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μ −Best One-Sided Approximation of Unbounded Functions in the Space $L_{p,\mu}(X)$

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ABSTRACT

The aim of this paper is to study the μ-best one-sided approximation of unbounded functions in the space $L_{p,\mu}[a,b]$, $(1 \le p < \infty)$ *by Spline polynomials ,we consider the point wise estimes in terms of Ditzian-Totic modulus of smoothness are true for spline approximation in the space* $L_{p,\mu}[a,b]$.

1. INTRODUCTION

Let X= $[a, b]$, Consider the space
$$\mathbb{P}_n = \{p(x): p(x) = \textstyle\sum_{i=0}^{n} c_i x^{i-1},\ c_1, c_2, \dots, c_n \ \ are\ reals\}$$
of polynomials of order n which has the attractive features [6]

Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ and write $\Delta = \{x_i\}_0^{k+1}$. The Δ partitions of the interval $[a, b]$ into $k+1$ subintervals $I_i = [x_i, x_{i+1}]$, $i = 0,1,\dots,k-1$ and $I_k = [x_k, x_{k+1}]$.

Let
$$\mathcal{P}_n(\Delta) = \left\{ \begin{matrix} f: there\ exist\ polynomials\ p_0, p_1, \dots, p_k\ in\ \mathbb{P}\ with \\ f(x) = p_i(x)\quad for\ \ x \in I_i,\ \ i = 0,1,\dots,k \end{matrix} \right\} \tag{1.1}$$

We call $\mathcal{P}_n(\Delta)$ the space of piecewise polynomials of order n with knots $x_1, x_2, \dots, x_k$. The termmology in (1.1) is perfectly descriptive-an element $f \in \mathcal{P}_n(\Delta)$ consists of k+1 polynomial pieces [9].

Let Δ be a partition of the interval $[a, b]$ as in (1.1) and let n be a positive integer.

Let $\mathcal{S}_n(\Delta) = \mathcal{P}_n(\Delta) \cap \mathrm{C}^{n-2}[a,b]$. We call $\mathcal{S}_n(\Delta)$ the space of polynomial splines of order n with simple knots at the points $x_1,\ x_2, \dots,\ x_k$.

Let $L_\infty(X)$, $(1 \le p < \infty)$ be the space of all bounded measurable functions with usual norm [8].
$$\|f\|_{L_\infty} = \|f\|_\infty = \sup\{|f(x)|\,,\ \ x \in X\} \le \infty, \tag{1.2}$$
$L_p(X)$ be the space of all of all bounded measurable function f on X, for which [3]
$$\|f\|_{L_p} = \|f\|_p = \left\{\left(\int_X |f(x)|^p\, dx\right)^{\frac{1}{p}} < \infty\right\}, \tag{1.3}$$
the locally global norm for $\delta > 0$ and $(1 \le p < \infty)$ of f is defined by
$$\|f\|_{\delta,p} = \left(\int_X (\sup\{\,|f(y)|^p : y \in \left[x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}\right]\} dy\right)^{\frac{1}{p}} \tag{1.4}$$

Let $L_{p,\mu}(X)$, $(1 \le p < \infty)$ be the space of all bounded μ-measurable functions f on X, for which [1]
$$\|f\|_{L_{p,\mu}} = \|f\|_{p,\mu} = \left\{\left(\int_X |f(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} < \infty\right\}, \tag{1.5}$$
where μ is the non-negative measureable function on set X.

For $\delta > 0$, the modulus of continuity of the function f on X [10] is defined by
$$\omega(f,\delta) = sup\{|f(x_1) - f(x_2)|: |x_1 - x_2| < \delta, x_1, x_2 \in X\} \tag{1.6}$$

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The moduli of smoothness form a natural generalization of the modulus of continuity.

