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ON THE TOPOLOGICAL INDICES OF THORNY-STAR GRAPHS

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ABSTRACT

Let G be the connected graph. The Wiener index $W(G)$ is the sum of all distances between vertices of G, whereas the hyper-Wiener index $WW(G)$ is defined as $WW(G) = W(G) + \frac{1}{2}\sum_{\{u,v\}\subseteq V\{G\}} d(u,v)^2$. In this paper we prove some general results on the topological indices of thorny-star graphs and some bounds on it.

Keywords: *Molecular graphs, Wiener index and hyper-Wiener index.*

2000 Mathematics subject classification: *05C12, 92E10.*

1. INTRODUCTION

In mathematical terms a graph is represented as $G = (V, E)$ where V is the set of vertices and E is the set of edges. Let G be an undirected connected graph without loops or multiple edges with n vertices, denoted by $1,2,\dots,n$. The topological distance between the vertices u and v of $V(G)$ is denoted by $d(u,v)$ or d_{uv} and it is defined as the number of edges in a minimal path connecting the vertices u and v.

The Wiener index $W(G)$ of a connected graph G is defined as the sum the distances between all unordered pairs of vertices of G. It was put forward by Harold Wiener. The Wiener index is a graph invariant intensively studied both in mathematics and chemical literature, see for details$[1, 6, 7, 8, 10\ and\ 12 - 14]$.

The hyper-Wiener index was proposed by Randic [11] for a tree and extended by Klein *et al.*. [2] to a connected graph. It is used to predict physicochemical properties of organic compounds. The hyper-Wiener index defined as,

$$WW(G) = \sum_{\{u,v\}\subseteq V(G)} \binom{d_{uv}+1}{2} = W(G) + \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d(u,v)^2$$

The hyper-Wiener index is studied both from a theoretical point of view and applications. We encourage the reader to consult $[4, 5, 10\ and\ 12 - 15]$ for further readings. The hyper-Wiener index of complete graph-K_p, path graph-P_n, star graph-$K_{1,(n-1)}$ and cycle graph C_n is given by the expressions

$$WW(K_n) = \frac{n(n-1)}{2}, WW(P_n) = \frac{n^4+2n^3-n^2-2n}{24},\ \ WW\left(K_{1,(n-1)}\right) = \frac{1}{2}(n-1)(3n-4)$$

And

$$WW(C_n) = \begin{cases} \frac{n^2(n+1)(n+2)}{48}, & if\ n\ is\ even \\ \frac{n(n^2-1)(n+3)}{48}, & if\ n\ is\ odd \end{cases}$$

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2. MAIN RESULTS

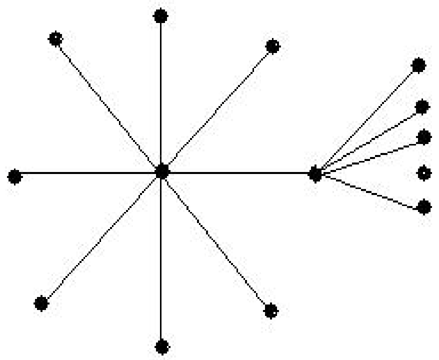
2.1 Wiener and hyper-Wiener indices of Thorny-star graphs in terms of number of vertices in a graph:

Theorem 1: Let H be the star graph on t vertices. The graph G obtained by attaching s-number of pendent vertices to any one pendent vertices of graph H with common vertex then its Wiener and hyper-Wiener indices given by

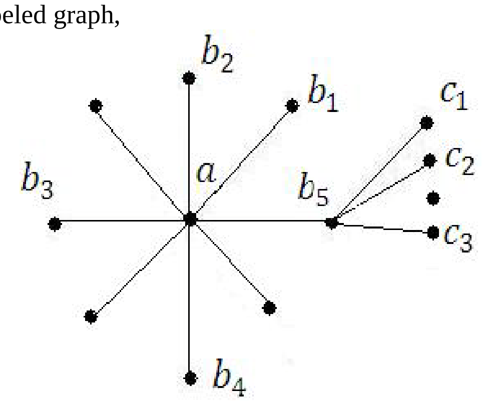
$$W(G) = \frac{1}{2}\{t(2t+3s-3) + S(3t+2s-5) + 2 - 3s - t\}$$

$$WW(G) = \frac{1}{2}\{t(3t+6s-5) + s(6t+3s-11) + 4 - 8s - 2t\}$$

where ‘n’ be the number of vertices in G and ‘s’ be the number of pendent vertices. The Schematic representation of G is shown as in below,



