SIMPLE GRAPHOIDAL COVERING NUMBER OF SNAKE GRAPHS

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ABSTRACT

A graphoidal cover of G is a collection ψ of (not necessarily open) paths in G, such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering of G and is denoted by η(G). If every two paths in ψ have at most one common vertex, then it is called simple graphoidal cover of G. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by ηₛ(G). Here we determine the simple graphoidal covering number of Snake graphs.

Keywords: Simple Graphoidal Cover, Simple Graphoidal Covering Number, Triangular Snake graph, Quadrilateral Snake graph.

Mathematics Subject Classification: 05C70.

1. INTRODUCTION

By a graph G = (V, E) we mean a finite undirected graph without loop or multiple edges. The order and size of the G are denoted by p and q respectively. For theoretical terminology of graph we refer Harary [1]. All the graphs considered in this paper are assumed to be connected and non-trivial. If P = (v₁, v₂, …, vₙ) be a path or cycle in a graph G, the vertices v₂, v₃, …, vₙ₋₁ are called internal vertices of P and v₁, vₙ are called external vertices of P. Two paths P and Q are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q. The concept of graphoidal cover was introduced by Dr. B.D. Acharya and Dr. E. Sampath Kumar [2]. The simple graphoidal cover was introduced by Dr. S. Arumugam and Dr. I. Shahul Hamid [3].

Definition 1.1 [1]: A graphoidal cover of G is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in ψ has at least two vertices.
(ii) Every vertex of G is an internal vertex of at most one path in ψ.
(iii) Every edge of G is in exactly one path in ψ.

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by η(G).

Definition 1.2 [3]: A simple graphoidal cover of a graph G is a graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple graphoidal cover of G is called simple graphoidal covering number of G and is denoted by ηₛ(G).

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Definition 1.3 [2]: Let $\psi$ be a collection of internally disjoint paths in $G$. A vertex of $G$ is said to be an interior vertex of $\psi$ if it is an internal vertex of some path in $\psi$. Any vertex which is not an interior vertex of $\psi$ is said to be an exterior vertex of $\psi$.

Theorem 1.4 [3]: For any simple graphoidal cover $\psi$ of a $(p, q)$ of graph $G$, let $t_\psi$ denote the number of exterior vertices of $\psi$. Let $t = \min t_\psi$, where the minimum is taken over all simple graphoidal covers $\psi$ of $G$. Then $\eta_h(G) = q - p + t$.

Theorem 1.5 [3]: For any graph $G$, $\eta_i(G) \geq q - p$. Moreover, the following are equivalent.

(i) $\eta_i(G) = q - p$.
(ii) There exists a simple graphoidal cover of $G$ without exterior vertices.
(iii) There exists a set of $P$ internally disjoint and edge disjoint induced paths without exterior vertices such that any two paths in $P$ have at most one vertex in common.

Definition 1.6: A triangular snake is obtained from a path of $P = (u_1, u_2, \ldots, u_n)$ by joining $u_i$ and $u_{i+1}$ with a new vertex $v$, $1 \leq i \leq n-1$. (i.e.) every edge of $P$ is replaced by a triangle $C_3$.

Definition 1.7: A double triangular snake consists of two triangular snake graphs that have a common path.

Definition 1.8: A triple triangular snake consists of three triangular snake graphs that have a common path.

Definition 1.9: An alternate triangular snake graph is obtained from a path of $P = (v_1, v_2, \ldots, v_n)$ by joining $v_i$ and $v_{i+1}$ (alternatively) to a new vertex $v$, $1 \leq i \leq n-1$.

Definition 1.10: An alternate double triangular snake graph consists of two alternate triangular snake graphs.

Definition 1.11: The quadrilateral snake is obtained from a path of $P = (u_1, u_2, \ldots, u_n)$ by joining $u_i$ and $u_{i+1}$ with two new vertices $v_i, w_i$, $1 \leq i \leq n-1$. (i.e.) every edge of $P$ is replaced by a cycle $C_4$.

Definition 1.12: The double quadrilateral snake consists of two quadrilateral snake graphs that have a common path.

Definition 1.13: The triple quadrilateral snake consists of three quadrilateral snake graphs that have a common path.

