μ – Best One-Sided Approximation of Unbounded Functions in the Space $L_{p,\mu}(X)$

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ABSTRACT

The aim of this paper is to study the μ-best one-sided approximation of unbounded functions in the space $L_{p,\mu}[a,b]$, $(1 \leq p < \infty)$ by Spline polynomials. We consider the point wise estimates in terms of Dititzan-Totic modulus of smoothness are true for spline approximation in the space $L_{p,\mu}[a,b]$.

1. INTRODUCTION

Let $X = [a, b]$, Consider the space

$$
P_n = \{ p(x): p(x) = \sum_{i=0}^{n} c_i x^{i-1}, c_1, c_2, ..., c_n \text{ are reals} \}
$$

of polynomials of order $n$ which has the attractive features [6].

Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ and write $\Delta = \{ x_i \}_{0}^{k+1}$. The $\Delta$ partitions of the interval $[a, b]$ into $k+1$ subintervals $I_i = [x_i, x_{i+1}], i = 0, 1, ..., k - 1$ and $l_k = [x_k, x_{k+1}]$.

Let

$$
P_n(\Delta) = \left\{ f: \text{there exist polynomials } p_0, p_1, ..., p_k \text{ in } P \text{ with } f(x) = p_i(x) \text{ for } x \in I_i, i = 0, 1, ..., k \right\}
$$

(1.1)

We call $P_n(\Delta)$ the space of piecewise polynomials of order $n$ with knots $x_1, x_2, ..., x_k$. The terminology in (1.1) is perfectly descriptive-an element $f \in P_n(\Delta)$ consists of $k+1$ polynomial pieces [9].

Let $\Delta$ be a partition of the interval $[a, b]$ as in (1.1) and let $n$ be a positive integer.

Let $S_n(\Delta) = P_n(\Delta) \cap C^\infty[a, b]$. We call $S_n(\Delta)$ the space of polynomial splines of order $n$ with simple knots at the points $x_1, x_2, ..., x_k$.

Let $L_\infty(X), (1 \leq p < \infty)$ be the space of all bounded measurable functions with usual norm [8].

$$
\| f \|_{L_\infty} = \| f \|_{\infty} = \sup \{|f(x)|, x \in X\} \leq \infty,
$$

(1.2)

$L_p(X)$ be the space of all of bounded measurable function $f$ on $X$, for which [3]

$$
\| f \|_{L_p} = \| f \|_p = \left( \int_X f(x)^p \, dx \right)^\frac{1}{p} < \infty,
$$

(1.3)

the locally global norm for $\delta > 0$ and $(1 \leq p < \infty)$ of $f$ is defined by

$$
\| f \|_{\delta,p} = \left( \int_X (\sup \{|f(y)|, y \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]})^p \, dy \right)^\frac{1}{p},
$$

(1.4)

Let $L_{p,\mu}(X), (1 \leq p < \infty)$ be the space of all bounded $\mu$-measurable functions $f$ on $X$, for which [1]

$$
\| f \|_{L_{p,\mu}} = \| f \|_{P,\mu} = \left( \int_X f(x)^p \, d\mu(x) \right)^\frac{1}{p} < \infty,
$$

(1.5)

where $\mu$ is the non-negative measurable function on set $X$.

For $\delta > 0$, the modulus of continuity of the function $f$ on $X$ [10] is defined by

$$
\omega(f, \delta) = \sup \{|f(x_1) - f(x_2)|: |x_1 - x_2| < \delta, x_1, x_2 \in X\}
$$

(1.6)

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The moduli of smoothness form a natural generalization of the modulus of continuity.

For every function \( f \) we define the \( k \)th difference with step \( h \) at a point \( x \) as follows:
\[
\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f(x + ih), \quad x, x + ih \in X
\]  
(1.7)

For \( \delta > 0 \), the modulus of smoothness of order \( k \) of function it following function [10]
\[
\omega_k(f, \delta) = \sup_{|h| < \delta} \left\{ \| \Delta_h^k f(x) \|, \ x, x + kh \in X \right\}
\]  
(1.8)

The \( k \)th ordinary modulus of continuity for \( f \in L_p(X) \) and \( f \in L_{p, \mu}(X) \) respectively by
\[
\omega_k(f, \delta)_p = \sup_{|h| < \delta} \left\{ \| \Delta_h^k f(.) \|_p, \delta > 0 \right\}
\]  
(1.9)

\[
\omega_k(f, \delta)_{p, \mu} = \sup_{|h| < \delta} \left\{ \| \Delta_h^k f(.) \|_{p, \mu}, \delta > 0 \right\}
\]  
(1.10)

