International Journal of Mathematical Archive-7(3), 2016, 153-156
Available online through www.ijma.info ISSN 2229 – 5046

STRONG ROMAN DOMINATION IN GRAPHS

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(Received On: 23-01-16; Revised & Accepted On: 11-03-16)

ABSTRACT

A Roman dominating function on a graph G = (V; E) is a function f: V$\mapsto$ {0, 1, 2} satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 2: The weight of a Roman dominating function is the value f(V) =$\sum_{u\in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the Roman dominating number of G: In this paper, we introduced the concept of strong Roman domination which is the generalization of Roman domination and we study the graph theoretic properties of this variant of the domination number of a graph.

Key Words: *Roman domination, Strong Roman domination, Strong Roman domination number.*

1. INTRODUCTION

Let G = (V; E) be a graph of order ∣V∣ = n: For any vertex v ∈ V; the open neighborhood of v is the set N(v) = {u∈V: uv ∈ E} and the closed neighborhood is the set N[v] = n(v) ∪{v}: For a set S⊆V; the open neighborhood S is N(S) = $\sum_{v\in S} N(v)$ and the closed neighborhood is N[S] = N(S)∪S:

Let v ∈ S⊆V: Vertex u is called a private neighbor of v with respect to S (denoted by u is an S pn of v) if u ∈ N[v] -N[S-{v}]: An S pn of v is external if it is a vertex of V S: The set pn(v; S) = N[v] -N[S-{v}] of all S -pn's of v is called the private neighborhood set of v with respect to S: The set S is said to be irredundant if every v ∈ S; pn(v; S) ≠ φ:

A set S V is a dominating set if N[S] = V; or equivalently, every vertex in V-S is adjacent to at least one vertex in S: The domination number γ (G) is the minimum cardinality of a dominating set in G; and a dominating set S of minimum cardinality is called a γ-set of G:

A set S of vertices is called a 2-packing if for every pair of vertices u; v ∈ S; N[u] \ N[v] = φ: The 2-packing number P_2(G) of a graph G is the maximum cardinality of a 2-packing in G: A set S of vertices is called a vertex cover if for every edge uv ∈ E; either u ∈ S or v ∈ S:

A Roman dominating function (RDF) on a graph G = (V; E) is defned in [7] as a function f: V→{0, 1, 2} satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 2: The weight of a RDF is the value f(V) = $\sum_{u\in V} f(u)$. The Roman dominating number of a graph G, denoted by γ_R(G), equals the minimum weight of an RDF on G.

Stated in other words, a Roman dominating function is a colouring of the vertices of a graph with the colours {0, 1, 2} such that every vertex coloured 0 is adjacent to at least one vertex coloured 2: The idea is that colours 1 and 2 represent either one or two Roman legions stationed at a given location (vertex v). A nearby location (an adjacent vertex u) is considered to be unsecured if no legions are stationed there (ie f(u) = 0). An unsecured location (u) can be secured by sending a legion to u from an adjacent location (v): But Emperor Constantine the Great, in the fourth century A:D:; decreed that a legion cannot be sent from a location v if doing so leaves that location unsecured (ie if f(v) = 1). Thus, two legions must be stationed at a location (f(v) = 2) before one of the legions can be sent to an adjacent location.

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In 2004, Cockayne *et al*. [1] studied the graph theoretic properties of Roman dominating sets. In recent years many authors studied the concept of Roman dominating functions and Roman domination numbers [1]-[8].

In this paper, first we introduced the concept of Strong Roman domination which is the generalization of Roman domination. A Strong Roman dominating function(SRDF) is a function $f : V \rightarrow \{0, 1, 2, 3\}$ satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 3 and every vertex u for which f(u) = 1 is adjacent to at least one vertex v for which f(v) = 2: The weight of a SRDF is the value $f(V) = \sum_{u\in V} f(u)$: Theminimum weight of a SRDF on a graph G is called the Strong Roman domination number of G: Then we study the graph theoretic properties of this variant of the domination number of a graph.