2. MAIN RESULTS

Theorem 2.1: Let $G$ be a triangular snake graph then $\eta_i(G) = q - p + 1$.

Proof: Let $\{u_1, u_2, u_3, u_4, \ldots, u_n\}$ be the underlined path of $G$ and $\{v_1, v_2, v_3, \ldots, v_{n-1}\}$ be the vertices in $G$ such that each $v_i$ is adjacent to $u_i, u_{i+1}$. The collection of path of $G$ given by

$$P_i = (u_i, v_i, u_{i+1}, u_i), \quad 1 \leq i \leq n-1.$$ 

is a simple graphoidal cover of $G$ in which $u_i$ is the only vertex which is not an internal. Therefore $\eta_i(G) \leq q - p + 1$. Now, let $\psi$ be any simple graphoidal cover of $G$. Then $\psi$ can contain at most $(n-1)$ triangles and paths of length 1. Since every triangle can make two vertices internal, at most $2(n-1)$ vertices can be made internal in $\psi$. Therefore $t_\psi \geq 1$, since $p = 2n - 1$. Hence $\eta_i(G) \geq q - p + 1$. Thus $\eta_i(G) = q - p + 1$.

Theorem 2.2: Let $G$ be a double triangular snake graph then $\eta_i(G) = q - p + n$.

Proof: Let $\{u_1, u_2, u_3, u_4, \ldots, u_n\}$ be the underlined path of $G$ and $\{v_1, v_2, v_3, \ldots, v_{n-1}, w_1, w_2, w_3, \ldots, w_{n-2}\}$ be the vertices in $G$ such that each $v_i, w_i$ is adjacent to $u_i, u_{i+1}$. The collection of paths of $G$ given by

$$P_i = (u_i, v_i, u_{i+1}, u_i)$$
$$Q_i = (u_i, w_i)$$
$$R_i = (w_i, u_{i+1}), \quad 1 \leq i \leq n-1.$$
is a simple graphoidal cover of $G$ in which $u_1,w_1,w_2,...,w_{n-1}$ are not an internal. Therefore $\eta_b(G) \leq q-p+n$ Now, let $\psi$ be any simple graphoidal cover of $G$. Now, let $\psi$ be any simple graphoidal cover of $G$. Then $\psi$ can contain at most $(n-1)$ triangles and paths of length 1. Since every triangle can make two vertices internal, at most $2(n-1)$ vertices can be made internal in $\psi$. Therefore $t_\psi \geq n$, since $p=3n-2$. Hence $\eta_b(G) \geq q-p+n$ Thus $\eta_b(G) = q-p+n$

Theorem 2.3: Let $G$ be a triple triangular snake graph, then $\eta_b(G) = q-p+2n-1$

Theorem 2.4: Let $G$ be a quadrilateral snake graph, then $\eta_b(G) = q-p+1$.

Proof: Let $\{u_1,u_2,u_3,u_4,...,u_n\}$ be the underlined path of $G$ and $\{v_1,v_2,...,v_{n-1},w_1,w_2,...,w_{n-1}\}$ be the vertices in $G$ such that by joining $u_i$ & $u_{i+1}$ with two new vertices $v_j$ and $w_j$ by the edges $(u_i,v_j), (v_j,w_j), (w_j,u_{i+1})$. The collection of paths of $G$ given by $P_i = (u_1,v_i,w_i,u_{i+1},u_i), 1 \leq i \leq n-1.$ is a simple graphoidal cover of $G$ in which $u_1$ is not an internal. Therefore $\eta_b(G) \leq q-p+1$. Now, let $\psi$ be any simple graphoidal cover of $G$. Then $\psi$ can contain at most $(n-1)$ cycles of length 4 and paths of length 1. Since every cycles of length 4 can make three vertices internal, at most $3(n-1)$ vertices can be made internal in $\psi$. Therefore $t_\psi \geq 1$ since $p=3n-2$. Hence $\eta_b(G) \geq q-p+1$. Thus $\eta_b(G) = q-p+1$.