The local modulus of smoothness of function \( f \) of order \( k \) at a point \( x \in X \) is following function of \( \delta > 0 \) [10]
\[
\omega_k(f, x, \delta) = \sup \left\{ \| \Delta_h^k f(t) \|, \ t, t + kh \in \left[ x - \frac{\delta}{2}, x + \frac{\delta}{2} \right] \cap X \right\}
\]  
(1.11)

The \( k \)th averaged modulus of smoothness of function \( f \) of order \( k \) (or \( \tau \)-modulus) of the function \( f \in L_p(X) \) is following function of \( \delta > 0 \) is given by [11]:
\[
\tau_k(f, \delta)_p = \| \omega_k(f, . , \delta) \|_p = \left( \int_X \| \omega_k(f, x, \delta) \|^p \, dx \right)^\frac{1}{p}
\]  
(1.12)

Further the \( k \)th averaged modulus of smoothness for \( f \in L_{p, \mu}(X) \) is given by
\[
\tau_k(f, \delta)_{p, \mu} = \| \omega_k(f, . , \delta) \|_{p, \mu}
\]  
(1.13)

The \( K \)-functional for \( f \in X_0 \) and \( g \in X_1 \) is given by
\[
K(f, \delta) = K(f, \delta, X_0, X_1) = \inf_{g \in X_1} \left\{ \| f - g \|_{X_0} + \delta \| g \|_{X_1}, \delta > 0 \right\}
\]  
(1.14)

Where \( X_0 \) and \( X_1 \) be two Banach spaces with \( X_1 \subset X_0 \). [6]

The inequality \( K(f, \delta) < \epsilon \) for some \( \delta > 0 \), \( \epsilon \) is a positive real number, implies that \( f \) has approximated with error \( \| f - g \| < \epsilon \) in \( X_0 \) by an element \( g \in X_1 \), whose norm is not too large (\( \| g \|_{X_1} < \epsilon \delta^{-1} \)).

The \( K \)-functional in \( L_p(X) \) space is given by [6]
\[
K_r(f, \delta^*)_p = \inf_{g \in W^{r}_{p}} \left\{ \| f - g \|_p + \delta^* \| g \|_{W^r_p}, \delta > 0 \right\}
\]  
(1.15)

Where \( X_0 = L_p(X) \) and \( X_1 = W^{r}_{p} \) and \( X_1 \subset X_0 \).

Now, we introduce \( K \)-functional of a function \( f \in L_{p, \mu}(X) \) such that [2]
\[
K_r(f, \delta^*)_{p, \mu} = \inf_{g \in W^{r}_{p}} \left\{ \| f - g \|_{p, \mu} + \delta^* \| g \|_{W^r_p}, \delta > 0 \right\}
\]  
(1.16)

The Ditzian-Totic modulus of smoothness for \( f \in L_p(X) \) as [4]
\[
\omega_k^p (f, \delta)_p = \sup_{|h| < \delta} \| \Delta_h^k f(.) \|_p
\]  
(1.17)

Where
\[
\Delta_h^k f(x) = \left\{ \begin{array}{ll}
\sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f(x + ih), x + ih \in X \\
0 & \text{otherwise}
\end{array} \right.
\]

Also, the locally \( \mu \)- Ditzian-Totic modulus of smoothness for \( f \in L_{p, \mu}(X) \) is defined by
\[
\omega_k^p (f, \delta)_{p, \mu} = \sup_{|h| < \delta} \| \Delta_h^k f(.) \|_{p, \mu}, \ where \ \mu(x) = (1 - x^2)^{\frac{1}{2}}
\]  
(1.18)

The degree of best approximation to a given continuous function with respect to a polynomial spline on interval \( X \) is given by [5]:
\[
E_{\mu}(f)_{\infty} = \inf \{ \| f - s \|_{\infty}; s \in S_{\mu}(\Delta) \}
\]  
(1.19)
While the degree of best approximation of a function \( f \in L_p(X) \) with respect to a polynomial spline of degree \( \leq n \) on \( X \) is given by
\[
E_n(f)_p = \inf \{ \| f - s \|_p; \ s \in S_n(\Delta) \}.
\] (1.20)

Also, the degree of \( \mu \)-best approximation to a given function \( f \in L_{p,\mu}(X) \) with respect to polynomial spline of degree \( \leq n \) on \( X \) is defined by
\[
E_n(f)_{p,\mu} = \inf \{ \| f - s \|_{p,\mu}; \ s \in S_n(\Delta) \}.
\] (1.21)

