternate quadrilateral snake starts at u_1

Subcase-(i): $n \equiv 0(\bmod 4)$ or $n \equiv 2(\bmod 4)$
The collection of paths of G given by
$P = (u_1, v_1, w_1, u_2, u_1)$,
$P_i = (u_{2i}, u_{2i+1})$,

$Q_i = (u_{2i+1}, v_{i+1}, w_{i+1}, u_{2i+2}, u_{2i+1})$, $1 \le i \le \left(\frac{n-2}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_3, u_5, \ldots\ldots u_{n-1}$ are not internal. Therefore $\eta_S(G) \le q - p + \left(\frac{n}{2}\right)$.

Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n}{2}\right)$ cycles of length 4 and each cycles of length 4 can make three vertices internal in ψ, at most $\left(\frac{3n}{2}\right)$ vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n}{2}\right)$, since $p = 2n$. Hence $\eta_s(G) \ge q - p + \left(\frac{n}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n}{2}\right)$.

Subcase-(ii): $n \equiv 1 (\mathrm{mod}\, 4)$ or $n \equiv 3 (\mathrm{mod}\, 4)$

The collection of paths of G given by

$P_i = (u_{2i-1}, v_i, w_i, u_{2i}, u_{2i-1})$,

$Q_i = (u_{2i}, u_{2i+1})$, $1 \le i \le \left(\frac{n-1}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_3, u_5, \ldots\ldots u_n$ are not internal. Therefore $\eta_S(G) \le q - p + \left(\frac{n+1}{2}\right)$. Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ cycles of length 4 and each cycle of length 4 can make three vertices internal in ψ, at most $\left(\frac{3n-3}{2}\right)$ vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n+1}{2}\right)$, since $p = 2n-1$. Hence $\eta_s(G) \ge q - p + \left(\frac{n+1}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+1}{2}\right)$.

Case-(ii): Alternate quadrilateral triangular snake starts at u_2

Subcase-(i): $n \equiv 0 (\mathrm{mod}\, 4)$ or $n \equiv 2 (\mathrm{mod}\, 4)$

The collection of paths of G given by

$P = (u_1, u_2)$,

$P_i = (u_{2i}, v_i, w_i, u_{2i+1}, u_{2i})$,

$Q_i = (u_{2i+1}, u_{2i+2})$, $1 \le i \le \left(\frac{n-2}{2}\right)$.

is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, u_8, \ldots\ldots, u_n$ are not internal. Therefore $\eta_S(G) \le q - p + \left(\frac{n+2}{2}\right)$.

Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-2}{2}\right)$ cycles of length 4 and paths of length 1. Each cycle of length 4 can make three vertices internal in ψ, at most $\left(\frac{3n-6}{2}\right)$ vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n+2}{2}\right)$, since $p = 2n-2$. Hence $\eta_s(G) \ge q - p + \left(\frac{n+2}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+2}{2}\right)$.

Subcase-(ii): $n \equiv 1 (\mathrm{mod}\, 4)$ or $n \equiv 3 (\mathrm{mod}\, 4)$

The collection of paths of G given by

$P_i = (u_{2i-1}, u_{2i})$

$Q_i = (u_{2i}, v_i, w_i, u_{2i+1}, u_{2i})$, $1 \le i \le \left(\frac{n-1}{2}\right)$

is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, u_8, \ldots\ldots, u_{n-1}$ are not internal. Therefore $\eta_S(G) \le q - p + \left(\frac{n+1}{2}\right)$.

Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ cycles of length 4 and each cycle of length 4 can make three vertices internal in ψ, at most $\left(\frac{3n-3}{2}\right)$ vertices can be made internal. Therefore $t_\psi \ge \left(\frac{n+1}{2}\right)$ since $p = 2n-1$. Hence $\eta_s(G) \ge q - p + \left(\frac{n+1}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+1}{2}\right)$.

Theorem 2.10: Let G be alternate double quadrilateral snake graph, then $\eta_s(G) = q - p + n$

Proof: Let $\{u_1, u_2, u_3, u_4 .... u_n\}$ be an underlined path in G and $\{v_1, v_2, v_3, ..., v_{n-1}, w_1, w_2, w_3, ..., w_{n-1}, x_1, x_2, x_3, ... x_{n-1}, y_1, y_2, y_3, ... y_{n-1}\}$ be the vertices in G by joining u_i & u_{i+1} with four new vertices v_i, w_i, x_i & y_i by the edges (u_i, v_i), (v_i, w_i), (w_i, u_{i+1}), (u_i, x_i), (x_i, y_i) & (y_i, u_{i+1}). Here we have two cases.