Theorem 2.5: Let $G$ be a double quadrilateral snake graph then $\eta_b(G) = q-p+n$

Proof: Let $\{u_1,u_2,u_3,u_4,...,u_n\}$ be the underlined path in $G$ and $\{v_1,v_2,v_3,...,v_{n-1},w_1,w_2,w_3,...,w_{n-1},x_1,x_2,x_3,...,x_{n-1},y_1,y_2,y_3,...,y_{n-1}\}$ be the vertices in $G$ such that by joining $u_i$ & $u_{i+1}$ with four new vertices $s,v_j,x_j & y_j$ by the edges $(u_i,v_j), (v_j,w_i), (w_i,u_{i+1}), (u_i,x_j), (x_j,y_j) & (y_j,u_{i+1})$. The collection of paths of $G$ given by $P_i = (u_1,v_i,w_i,u_{i+1},u_i)$ $Q_i = (u_1,x_i,y_i)$ $R_i = (y_i,u_{i+1}), 1 \leq i \leq n-1.$ is a simple graphoidal cover of $G$ in which $u_1,v_1,y_2,...,y_{n-1}$ are not internal. Therefore $\eta_b(G) \leq q-p+n$ Now, let $\psi$ be any simple graphoidal cover of $G$. Then $\psi$ can contain at most $(n-1)$ cycle of length 4 and at most $(n-1)$ paths of length 2. Since every cycle of length 4 can make three vertices internal and every path of length 2, can make one vertex internal. Therefore at most $4(n-1)$ vertices can be made internal in $\psi$. Therefore $t_\psi \geq n$ since $p=5n-4$. Hence $\eta_b(G) \geq q-p+n$. Thus $\eta_b(G) = q-p+n$.

Theorem 2.6: Let $G$ be triple quadrilateral snake graph then $\eta_b(G) = q-p+2n-1$

Theorem 2.7: Let $G$ be an alternate triangle snake graph then

$$\eta_b(G) = \begin{cases} 
q-p+\left(\frac{n}{2}\right), & \text{triangular paths start at } u_1 \text{ and } n = 0 \pmod{4} \text{ or } n = 2 \pmod{4} \\
q-p+\left(\frac{n+1}{2}\right), & \text{triangular paths start at } u_1 \text{ or } u_2 \text{ and } n = 1 \pmod{4} \text{ or } n = 3 \pmod{4} \\
q-p+\left(\frac{n+2}{2}\right), & \text{triangular paths start at } u_2 \text{ and } n = 0 \pmod{4} \text{ or } n = 2 \pmod{4} 
\end{cases}$$

Proof: Let $\{u_1,u_2,u_3,u_4,...,u_n\}$ be the underlined path and $\{v_1,v_2,v_3,...,v_{n-1}\}$ be the vertices in $G$ such that each $v_i$ is adjacent to $u_{i+1}$.

Case-(i): alternate triangular snake starts at $u_1$

Subcase-(i): $n = 0 \pmod{4}$ or $n = 2 \pmod{4}$
The collection of paths of G given by
\[ P_0 = (u_1, v_1, u_2, u_t), \]
\[ P_i = (u_{2i}, u_{2i+1}), \]
\[ Q_i = (u_{2i+1}, v_i, u_{2i+2}, u_{2i+3}), \]
es the simple graphoidal cover of G in which \( u_i, u_{i+1}, u_{i+2}, \ldots, u_t \) are not internal. Therefore \( \eta_n(G) \leq q - p + \left( \frac{n+1}{2} \right) \).

Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n}{2} \right) \) triangles and each triangle can make two vertices internal in \( \psi \), at most \( n \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n}{2} \right) \), since \( p = \left( \frac{3n}{2} \right) \).

Hence \( \eta_n(G) \geq q - p + \left( \frac{n}{2} \right) \). Thus \( \eta_n(G) = q - p + \left( \frac{n}{2} \right) \).

**Subcase-(ii):** \( n \equiv 1 \pmod{4} \) or \( n \equiv 3 \pmod{4} \)

The collection of paths of G given by
\[ P_i = (u_{2i-1}, v_i, u_{2i}, u_{2i+1}), \]
\[ Q_i = (u_{2i+1}, u_{2i+2}), \]
is the simple graphoidal cover of G in which \( u_i, u_{i+1}, u_{i+2}, \ldots, u_t \) are not internal. Therefore \( \eta_n(G) \leq q - p + \left( \frac{n+1}{2} \right) \). Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n-1}{2} \right) \) triangles and each triangle can make two vertices internal in \( \psi \), at most \( n-1 \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n-1}{2} \right) \), since \( p = \left( \frac{3n-1}{2} \right) \).