The degree of best one-sided approximation of function \( f \in L_p(X) \) with respect to polynomial spline of degree \( \leq n \) on interval \( X \) is given by
\[
\tilde{E}_n(f)_p = \inf \{ \| \tilde{s} - \tilde{f} \|_p; \ \tilde{s}, \tilde{f} \in S_n(\Delta) \text{ and } \tilde{s}(x) \leq f(x) \leq \tilde{f}(x) \}.
\] (2.1)

The degree of \( \mu \)-best one-sided approximation of function \( f \in L_{p,\mu}(X) \) with respect to polynomial spline of degree \( \leq n \) on interval \( X \) is given by
\[
\tilde{E}_n(f)_{p,\mu} = \inf \{ \| \tilde{s} - \tilde{f} \|_{p,\mu}; \ s, \tilde{f} \in S_n(\Delta) \text{ and } \tilde{s}(x) \leq f(x) \leq \tilde{f}(x) \}.
\] (2.2)

2. AUXILIARY LEMMAS

**Lemma I [7]:** For \( f \in L_p(X), (0 < p \leq \infty) \), we have
\[
\omega_k(f, \delta)_p \leq c(p)\omega_k^\mu(f, \delta)_p
\] (2.1)

where \( c \) is constant depending on \( p \).

**Lemma II [7]:** For \( f \in L_p(X), (0 < p \leq \infty) \), we have
\[
\omega_k(f, \delta)_p \leq \omega_k(f, \delta)_\infty, \ \ \ \delta > 0.
\] (2.2)

**Lemma III [9]:** If \( f \) is a bounded measurable function on the interval \( [a,b], \ a, b \in \mathbb{R} \), then
\[
\int_a^b f(x) \, dx \leq \frac{b-a}{n} \sum_{i=1}^n f(x_i)
\] (2.3)

where
\[
x_i = a + \frac{(b-a)(i-1)}{2n}.
\]

**Lemma IV [1]:** Let \( f \) be a bounded \( \mu \)-measurable function and \( (1 \leq p \leq \infty) \), then we have
\[
\| f \|_p \leq c(p)\| f \|_{p,\mu}
\] (2.4)

3. MAIN RESULTS

In this section, we will get an estimation for \( \tilde{E}_n(f)_{p,\mu} \). The estimation will be given in terms of \( k^{th} \) local modulus of continuity and Ditzian-Totic modulus of smoothness.

Now, we need the following lemmas:

**Lemma 1:** Let \( f \in L_{p,\mu}(X), (1 \leq p < \infty) \) Then
\[
\omega_k(f, \delta)_{\omega,\mu} \leq c(p)\omega_k^\mu(f, \delta)_{p,\mu}.
\] (3.1)

**Proof:** From (2.3) and (2.1)
\[
\omega_k(f, \delta)_{\omega,\mu} = \sup_{|h| < \delta} \| \Delta_k^h f(.) \|_{\omega,\mu} = \sup_{|h| < \delta} \| \Delta_k^h f(.) \, d\mu(.) \|_\infty
\]
\[
= \sup_{|h| < \delta} \{ \sup_{x \in X} | \Delta_k^h f(x) \, d\mu(x) | \}_p
\]
\[
\leq \frac{1}{n} \sum_{i=1}^n | \Delta_k^h f(x_i) \, d\mu(x_i) |_p
\]
\[
= \int_X | \Delta_k^h f(x) |_p \, d\mu(x)
\]

Implies that
\[
\omega_k(f, \delta)_{\omega,\mu} \leq (\int_X | \Delta_k^h f(x_i) |_p^p \, d\mu(x_i))^{\frac{1}{p}}
\]
\[
= \| \Delta_k^h f(.) \, d\mu(.) \|_p \leq \sup \| \Delta_k^h f(.) \, d\mu(.) \|_p = \omega_k(f \, d\mu, \delta)_p
\]
\[
\leq c(p) \omega_k^\mu(f \, d\mu, \delta)_p = c(p) \omega_k^\mu(f, \delta)_{p,\mu}.
\]
Theorem 1: Let $f \in L_{p,\mu}(X)$, ($1 \leq p < \infty$). Then 
$$E_n(f)_{p,\mu} \leq c(p)\omega_k(f, \delta)_{p,\mu}. \quad (3.2)$$

**Proof:** Consider $s \in S_\mu(\Delta)$ is a best approximation of a function $f$.

From (1.21), we have
$$E_n(f)_{p,\mu} = \|f - s\|_{p,\mu} \leq \left( \int_X |f - s(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \leq \left( \int_X \sup |f(x) - s(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \leq c(p)\sup \left( \int_X |f(x) - s(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} = c(p)\left( \int_X |\Delta_k f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} = c(p)\omega_k(f)_{p,\mu}.$$ 

Now, we want to find a relation between $\mu$-best approximation and $\mu$-best one-sided approximation.