For every function *f* we define the k^{th} difference with step *h* at a point *x* as follows:

$$\Delta_h^k f(x) = \sum_{i=0}^{k}(-1)^{k+i}\binom{k}{i} f(x+i\mathrm{h}), x, x+ih \in X \tag{1.7}$$

For $\delta > 0$, the modulus of smoothness of order *k* of function it following function [10]

$$\omega_k(f,\delta) = \underbrace{sup}_{|h|<\delta}\left\{\left|\Delta_h^k f(x)\right|,\ \ x, x+kh \in X\right\} \tag{1.8}$$

The $k^{\ th}$ ordinary modulus of continuity for $f \in L_p(X)$ and $f \in L_{p,\mu}(X)$ respectively by

$$\omega_k(f,\delta)_p = \underbrace{sup}_{|h|<\delta}\left\{\left\|\Delta_h^k f(.)\right\|_p\right\}, \delta > 0 \tag{1.9}$$

$$\omega_k(f,\delta)_{p,\mu} = \underbrace{sup}_{|h|<\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\mu}\right\}, \delta > 0 \tag{1.10}$$

The local modulus of smoothness of function *f* of order *k* at a point $x \in X$ is following function of $\delta > 0$ [10]

$$\omega_k(f,x,\delta) = sup\left\{\left|\Delta_h^k f(t)\right|,\ \ t, t+kh \in [x-\tfrac{\delta}{2}, x+\tfrac{\delta}{2}] \cap X\right\} \tag{1.11}$$

The k^{th} averaged modulus of smoothness of function f of order *k* (or τ-modulus) of the function $f \in L_p\,(X)$ is following function of $\delta > 0$ is given by [11]:

$$\tau_k(f,\delta)_p = \|\omega_k(f,.,\delta)\|_p = \left(\int_X |\omega_k(f,x,\delta)|^p\ dx\right)^{\frac{1}{p}} \tag{1.12}$$

Further the k^{th} averaged modulus of smoothness for $f \in L_{p,\mu}(X)$ is given by

$$\tau_k(f,\delta)_{p,\mu} = \|\omega_k(f,.,\delta)\|_{p,\mu} \tag{1.13}$$

The K-functional for $f \in X_0$ and $g \in X_1$ is given by

$$K(f,\delta) = K(f,\delta,X_0,X_1) = \underbrace{inf}_{g\in X_1}\left\{\|f-g\|_{X_0} + \delta\|g\|_{X_1}, \delta > 0\right\} \tag{1.14}$$

Where X_0 and X_1 be two Banach spaces with $X_1 \subset X_0$. [6]

The inequality $K(f,\delta) < \epsilon$ for some $\delta > 0$, ϵ is a positive real number, implies that *f* has approximated with error $\|f-g\| < \epsilon$ in X_0 by an element $g \in X_1$, whose norm is not too large ($\|g\|_{X_1} < \epsilon\,\delta^{-1}$).

The *K*-functional in $L_p(X)$ space is given by [6]

$$K_r(f,\delta^r)_p = \underbrace{inf}_{g\in W_p^r}\left\{\|f-g\|_p + \delta^r\left\|g^{(r)}\right\|_p, \delta > 0\right\} \tag{1.15}$$

Where $X_0 = L_p(X)$ and $X_1 = W_p^r$ and $X_1 \subset X_0$,

Now, we introduce K-functional of a function $f \in L_{p,\mu}(X)$ such that [2]

$$K_r(f,\delta^r)_{p,\mu} = \underbrace{inf}_{g\in W_p^r}\left\{\|f-g\|_{p,\mu} + \delta^r\left\|g^{(r)}\right\|_{p,\mu}, \delta > 0\right\} \tag{1.16}$$

The Ditzian-Totic modulus of smoothness for $f \in L_p(X)$ as [4]

$$\omega_k^{\varphi}\ (f.\delta)_p = \underbrace{sup}_{|h|<\delta}\left\|\Delta_{h\varphi}^k f(.)\right\|_p \tag{1.17}$$

Where $$\Delta_{h\varphi}^k f(x) = \begin{Bmatrix}\sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i} f(x+i\varphi h), x+\varphi h \in X \\ 0 \qquad\qquad otherwise\end{Bmatrix}$$

Also, the locally µ- Ditzian-Totic modulus of smoothness for $f \in L_{p,\mu}(X)$ is defined by

$$\omega_k^{\varphi}\ (f.\delta)_{p,\mu} = \underbrace{sup}_{|h|<\delta}\left\|\Delta_{h\varphi}^k f(.)\right\|_{p,\mu}, \text{ where } \ \varphi(x) = (1-x^2)^{\frac{1}{2}} \tag{1.18}$$