Hence \( \eta_n(G) \geq q - p + \left( \frac{n+1}{2} \right) \). Thus \( \eta_n(G) = q - p + \left( \frac{n+1}{2} \right) \).

**Case-(ii): alternate triangular snake starts at \( u_2 \)**

**Subcase-(i):** \( n \equiv 0 \pmod{4} \) or \( n \equiv 2 \pmod{4} \)

The collection of paths of G given by
\[ P_i = (u_{2i}, v_i, u_{2i+1}, u_{2i+2}), \]
\[ Q_i = (u_{2i+1}, u_{2i+2}), \]
is the simple graphoidal cover of G in which \( u_i, u_{i+1}, u_{i+2}, \ldots, u_t \) are not internal. Therefore \( \eta_n(G) \leq q - p + \left( \frac{n+2}{2} \right) \). Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n-2}{2} \right) \) triangles and each triangle can make two vertices internal in \( \psi \), at most \( n-2 \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n-2}{2} \right) \), since \( p = \left( \frac{3n-2}{2} \right) \). Hence \( \eta_n(G) \geq q - p + \left( \frac{n+2}{2} \right) \). Thus \( \eta_n(G) = q - p + \left( \frac{n+2}{2} \right) \).

**Subcase-(ii):** \( n \equiv 1 \pmod{4} \) or \( n \equiv 3 \pmod{4} \)

The collection of paths of G given by
\[ P_i = (u_{2i-1}, u_{2i}), \]
\[ Q_i = (u_{2i}, v_i, u_{2i+1}, u_{2i+2}), \]
is the simple graphoidal cover of G in which \( u_i, u_{i+1}, u_{i+2}, \ldots, u_t \) are not internal. Therefore \( \eta_n(G) \leq q - p + \left( \frac{n+2}{2} \right) \). Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n-1}{2} \right) \) triangles and each triangle can make two vertices internal in \( \psi \), at most \( n-1 \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n-1}{2} \right) \), since \( p = \left( \frac{3n-1}{2} \right) \). Hence \( \eta_n(G) \geq q - p + \left( \frac{n+2}{2} \right) \). Thus \( \eta_n(G) = q - p + \left( \frac{n+2}{2} \right) \).
is a simple graphoidal cover of $G$ in which $u_1, u_2, u_4, u_6, u_8, u_{10}, \ldots, u_{n-1}$ are not internal. Therefore $\eta_s(G) \leq q - p + \left(\frac{n+1}{2}\right)$. Now, let $\psi$ be any simple graphoidal cover of $G$. Since $\psi$ can contain at most \(\left(\frac{n-1}{2}\right)\) triangles and each triangle can make two vertices internal in $\psi$, at most $(n-1)$ vertices can be made internal. Therefore $t_{\psi} \geq \left(\frac{n+1}{2}\right)$ since $p = \left(\frac{3n-1}{2}\right)$. Hence $\eta_s(G) \geq q - p + \left(\frac{n+1}{2}\right)$. Thus $\eta_s(G) = q - p + \left(\frac{n+1}{2}\right)$.

**Theorem 2.8:** Let $G$ be an alternate double triangle snake graph, then $\eta_s(G) = q - p + n$.

**Proof:** Let $\{u_1, u_2, u_3, u_4, \ldots, u_n\}$ be the underlined path in $G$ and $\{v_1, v_2, v_3, \ldots, v_{n-1}, w_1, w_2, w_3, \ldots, w_{n-1}\}$ be the vertices in $G$ such that each $v_i$ & $w_i$ is adjacent to $u_i, u_{i+1}$. Here we have two cases.