**Theorem 2:** Let $f \in L_{p,\mu}(X)$, ($1 \leq p < \infty$) and $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$. Then 
$$E_n(f)_{p,\mu} \leq c(p)\tilde{E}_n(f)_{p,\mu} \leq c(p)E_n(f)_{p,\mu} \quad (3.3)$$
where $c$ constant depending on $p$.

**Proof:** Consider $s \in S_\mu(\Delta)$ is the best approximation of $f \in L_{p,\mu}(X)$ and $\bar{s}, \tilde{s} \in S_\mu(\Delta)$ are the best one-sided approximation of $f \in L_{p,\mu}(X)$ such that $\bar{s}(x) \leq f(x) \leq \tilde{s}(x)$; $x \in X$

We want to prove
$$E_n(f)_{p,\mu} \leq c(p)\tilde{E}_n(f)_{p,\mu} \leq c(p)\tilde{E}_n(f)_{p,\mu}.$$ 

Now to prove 
$$\tilde{E}_n(f)_{p,\mu} \leq c(p)\tilde{E}_n(f)_{p,\mu}.$$ 

Hence 
$$E_n(f)_{p,\mu} \leq c(p)\tilde{E}_n(f)_{p,\mu}.$$
Then we get

\[ E_n(f)_{p,\mu} \leq c(p) \overline{E}_n(f)_{p,\mu} \leq c(p) E_n(f)_{p,\mu} \]

**Theorem 3:** If \( f \in L_{p,\mu}(X), (1 \leq p < \infty) \), then

\[ \overline{E}_n(f)_{p,\mu} \leq \tau_k(f, \Delta_n)_{p,\mu}. \]

**Proof:** Let \( \Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}, \Delta_n = \max |x_i - x_{i-1}|, i = 0,1, \ldots, n \)

Set \( \overline{s}_k(x) = \sup f(t), x \in [x_{i-1}, x_i], t \in [x_{i-1}, x_i] \)

\( \underline{s}_k(x) = \inf f(t), x \in [x_{i-1}, x_i], t \in [x_{i-1}, x_i] \)

and \( \overline{S}(f, x, \delta) = \sup f(t) \) where \( |t - x| \leq \delta/2 \)

\( \underline{S}(f, x, \delta) = \inf f(t) \) where \( |t - x| \leq \delta/2 \)

then, \( \overline{S}(f, x, \delta) \leq \overline{s}(x) \leq \underline{S}(f, x, \delta) \)

we have

\[ \overline{E}_n(f)_{p,\mu} = \| \overline{s}_k - \underline{s}_k \|_{p,\mu} = \left( \int_X (|\overline{x}_k - \underline{x}_k|(x))^p \mu(x) \right)^{1/p} \]

\[ \leq \left( \int_X |\overline{s}(f, x, \delta) - \underline{S}(f, x, \delta)|^p \mu(x) \right)^{1/p} \]

\[ \leq \sup \left( \int_X |\overline{s}(f, x, \delta) - \underline{S}(f, x, \delta)|^p \mu(x) \right)^{1/p} \]

\[ = \tau_k(f, \Delta_n)_{p,\mu}. \]

**Theorem 4:** Let \( f \in L_{p,\mu}(X), (1 \leq p < \infty) \). Then

\[ \overline{E}_n(f)_{p,\mu} \leq c(p) \omega_k^p(f, \delta)_{p,\mu}. \]

**Proof:** By using (2.1), (2.2), (3.1), (2.4) and (3.2) we get

\[ \overline{E}_n, k(f)_{p,\mu} = \inf \| \overline{x} - \underline{x} \|_{p,\mu} \]

\[ \leq \inf \| \overline{s} - \underline{s} \|_{p,\mu} \]

\[ = \overline{E}_n(f)_{p,\mu} \]

\[ \leq \omega_k^p(f, \delta)_{p,\mu} \]

\[ = \sup \| \Delta_k \nu f(\cdot) \|_{p,\mu} \]

\[ \leq c(p) \sup \| \Delta_k \nu f(\cdot) \|_{p,\mu} \]

\[ = c(p) \omega_k^p(f, \delta)_{p,\mu}. \]

**REFERENCES**


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