2. PROPERTIES OF STRONG ROMAN DOMINATING SETS

For a graph G = (V; E); let $f:V\rightarrow\{0, 1, 2, 3\}$ and let (V_0, V_1, V_2, V_3) be the ordered partition of V induced by f; where $V_i = \{v \in V: f(v) = i\}$ and $|V_i| = n_i$ for i = 0, 1, 2, 3: Note that there exists a correspondence between the function $f: V \rightarrow \{0, 1, 2, 3\}$ and the ordered partition (V_0, V_1, V_2, V_3) of V: Thus we will write $f = (V_0, V_1, V_2, V_3)$: A function $f = (V_0, V_1, V_2, V_3)$ is a Strong Roman dominating function (SRDF) if $V_3 \succ V_0$ and $V_2 \succ V_1$ (where $A \succ B$ means that the set A dominates the set B). (ie) $V_0 \subseteq N[V_3]$ and $V_1 \subseteq N[V_2]$: The weight of f is $f(V) = \sum_{u\in V} f(u)$. The Strong Roman domination number, denoted by $\gamma_{SR}(G)$, equals the minimum weight of an SRDF of G. we say that a function $f = (V_0, V_1, V_2, V_3)$ is a γ_{SR} -function if it is an SRDF and $f(V) = \gamma_{SR}(G)$:

Theorem 2.1: For any graph G; $\gamma(G) \le \gamma_{SR}(G) \le \gamma(G)$.

Proof: Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{SR}-function. Since $V_3 \succ V_0$ and $V_2 \succ V_1$; $(V_2 \cup V_3)$ is a dominating set of G: Therefore $\gamma(G) \le | |V_2 [V_3| = | V_2 | + | V_3| \; 2| V_2| + 2| V_3| = \gamma_{SR}(G)$:

Now, let S be a -set of G: Define $(V_0, V_1, V_2, V_3) = (V\backslash S; \phi; \phi; S)$ and let $f = (V_0, V_1, V_2, V_3)$: Since $V_3 \succ V_0$ and $V_2 \succ V_1$; f is an SRDF and $\gamma_{SR}(G) = f(V) = 2|S| = 2\gamma(G)$:

Theorem 2.2: For any graph G; $\gamma(G) \neq \gamma_{SR}(G)$:

Proof: Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{SR}-function. The equality $\gamma(G) = \gamma_{SR}(G)$ implies that we have an equality in $\gamma(G) \le |V_2| + |V_3| \le 2|V_2| + 2|V_3| = \gamma_{SR}(G)$: Hence, $|V_2| = 0$ and $|V_3| = 0$ which implies $V_0 = \varphi$ and $V_1 = \varphi$: Therefore there is no vertices in G which is a contradiction.

Theorem 2.3: For any graph G; $\gamma_R(G) \le \gamma_{SR}(G)$:

Proof: Let $f = (V_0, V_1, V_2, V_3)$ be an γ_{SR}-function. Since $V_3 \succ V_0$ and $V_2 \succ V_1$; $f = (V_0, V_1, V_2, V_3)$ is a Roman dominating set of G: Therefore,

$$\begin{aligned}\gamma_R(G) &\le |V_0 \cup V_2 \cup V_3| \\ &= |V_0| + |V_2| + |V_3| \\ &= 2|V_3| + |V_2| \\ &\le 2|V_3| + 2|V_2| \\ &= \gamma_{SR}(G):\end{aligned}$$

Theorem 2.4: Let $f = (V_0, V_1, V_2, V_3)$ be any γ_{SR}-function. Then,

(i) $G[V_2]$; the subgraph induced by V_2 has maximum degree 0. (ie.) There is no edge in $G[V_2]$:
(ii) No edge of G joins V_2 and V_3:
(iii) No edge of G joins V_1 and V_3:

Proof:

(i) Suppose uv is an edge in $G[V_2]$: Form a new function f^0 by changing (f(u); f(v)) = (2; 2) to (0; 3): Then f^0 is an SRDF with $f^0(V) < f(V)$; a contradiction.
(ii) Let $uv \in E(G)$ where f(u) = 2 and f(v) = 3: Form a new function f^0 by changing f(u) = 0: Then f^0 is an SRDF with $f^0(V) < f(V)$; a contradiction.
(iii) Let $uv \in E(G)$ where f(u) = 1 and f(v) = 3: Form a new function f^0 by changing f(u) = 0: Then f^0 is an SRDF with $f^0(V) < f(v)$; a contradiction.