**Case-(i):** Alternate double triangular snake starts at $u_1$

**Subcase-(i):** $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$

The collection of paths of $G$ given by

$P = (u_1, v_1, u_2, u_1)$,
$Q = (u_1, w_1)$,
$R = (w_1, u_2)$,
$P_i = (u_{2i-1}, u_{2i+1})$,
$Q_i = (u_{2i-1}, v_{2i-1}, u_{2i+2}, u_{2i+1})$,
$R_i = (u_{2i-1}, w_{2i+1})$,
$S_i = (w_{2i+1}, u_{2i+2})$, \(1 \leq i \leq \left(\frac{n-2}{2}\right)\)

is a simple graphoidal cover of $G$ in which $u_1, u_2, u_3, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{\frac{n-1}{2}}$ are not internal. Therefore $\eta_s(G) \leq q - p + n$. Now, let $\psi$ be any simple graphoidal cover of $G$. Since $\psi$ can contain at most \(\left(\frac{n}{2}\right)\) triangles and each triangle can make two vertices internal in $\psi$, at most $n$ vertices can be made internal. Therefore $t_{\psi} \geq n$, since $p = 2n$. Hence $\eta_s(G) \geq q - p + n$. Thus $\eta_s(G) = q - p + n$.

**Subcase-(ii):** $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$

The collection of paths of $G$ given by

$P_i = (u_{2i-1}, v_{2i-1}, u_{2i+1})$,
$Q_i = (u_{2i-1}, w_{2i})$
$R_i = (w_{2i+1}, u_{2i})$
$S_i = (u_{2i}, u_{2i+1})$, \(1 \leq i \leq \left(\frac{n-1}{2}\right)\)

is a simple graphoidal cover of $G$ in which $u_1, u_3, u_5, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{\frac{n-1}{2}}$ are not internal. Therefore $\eta_s(G) \leq q - p + n$. Now, let $\psi$ be any simple graphoidal cover of $G$. Since $\psi$ can contain at most \(\left(\frac{n-1}{2}\right)\) triangles and each triangle can make two vertices internal in $\psi$, at most $(n-1)$ vertices can be made internal. Therefore $t_{\psi} \geq n$, since $p = 2n-1$. Hence $\eta_s(G) \geq q - p + n$. Thus $\eta_s(G) = q - p + n$.

**Case-(ii):** Alternate double triangular snake starts at $u_2$
Subcase-(i): \( n \equiv 0 \pmod{4} \) or \( n \equiv 2 \pmod{4} \)

The collection of paths of \( G \) given by

\[
P_i = (u_{2i}, u_{2i+1}, u_{2i+2}),
\]

\[
Q_j = (u_{2j}, w_j),
\]

\[
R_i = (w_j, u_{2i+1}),
\]

\[
S_i = (u_{2i+1}, u_{2i+2}), 1 \leq i \leq \left( \frac{n-2}{2} \right).
\]

is a simple graphoidal cover of \( G \) in which \( u_1, u_2, u_4, \ldots, u_n, w_1, w_2, \ldots, w_{\left( \frac{n-2}{2} \right)} \) are not internal. Therefore

\[
\eta_b(G) \leq q - p + n
\]

Now, let \( \psi \) be any simple graphoidal cover of \( G \). Since \( \psi \) can contain at most \( \left( \frac{n-2}{2} \right) \) triangles and each triangle can make two vertices internal in \( \psi \), at most \( (n-2) \) vertices can be made internal.

Therefore \( t_\psi \geq n \), since \( p = 2n - 2 \). Hence \( \eta_b(G) = q - p + n \)

Subcase-(ii): \( n \equiv 1 \pmod{4} \) or \( n \equiv 3 \pmod{4} \)

The collection of paths of \( G \) is given by

\[
P_i = (u_{2i-1}, u_{2i}),
\]

\[
Q_j = (u_{2j}, v_j, u_{2j+1}, u_{2j+2}),
\]

\[
R_i = (u_{2i}, w_j),
\]

\[
S_i = (w_j, u_{2i+1}), 1 \leq i \leq \left( \frac{n-1}{2} \right).
\]

is a simple graphoidal cover of \( G \) in which \( u_1, u_2, u_4, \ldots, u_{n-1}, w_1, w_2, \ldots, w_{\left( \frac{n-1}{2} \right)} \) are not internal. Therefore

\[
\eta_b(G) \leq q - p + n
\]

Now, let \( \psi \) be any simple graphoidal cover of \( G \). Since \( \psi \) can contain at most \( \left( \frac{n-1}{2} \right) \) triangles and each triangle can make two vertices internal in \( \psi \), at most \( (n-1) \) vertices can be made internal.