The degree of best approximation to a given continuous function with respect to a polynomial spline on interval *X* is given by [5]:

$$E_n(f)_\infty = inf\{\|f-s\|_\infty;\ s \in \mathcal{S}_n(\Delta)\}. \tag{1.19}$$

While the degree of best approximation of a function $f \in L_p(X)$ with respect to a polynomial spline of degree ≤ n on X is given by

$$E_n(f)_p = inf\{\|f-s\|_p;\ s \in \mathcal{S}_n(\Delta)\}. \tag{1.20}$$

Also, the degree of μ- best approximation to a given function $f \in L_{p,\mu}(X)$ with respect to polynomial spline of degree $\leq n$ on X is defined by

$$E_n(f)_{p,\mu} = inf\{\|f-s\|_{p,\mu};\ s \in \mathcal{S}_n(\Delta)\}. \tag{1.21}$$

The degree of best one-sided approximation of function $f \in L_p(X)$with respect to polynomial spline of degree $\leq n$ on interval X is given by

$$\tilde{\mathrm{E}}_n(f)_p = inf\left\{\begin{matrix}\|\bar{s}-\bar{\bar{s}}\|_p;\ \bar{s},\bar{\bar{s}} \in \mathcal{S}_n(\Delta)\ and \\ \bar{\bar{s}}(x) \leq f(x) \leq \bar{s}(x)\end{matrix}\right\}. \tag{1.22}$$

The degree of μ-best one-sided approximation of function $f \in L_{p,\mu}(X)$ with respect to polynomial spline of degree $\leq n$ on interval X is given by

$$\tilde{\mathrm{E}}_n(f)_{p,\mu} = inf\left\{\begin{matrix}\|\bar{s}-\bar{\bar{s}}\|_{p,\mu};\ s,\bar{\bar{s}} \in \mathcal{S}_n(\Delta)\ and \\ \bar{\bar{s}}(x) \leq f(x) \leq \bar{s}(x)\end{matrix}\right\}. \tag{1.23}$$

2. AUXILIARY LEMMAS

Lemma I [7]: For $f \in L_p(X)$, $(0 < p \leq \infty)$, we have

$$\omega_k(f,\delta)_p \leq c(p)\omega_k^{\varphi}(f,\delta)_p \tag{2.1}$$

where *c* is constant depending on *p*.

Lemma II [7]: For $f \in L_p(X)$, $(0 < p \leq \infty)$, we have

$$\omega_k(f,\delta)_p \leq \omega_k(f,\delta)_\infty, \qquad \delta > 0. \tag{2.2}$$

Lemma III [9]: If f is a bounded measurable function on the interval *[a,b]*, *a,b*$\in\mathbb{R}$, then

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^n f(x_i) \tag{2.3}$$

$where \qquad x_i = a + \frac{(b-a)(2i-1)}{2n}$.

Lemma IV [1]: Let f be a bounded μ –measurable function and $(1 \leq p < \infty)$, then we have

$$\|f\|_p \leq c(p)\|f\|_{p,\mu} \tag{2.4}$$

3. MAIN RESULTS

In this section, we will get an estimation for $\tilde{\mathrm{E}}_n(f)_{p,\mu}$. The estimation will be given in terms of k^{th} local modulus of continuity and Ditzian-Totic modulus of smoothness.

Now, we need the following lemmas:

Lemma 1: Let $f \in L_{p,\mu}(X)$, $(1 \leq p < \infty)$ Then

$$\omega_k(f,\delta)_{\infty,\mu} \leq c(p)\ \omega_k^{\varphi}(f,\delta)_{p,\mu}\ . \tag{3.1}$$

Proof: From **(2.3)** and **(2.1)**

$$\begin{aligned}\omega_k(f,\delta)_{\infty,\mu}^p &= sup_{|h|<\delta}\left\|\Delta_h^k f(.)\right\|_{\infty,\mu}^p = sup_{|h|<\delta}\left\|\ \Delta_h^k f(.)d\mu(.)\right\|_\infty^p \\ &= sup_{|h|<\delta}\left\{\sup\left|\Delta_h^k f(x)d\mu(x)\right|^p, x \in X\right\} \\ &\leq \frac{1}{n}\sum_{i=1}^n\left|\ \Delta_h^k f(x_i)d\mu(x_i)\right|^p \\ &\cong \int_X \left|\Delta_h^k f(x_i)\right|^p d\mu(x_i)\end{aligned}$$