Case-(i): Alternate double quadrilateral snake starts at u_1

Subcase-(i): $n \equiv 0 \pmod 4$ or $n \equiv 2 \pmod 4$
The collection of paths of G is given by
$P = (u_1, v_1, w_1, u_2, u_1)$
$Q = (u_1, x_1, y_1)$
$R = (y_1, u_2)$
$P_i = (u_{2i}, u_{2i+1})$
$Q_i = (u_{2i+1}, v_{i+1}, w_{i+1}, u_{2i+2}, u_{2i+1})$,
$R_i = (u_{2i+1}, x_{i+1}, y_{i+1})$,
$S_i = (y_{i+1}, u_{2i+2})$, $1 \le i \le \left(\frac{n-2}{2}\right)$.
is a simple graphoidal cover of G in which $u_1, u_3, u_5, ... u_{n-1}, y_1, y_2, y_3 ...., y_{\left(\frac{n}{2}\right)}$ are not internal.

Therefore $\eta_S(G) \le q - p + n$. Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n}{2}\right)$ cycles of length 4 and $\left(\frac{n}{2}\right)$ paths of length 2. In each cycle of length 4 can make three vertices internal and each paths of length 2 can make one vertex internal in ψ, at most $2n$ vertices can be made internal. Therefore $t_\psi \ge n$, since $p = 3n$. Vertices. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Subcase-(ii): $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$
The collection of paths of G given by
$P_i = (u_{2i-1}, v_i, w_i, u_{2i}, u_{2i-1})$,
$Q_i = (u_{2i-1}, x_i, y_i)$,
$R_i = (y_i, u_{2i})$,
$S_i = (u_{2i}, u_{2i+1})$, $1 \le i \le \left(\frac{n-1}{2}\right)$.
is a simple graphoidal cover of G in which $u_1, u_3, u_5, ... u_n, y_1, y_2, y_3 ...., y_{\left(\frac{n-1}{2}\right)}$ are not internal. Therefore $\eta_S(G) \le q - p + n$. Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ cycles of length 4 and $\left(\frac{n-1}{2}\right)$ paths of length 2. In each cycle of length 4 can make three vertices internal and each paths of length 2 can make one vertex internal in ψ, at most $2n-2$ vertices can be made internal. Therefore $t_\psi \ge n$ since $p = 3n-2$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Case-(ii): Alternate double quadrilateral snake starts at u_2

Subcase-(i): $n \equiv 0 (\mathrm{mod}\, 4)$ or $n \equiv 2 (\mathrm{mod}\, 4)$
The collection of paths of G given by
$P = (u_1, u_2)$
$P_i = (u_{2i}, v_i, w_i, u_{2i+1}, u_{2i}),$
$Q_i = (u_{2i}, x_i, y_i),$
$R_i = (y_i, u_{2i+1}),$
$S_i = (u_{2i+1}, u_{2i+2}),\ 1 \le i \le \left(\frac{n-2}{2}\right).$
is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, \ldots u_n, y_1, y_2, y_3 \ldots\ldots\ldots, y_{\left(\frac{n-2}{2}\right)}$ are not internal. Therefore $\eta_S(G) \le q - p + n.$ Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-2}{2}\right)$ cycles of length 4 and $\left(\frac{n-2}{2}\right)$ paths of length 2. Each cycle of length 4 can make three vertices internal and each paths of length 2 can make one vertex internal in ψ , at most $2n-4$ vertices can be made internal. Therefore $t_\psi \ge n$, since $p = 3n-4$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

Subcase-(ii): $n \equiv 1 (\mathrm{mod}\, 4)$ or $n \equiv 3 (\mathrm{mod}\, 4)$
The collection of paths of G given by
$P_i = (u_{2i-1}, u_{2i}),$
$Q_i = (u_{2i}, v_i, w_i, u_{2i+1}, u_{2i}),$
$R_i = (u_{2i}, x_i, y_i),$
$S_i = (y_i, u_{2i+1}),\ 1 \le i \le \left(\frac{n-1}{2}\right).$
is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_6, \ldots u_{n-1}, y_1, y_2, y_3 \ldots\ldots\ldots, y_{\left(\frac{n-1}{2}\right)}$ are not internal. Therefore $\eta_S(G) \le q - p + n.$ Now, let ψ be any simple graphoidal cover of G. Since ψ can contain at most $\left(\frac{n-1}{2}\right)$ cycles of length 4 and $\left(\frac{n-1}{2}\right)$ paths of length 2. Each cycle of length 4 can make three vertices internal and each path of length 2 can make one vertex internal in ψ , at most $2n-2$ vertices can be made internal. Therefore $t_\psi \ge n$, since $p = 3n-2$. Hence $\eta_s(G) \ge q - p + n$ Thus $\eta_s(G) = q - p + n$

REFERENCES

1. F. Harary, Graph theory Addison- Wesley, Reading, MA, 1969.
2. B.D. Acharya and E. Sampath Kumar, Graphoidal covers and Graphoidal Covering number of a Graph, Indian Journal Pure Appl. Math18 (10) (1987), 882-890.
3. S. Arumugam and I. Sahul Hamid, Simple Graphoidal Covers in a Graph, J. Combin. Math. Combin. Comput. 64(2008), 79-95.

Source of support: Nil, Conflict of interest: None Declared