Therefore \( t_\psi \geq n \), since \( p = 2n - 1 \). Hence \( \eta_b(G) = q - p + n \)

Theorem 2.9: Let \( G \) be an alternate quadrilateral snake graph, then

\[
\eta_b(G) = \begin{cases} 
q - p + \left( \frac{n}{2} \right), & \text{triangular paths start at } u_1 \text{ and } n \equiv 0 \pmod{4} \text{ or } n \equiv 2 \pmod{4} \\
q - p + \left( \frac{n+1}{2} \right), & \text{triangular paths start at } u_1 \text{ or } u_2 \text{ and } n \equiv 1 \pmod{4} \text{ or } n \equiv 3 \pmod{4} \\
q - p + \left( \frac{n+2}{2} \right), & \text{triangular paths start at } u_2 \text{ and } n \equiv 0 \pmod{4} \text{ or } n \equiv 2 \pmod{4}
\end{cases}
\]

Proof: Let \( \{u_1, u_2, u_3, u_4, \ldots, u_n\} \) be the underlined path in \( G \) and \( \{v_1, v_2, \ldots, v_{n-1}, w_1, w_2, \ldots, w_{n-1}\} \) be the vertices in \( G \) such that joining \( u_i \) \& \( u_{i+1} \) with two new vertices \( v_i, w_j \) by the edges \( (u_i, v_j), (v_j, w_j), (w_j, u_{i+1}) \). Here we have two cases.

Case-(i): Alternate quadrilateral snake starts at \( u_1 \)

Subcase-(i): \( n \equiv 0 \pmod{4} \) or \( n \equiv 2 \pmod{4} \)

The collection of paths of \( G \) given by

\[
P_i = (u_1, v_1, u_2, u_1),
\]

\[
P_i = (u_{2i}, u_{2i+1}).
\]
A simple graphoidal cover of G in which \( u_1, u_3, u_5, \ldots, u_{n-1} \) are not internal. Therefore \( \eta_C(G) \leq q - p + \left( \frac{n}{2} \right) \).

Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n}{2} \right) \) cycles of length 4 and each cycle of length 4 can make three vertices internal in \( \psi \), at most \( \left( \frac{3n}{2} \right) \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n}{2} \right) \), since \( p = 2n \). Hence \( \eta_C(G) \geq q - p + \left( \frac{n}{2} \right) \). Thus \( \eta_C(G) = q - p + \left( \frac{n}{2} \right) \).

Subcase-(ii): \( n \equiv 1 (\mod 4) \) or \( n \equiv 3 (\mod 4) \)

The collection of paths of G given by
\[
P_i = (u_{2i-1}, v_i, w_i, u_{2i+1}, u_{2i}),
\]
\[
Q_i = (u_{2i+1}, u_{2i+2}), \quad 1 \leq i \leq \left( \frac{n-1}{2} \right).
\]
is a simple graphoidal cover of G in which \( u_1, u_3, u_5, \ldots, u_{n} \) are not internal. Therefore \( \eta_C(G) \leq q - p + \left( \frac{n+1}{2} \right) \). Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n-1}{2} \right) \) cycles of length 4 and each cycle of length 4 can make three vertices internal in \( \psi \), at most \( \left( \frac{3n-3}{2} \right) \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n+1}{2} \right) \), since \( p = 2n - 1 \). Hence \( \eta_C(G) \geq q - p + \left( \frac{n+1}{2} \right) \). Thus \( \eta_C(G) = q - p + \left( \frac{n+1}{2} \right) \).

Case-(ii): Alternate quadrilateral triangular snake starts at \( u_2 \)

Subcase-(i): \( n \equiv 0 (\mod 4) \) or \( n \equiv 2 (\mod 4) \)

The collection of paths of G given by
\[
P = (u_1, u_2),
\]
\[
P_i = (u_{2i}, v_i, w_i, u_{2i+2}, u_{2i+1}),
\]
\[
Q_i = (u_{2i+1}, u_{2i+2}), \quad 1 \leq i \leq \left( \frac{n-2}{2} \right).
\]
is a simple graphoidal cover of G in which \( u_1, u_2, u_4, u_6, u_8, \ldots, u_{n} \) are not internal. Therefore \( \eta_C(G) \leq q - p + \left( \frac{n+2}{2} \right) \).