Theorem 2.5: Let $f = (V_0, V_1, V_2, V_3)$ be any γ_{SR}-function. Then,

(i) Each vertex of V_0 is adjacent to at most one vertex of V_2: (ie.) V_2 is a 2-packing.
(ii) V_3 is a γ-set of $H_1 = [V_0 \cup V_3]$:
(iii) V_2 is a γ-set of $H_2 = [V_1 \cup V_2]$:

Proof:

(i) Suppose $v_0 \in V_0$ is adjacent to $a_2, b_2 \in V_2$:
Let $f^0 = (W_0, W_1, W_2, W_3)$ where
$W_0 = (V_0 \cup \{a_2; b_2\}) \setminus \{v_0\}$
$W_1 = V_1$
$W_2 = V_2 \setminus \{a_2; b_2\}$
$W_3 = V_3 \cup \{v_0\}$:

It follows that $f^0(V) = 2(n_2 - 2) + 2(n_3 + 1) = 2n_2 + 2n_3 - 2$
$< f(V) = \gamma_{SR}(G)$:

But Since $W_3 \succ W_0$ and $W_2 \succ W_1$; f^0 is an SRDF with $f^0(V) < f(V)$; a contradiction.

(ii) Suppose that S is a dominating set of $G[V_0 \cup V_3]$ and $|S| < |V_3|$: Define $f^0 = (W_0, W_1, W_2, W_3)$ by
$W_0 = (V_0 \cup V_3) \setminus S$
$W_1 = V_1$
$W_2 = V_2$
$W_3 = S$:

Then $W_3 \succ W_0$; $W_2 \succ W_1$ and f^0 is an SRDF.

However; $f^0(V) = 2|W_2| + 2|W_3|$
$= 2|V_2| + 2|S|$
$< 2n_2 + 2n_3$
$= f(V)$; which is a contradiction:

(iii) Suppose that S is a dominating set of $G[V_1 \cup V_2]$ and $|S| < |V_2|$: Define $f^0 = (W_0, W_1, W_2, W_3)$ by
$W_0 = V_0$
$W_1 = (V_1 \cup V_2) \setminus S$
$W_2 = S$
$W_3 = V_3$:

Then $W_3 \succ W_0$; $W_2 \succ W_1$ and f^0 is an SRDF.

However; $f^0(V) = 2|W_2| + 2|W_3|$
$= 2|S| + 2|V_3|$
$< 2n_2 + 2n_3$
$= f(V)$; a contradiction:

Theorem 2.6: Let $f = (V_0, V_1, V_2)$ be any γ_{SR}-function. Then,

(i) Each vertex $v \in V_3$ has at least two V_3 - pns in $H_1 = G[V_0 \cup V_3]$:
(ii) If v is isolated in $G[V_3]$ and has precisely one external V_3 - pn (in $H = G[V_0 \cup V_3]$) say $w \in V_0$: Then $N(w) \setminus V_1 = \varphi$:

Proof:

(i) By Theorem 2.5 (ii), V_3 is a -set of H_1 and hence is a maximal irredundant set in H_1: Therefore each $v \in V_3$ has at least one V_3 -pn in H_1:

Let v be isolated in $G[V_3]$: Then v is a V_3 -pn of v: Suppose that v has no external V_3- pn: Then the function produced by changing f(v) from 3 to 2 is an SRDF of smaller weight, a contradiction. Hence v has at least two V_3 -pns in H_1:

Suppose that v is not an isolated in $G[V_3]$ and has precisely one V_3 - pn(in H_1) say w; consider the function produced by changing f(v) to 0 and f(w) to 1 or 2. The vertex v is still defended because it has a neighbor in V_3: All of v^0s neighbors in V_0 are also defended, since every vertex in V_0 has another neighbor in V_3 except w; which is now in V_1 or V_2: Therefore, this new function is a SRDF of smaller weight, which is a contradiction. Again we can conclude that v has at least two V_3 pns in H_1:

(ii) Suppose that $N(w) \setminus V_1 \neq \phi$: Let $y \in N(w) \setminus V_1$: Define a new function f^0 with $f^0(v) = 0$; $f^0(y) = 0$; $f^0(w) = 3$ and $f^0(x) = f(x)$ for all other vertices x: Therefore $f^0(V) = f(V)$, $|N(w) \setminus V_1| < f(V)$; which is a contradiction to the minimality of f:

Theorem 2.7: For any non-trivial connected graph G;

$$\gamma_{SR}(G) = \min\{2\,\gamma\,(G \setminus S) + |S| \;:\; S \text{ is a 2 packing}\}$$

Proof: Let f = (V_0, V_1, V_2, V_3) be a γ_{SR}-function of a graph G: From Theorem 2.5(i), we can assume that V_2 is 2-packing. It follows from Theorem 2.5(ii) that V_3 is a γ -set of the graph G \S obtained from G by deleting all vertices in V_1 and V_2: Thus $\gamma_{SR}(G) = \min\{2\gamma(G\setminus S) + |S| : S$ is a 2-packing$\}$. Conversely, let V_1 and V_2 are 2-packings for which 2 (G\ S) + |S| is minimum, and let V_3 be a γ -set of G ($V_1 \cup V_2$): Then ($V \setminus (V_1 \cup V_2)$; V_1; V_2; V_3) is an SRDF, and $\gamma_{SR}(G) = 2|V_3| + 2|V_2| = \min\{2\,\gamma(G \setminus S) + |S| : S$ is a 2-packing$\}$.

3. CONCLUSION AND SCOPE

In this paper, we introduced a new variation of domination is called Strong Roman dominating function. Then we studied some basic properties of Strong Roman dominating sets.

Among the many questions raised by this research and the particular interest of the authors, we propose the following open problems.

Problem 3.1: Can you nd other properties of Strong Roman dominating sets and functions?

Problem 3.2: Can you determine the bounds for Strong Roman domination number?

Problem 3.3: What can you say about the minimum and maximum values of n_0, n_1, n_2 and n_3 for a γ_{SR} function f = (V_0, V_1, V_2, V_3) of a graph G?

REFERENCES

1. E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi, Roman domination in graphs, Discrete Math., 278 (2004), 11-22.
2. P. A. Dreyer, Applications and variations of domination in graphs, Ph.D. Thesis, New Brunswick, New Jersey, 2000.
3. O. Favaron, H. Karamic, R. Khoeilar and S. M. Sheikholeslami, On the Roman domination number of a graph, Discrete Math., 309 (2009), 3447- 3451.
4. M. A. Henning, Defending the Roman empire from multiple attacks, Discrete Math., 271 (2003), 101-115.
5. T. Kraner Sumenjak, P. Parlic and A. Tepeh, On the Roman domination in the lexicographic product of graphs, Discrete Appl. Math., 160 (2012), 2030-2036.
6. P. Parlic and J. Zerovnik, Roman domination number of the Cartesian prod-ucts of paths and cycles, Electron. J. Combin., 19(3) (2012), #P19.
7. I. Strewart, Defend the Roman Empire, Sci. Amer., 28(6) (1999), 136-139.
8. F. Xueliang, Y. Yuausheng and J. Bao, Roman domination in regular graphs, Discrete Math., 309(6) (2009), 1528-1537.

Source of support: Nil, Conflict of interest: None Declared