Implies that

$$\begin{aligned}\omega_k(f,\delta)_{\infty,\mu} &\leq \left(\int_X\left|\ \Delta_h^k f(x_i)\right|^p d\mu(x_i)\right)^{\frac{1}{p}} \\ &= \left\|\ \Delta_h^k f(.)d\mu(.)\right\|_p \leq sup\left\|\Delta_h^k f(.)d\mu(.)\right\|_p = \omega_k(\,fd\mu,\delta)_p \\ &\leq c(p)\ \omega_k^{\varphi}(fd\mu\,,\delta)_p = c(p)\ \ \omega_k^{\varphi}(f,\delta)_{p,\mu}\,.\end{aligned}$$

Theorem 1: Let $f \in L_{p,\mu}(X)$, $(1 \le p < \infty)$. Then

$$E_n(f)_{p,\mu} \le c(p)\omega_k(f,\delta)_{p,\mu}. \tag{3.2}$$

Proof: Consider $s \in \mathcal{S}_n(\Delta)$ is a best approximation of a function f.

From **(1.21)**, we have

$$\begin{aligned} E_n(f)_{p,\mu} &= \|f-s\|_{p,\mu} \\ &= \left(\int_X |(f-s)(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &\le \left(\int_X sup|(f-s)(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &\le c(p)sup\left(\int_X |f(x)-s(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &= c(p)\, sup\left(\int_X \left|\Delta_h^k f(x)\right|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &= c(p)sup\left\|\Delta_h^k f(.)\right\|_{p,\mu} = c(p)\omega_k(f,\delta)_{p,\mu}. \end{aligned}$$

Now, we want to find a relation between μ-best approximation and μ-best one-sided approximation.

Theorem 2: Let $f \in L_{p,\mu}(X)$, $(1 \le p < \infty)$ and $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$. Then

$$E_n(f)_{p,\mu} \le c(p)\tilde{E}_n(f)_{p,\mu} \le c(p)\, E_n(f)_{p,\mu} \tag{3.3}$$

where c constant depending on p.

Proof: Consider $s \in \mathcal{S}_n(\Delta)$ is the best approximation of $f \in L_{p,\mu}(X)$ and $\bar{s}, \bar{\bar{s}} \in \mathcal{S}_n(\Delta)$ are the best one-sided approximation of $f \in L_{p,\mu}(X)$ such that $\bar{\bar{s}}(x) \le f(x) \le \bar{s}(x)$; $x \in X$

We want to prove

$$E_n(f)_{p,\mu} \le c(p)\tilde{E}_n(f)_{p,\mu}$$

$$\begin{aligned} E_n(f)_{p,\mu} &= \|f-s\|_{p,\mu} \\ &= \left(\int_X |(f-s)(x)|^p d\mu(x)\right)^{\frac{1}{p}} \\ &= \left(\int_X |f(x)-s(x)|^p d\mu(x)\right)^{\frac{1}{p}} \\ &= \left(\int_X |f(x)-\bar{s}(x)+\bar{s}(x)-\bar{\bar{s}}(x)+\bar{\bar{s}}(x)-s(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &\le \left(\int_X |f(x)-\bar{s}(x)|^p d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X |\bar{s}(x)-\bar{\bar{s}}(x)|^p d\mu(x)\right)^{\frac{1}{p}} + \left(\int_X |\bar{\bar{s}}(x)-s(x)|^p d\mu(x)\right)^{\frac{1}{p}} \\ &= \|f-\bar{s}\|_{p,\mu} + \|\bar{s}-\bar{\bar{s}}\|_{p,\mu} + \|\bar{\bar{s}}-s\|_{p,\mu} \\ &\le \|f-s\|_{p,\mu} + \mathrm{c(p)}\,\tilde{E}_n(f)_{p,\mu} \\ &= E_n(f)_{p,\mu} + \mathrm{c(p)}\,\tilde{E}_n(f)_{p,\mu} \\ &\le \tilde{E}_n(f)_{p,\mu} + \mathrm{c\,(p)}\,\tilde{E}_n(f)_{p,\mu} \\ &= \mathrm{c\,(p)}\,\tilde{E}_n(f)_{p,\mu} \end{aligned}$$