Now, let \( \psi \) be any simple graphoidal cover of G. Since \( \psi \) can contain at most \( \left( \frac{n-2}{2} \right) \) cycles of length 4 and paths of length 1. Each cycle of length 4 can make three vertices internal in \( \psi \), at most \( \left( \frac{3n-6}{2} \right) \) vertices can be made internal. Therefore \( t_\psi \geq \left( \frac{n+2}{2} \right) \), since \( p = 2n - 2 \). Hence \( \eta_C(G) \geq q - p + \left( \frac{n+2}{2} \right) \). Thus \( \eta_C(G) = q - p + \left( \frac{n+2}{2} \right) \).

Subcase-(ii): \( n \equiv 1 (\mod 4) \) or \( n \equiv 3 (\mod 4) \)

The collection of paths of G given by
\[
P_i = (u_{2i-1}, u_{2i}) \]
\[
Q_i = (u_{2i}, v_i, w_i, u_{2i+2}, u_{2i+1}), \quad 1 \leq i \leq \left( \frac{n-1}{2} \right)
\]
is a simple graphoidal cover of G in which \( u_1, u_2, u_4, u_6, u_8, \ldots, u_{n-1} \) are not internal. Therefore \( \eta_C(G) \leq q - p + \left( \frac{n+1}{2} \right) \).
Now, let $\psi$ be any simple graphoidal cover of $G$. Since $\psi$ can contain at most $\left(\frac{n-1}{2}\right)$ cycles of length 4 and each cycle of length 4 can make three vertices internal in $\psi$, at most $\left(\frac{3n-3}{2}\right)$ vertices can be made internal. Therefore $t_{\psi} \geq \left(\frac{n+1}{2}\right)$ since $p = 2n - 1$. Hence $\eta_{b}(G) \geq q - p + \left(\frac{n+1}{2}\right)$. Thus $\eta_{b}(G) = q - p + \left(\frac{n+1}{2}\right)$.

**Theorem 2.10:** Let $G$ be alternate double quadrilateral snake graph, then $\eta_{b}(G) = q - p + n$

**Proof:** Let $\{u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}, \ldots, u_{i_n}\}$ be an underlined path in $G$ and $\{v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}, \ldots, v_{j_{n-1}}, w_1, w_2, w_3, \ldots, w_{n-1}, x_1, x_2, x_3, \ldots, x_{n-1}, y_1, y_2, y_3, \ldots, y_{n-1}\}$ be the vertices in $G$ by joining $u_i \& u_{i+1}$ with four new vertices $v_j, w_i, x_i, y_i$ by the edges $(u_i, v_j), (v_j, w_i), (w_i, u_{i+1}), (u_{i+1}, x_i), (x_i, y_i), (y_i, u_{i+1})$. Here we have two cases.

**Case-(i):** Alternate double quadrilateral snake starts at $u_1$

**Subcase-(i):** $n = 0 \pmod{4}$ or $n = 2 \pmod{4}$

The collection of paths of $G$ is given by

\[
P = (u_1, v_1, w_1, u_2, u_1)
\]

\[
Q = (u_1, x_1, y_1)
\]

\[
R = (y_1, u_2)
\]

\[
P_i = (u_{2i-1}, u_{2i+1})
\]

\[
Q_i = (u_{2i-1}, v_{i+1}, w_{i+1}, u_{2i+1}, u_{2i+2}, u_{2i+1})
\]

\[
R_i = (u_{2i-1}, x_{i+1}, y_{i+1})
\]

\[
S_i = (y_{i+1}, u_{2i+2}), \quad 1 \leq i \leq \left(\frac{n-2}{2}\right)
\]

is a simple graphoidal cover of $G$ in which $u_1, u_3, u_5, \ldots, u_{n-1}, y_1, y_2, y_3, \ldots, y_{n-1}$ are not internal.

Therefore $\eta_{b}(G) \leq q - p + n$. Now, let $\psi$ be any simple graphoidal cover of $G$. Since $\psi$ can contain at most $\left(\frac{n}{2}\right)$ cycles of length 4 and $\left(\frac{n}{2}\right)$ paths of length 2. In each cycle of length 4 can make three vertices internal and each path of length 2 can make one vertex internal in $\psi$, at most $2n$ vertices can be made internal. Therefore $t_{\psi} \geq n$, since $p = 3n$. Vertices. Hence $\eta_{b}(G) \geq q - p + n$. Thus $\eta_{b}(G) = q - p + n$.

**Subcase-(ii):** $n = 1 \pmod{4}$ or $n = 3 \pmod{4}$

The collection of paths of $G$ given by

\[
P_i = (u_{2i-1}, v_{i}, w_{i}, u_{2i}, u_{2i-1})
\]

\[
Q_i = (u_{2i-1}, x_{i}, y_{i})
\]

\[
R_i = (y_{i}, u_{2i})
\]

\[
S_i = (u_{2i}, u_{2i+1}), \quad 1 \leq i \leq \left(\frac{n-1}{2}\right)
\]

is a simple graphoidal cover of $G$ in which $u_{i_1}, u_{i_3}, u_{i_5}, \ldots, u_{i_{n-1}}, y_{j_1}, y_{j_2}, y_{j_3}, \ldots, y_{j_{n-1}}$ are not internal. Therefore $\eta_{b}(G) \leq q - p + n$. Now, let $\psi$ be any simple graphoidal cover of $G$. Since $\psi$ can contain at most $\left(\frac{n-1}{2}\right)$ cycles of length 4 and $\left(\frac{n-1}{2}\right)$ paths of length 2. In each cycle of length 4 can make three vertices internal and each path of length 2 can make one vertex internal in $\psi$, at most $2n - 2$ vertices can be made internal. Therefore $t_{\psi} \geq n$ since $p = 3n - 2$. Hence $\eta_{b}(G) \geq q - p + n$. Thus $\eta_{b}(G) = q - p + n$. 

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Case-(ii): Alternate double quadrilateral snake starts at $u_2$

**Subcase-(i):** $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$

The collection of paths of G given by

$P_i = (u_{2i}, v_i, w_i, u_{2i+1}, u_{2i+2})$,

$Q_i = (u_{2i}, x_i, y_i)$,

$R_i = (y_i, u_{2i+1})$.

$S_i = (u_{2i+1}, u_{2i+2})$, $1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor$.

is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_5, \ldots, u_n, y_1, y_2, y_3, \ldots, y_{\frac{n-2}{2}}$ are not internal. Therefore $\eta_s(G) \leq q-p+n$. Now, let $\psi$ be any simple graphoidal cover of G. Since $\psi$ can contain at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ cycles of length 4 and $\left\lfloor \frac{n-2}{2} \right\rfloor$ paths of length 2. Each cycle of length 4 can make three vertices internal and each path of length 2 can make one vertex internal in $\psi$, at most $2n-4$ vertices can be made internal. Therefore $\ell_\psi \geq n$, since $p = 3n - 4$.

Hence $\eta_s(G) \geq q - p + n$ . Thus $\eta_s(G) = q-p+n$

**Subcase-(ii):** $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$

The collection of paths of G given by

$P_i = (u_{2i-1}, u_{2i})$,

$Q_i = (u_{2i}, v_i, w_i, u_{2i+1}, u_{2i+2})$,

$R_i = (u_{2i}, x_i, y_i)$,

$S_i = (y_i, u_{2i+1})$, $1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

is a simple graphoidal cover of G in which $u_1, u_2, u_4, u_5, \ldots, u_{n-1}, y_1, y_2, y_3, \ldots, y_{\frac{n-1}{2}}$ are not internal. Therefore $\eta_s(G) \leq q-p+n$. Now, let $\psi$ be any simple graphoidal cover of G. Since $\psi$ can contain at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ cycles of length 4 and $\left\lfloor \frac{n-1}{2} \right\rfloor$ paths of length 2. Each cycle of length 4 can make three vertices internal and each path of length 2 can make one vertex internal in $\psi$, at most $2n-2$ vertices can be made internal. Therefore $\ell_\psi \geq n$, since $p = 3n - 2$.

Hence $\eta_s(G) \geq q - p + n$ . Thus $\eta_s(G) = q-p+n$

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