Hence $\quad E_n(f)_{p,\mu} \le c(p)\tilde{E}_n(f)_{p,\mu}$

Now to prove $\quad \tilde{E}_n(f)_{p,\mu} \le c(p)\, E_n(f)_{p,\mu}$

$$\begin{aligned} \tilde{E}_n(f)_{p,\mu} &= \|\bar{s}-\bar{\bar{s}}\|_{p,\mu} = \left(\int_X |(\bar{s}-\bar{\bar{s}})(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &= \left(\int_X |\bar{s}(x)-\bar{\bar{s}}(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &\le \left(\int_X \left|s(x)+\left(f(x)-s(x)\right)\right) - \left(s(x)-\left(f(x)-s(x)\right)\right)\right|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &= \left(\int_X |(f(x)-2s(x)+f(x))|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &= 2\left(\int_X |f(x)-s(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} \\ &\le c(p)\|f-s\|_{p,\mu} = c(p)\, E_n(f)_{p,\mu}. \end{aligned}$$

Hence $\quad \tilde{E}_n(f)_{p,\mu} \le c(p)\, E_n(f)_{p,\mu}$

Then we get

$$E_n(f)_{p,\mu} \le c(p)\tilde{E}_n(f)_{p,\mu} \le c(p)\, E_n(f)_{p,\mu}$$

Theorem 3: If $f \in L_{p,\mu}(X)$, $(1 \le p < \infty)$, then

$$\tilde{E}_n(f)_{p,\mu} \le \tau_k(f,\Delta_n)_{p,\mu}.$$

Proof: Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$, $\Delta_n = max|x_i - x_{i-1}|$, $i = 0,1,\dots,n$

Set $\quad \bar{s}_k(x) = supf(t),\ x \in [x_{i-1}, x_i),\ t \in [x_{i-1}, x_i]$

$\quad \bar{\bar{s}}_k(x) = inf\ f(t),\ x \in [x_{i-1}, x_i),\ t \in [x_{i-1}, x_i]$

and $\quad \bar{S}(f,x,\delta) = \sup f(t) \quad where\ |t-x| \le \delta/2$

$\quad \bar{\bar{S}}(f,x,\delta) = \inf f(t) \quad where\ |t-x| \le \delta/2$

then, $\quad \bar{\bar{S}}(f,x,\delta) \le \bar{\bar{s}}(x) \le \bar{s}(x) \le \bar{S}(f,x,\delta)$

we have

$$\begin{aligned}\tilde{E}_n(f)_{p,\mu} = \|\bar{s}_k - \bar{\bar{s}}_k\|_{p,\mu} &= \left(\int_X |(\bar{s}_k - \bar{\bar{s}}_k)(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} \\ &= \left(\int_X |(\bar{s}_k - \bar{\bar{s}}_k)(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} \\ &\le \left(\int_X \left|\bar{S}(f,x,\delta) - \bar{\bar{S}}(f,x,\delta)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\ &\le sup\left(\int_X \left|\bar{S}(f,x,\delta) - \bar{\bar{S}}(f,x,\delta)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\ &= \tau_k(f,\Delta_n)_{p,\mu}.\end{aligned}$$

Theorem 4: Let $f \in L_{p,\mu}(X)$, $(1 \le p < \infty)$. Then

$$\tilde{E}_n(f)_{p,\mu} \le c(p)\omega_k^{\varphi}(f.\delta)_{p,\mu}.$$

Proof: By using (2.1), (2.2), (3.1), (2.4) and (3.2) we get

$$\begin{aligned}\tilde{E}_{n,k}(f)_{p,\mu} &= inf\|\bar{s} - \bar{\bar{s}}\|_{p,\mu} \\ &\le inf\|\bar{s} - \bar{\bar{s}}\|_p \\ &= \tilde{E}_n(f)_p \\ &\le \omega_k^{\varphi}(f.\delta)_p \\ &= sup\left\|\Delta_{h\varphi}^k f(.)\right\|_p \\ &\le c(p)sup\left\|\Delta_{h\varphi}^k f(.)\right\|_{p,\mu} \\ &= c(p)\omega_k^{\varphi}(f.\delta)_{p,\mu}.\end{aligned}$$

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