Proof: Let us consider the vertex labeled graph,



To find Wiener index of the graph,

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(u_i, u_j)$$

$$W(G) = \frac{1}{2}\{d(a \mid G) + \sum_{i=1}^{t-1} d(b_i \mid G) + \sum_{j=1}^{s} d(c_j \mid G)\}$$

$$W(G) = \frac{1}{2}\left[\begin{array}{l}\{\sum_{i=1}^{t-1} d(a \mid b_i) + \sum_{j=1}^{s} d(a \mid c_j)\} + (t-2)\{1 + \sum_{i=2}^{t-2} d(b_1 \mid b_i) + \sum_{j=1; b_5 \notin G}^{s} d(b_1 \mid c_j)\} \\ + \{\sum_{i=1}^{s} d(b_5 \mid c_i) + 1 + \sum_{j=1}^{t-2} d(b_5 \mid b_j)\} + s\{1 + \sum_{i=1}^{s} d(c_i \mid c_1) + \sum_{j=1; b_5 \notin G}^{t-2} d(c_1 \mid b_j)\}\end{array}\right]$$

$$W(G) = \frac{1}{2}\left[\begin{array}{c}\{(t-1)\times 1 + s \times 2\} + (t-2)\{1 + (t-2)\times 2 + s \times 3\} + \{s \times 1 + 1 + (t-2)\times 2\} \\ + s\{1 + s \times 2 + (t-2)\times 3\}\end{array}\right]$$

Where $d(a \mid b_1) = 1$; $d(a \mid c_1) = 2$; $d(b_1 \mid b_2) = 2$; $d(b_1 \mid c_1) = 3$;
$d(b_5 \mid c_1)$=1; $d(b_5 \mid b_1) = 2$; $d(c_1 \mid c_2) = 2$; $d(c_1 \mid b_1) = 3$.

$$W(G) = \frac{1}{2}[t - 1 + 2s + (t-2)(2t+3s-3) + (2t+s-3) + s(3t+2s-5)]$$

$$W(G) = \frac{1}{2}[t - 1 + 2s + t(2t+3s-3) - 4t - 6s + 6 + 2t + s - 3 + s(3t+2s-5)]$$

$$W(G) = \frac{1}{2}\{t(2t+3s-3) + S(3t+2s-5) + 2 - 3s - t\}$$

To find hyper-Wiener index of the graph,

$$WW(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(u_i, u_j)^2$$

$$WW(G) = \frac{1}{2}\{d(\,a \mid G\,) + \sum_{i=1}^{t-1} d(\,b_i \mid G\,)^2 + \sum_{j=1}^{s} d(c_j \mid G)^2\}$$

$$WW(G) = \frac{1}{2}\left[\begin{array}{l} \left\{\sum_{i=1}^{t-1} d(\,a \mid b_i\,)^2 + \sum_{j=1}^{s} d\left(\,a \mid c_j\,\right)^2\right\} + (t-2)\left\{1 + \sum_{i=2}^{t-2} d(\,b_1 \mid b_i\,)^2 + \sum_{j=1;b_5 \notin G}^{s} d\left(\,b_1 \mid c_j\,\right)^2\right\} \\ + \left\{\sum_{i=1}^{s} d(\,b_5 \mid c_i\,)^2 + 1 + \sum_{j=1}^{t-2} d\left(\,b_5 \mid b_j\,\right)^2\right\} + s\left\{1 + \sum_{i=1}^{s} d(\,c_i \mid c_1\,)^2 + \sum_{j=1;b_5 \notin G}^{t-2} d\left(\,c_1 \mid b_j\,\right)^2\right\} \end{array}\right]$$

$$WW(G) = \frac{1}{2}\left[\begin{array}{c} \{(t-1) \times 1 + s \times 3\} + (t-2)\{1 + (t-2) \times 3 + s \times 6\} + \{s \times 1 + 1 + (t-2) \times 3\} \\ +s\{1 + s \times 3 + (t-2) \times 6\} \end{array}\right]$$

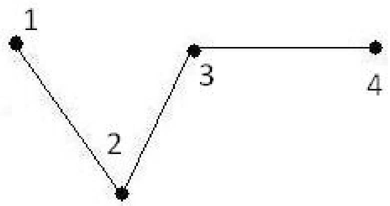
where $d(\,a \mid b_1\,)^2 = 1;\;\; d(\,a \mid c_1\,)^2 = 3;\;\; d(\,b_1 \mid b_2\,)^2 = 3;\;\; d(\,b_1 \mid c_1\,)^2 = 6;$
$d(\,b_5 \mid c_1\,)^2 {=} 1;\; d(\,b_5 \mid b_1\,)^2 = 3;\;\; d(\,c_1 \mid c_2\,)^2 = 3;\;\; d(\,c_1 \mid b_1\,)^2 = 6.$

$$WW(G) = \frac{1}{2}\{(t - 1 + 3s) + (t-2)(1 + 3t - 6 + 6s) + (s + 1 + 3t - 6) + s(1 + 3s + 6t - 12)\}$$

$$WW(G) = \frac{1}{2}\{t + 3s - 1 + (t-2)(3t + 6s - 5) + s(6t + 3s - 11) + 3t + s - 5\}$$

$$WW(G) = \frac{1}{2}\{t(3t + 6s - 5) + s(6t + 3s - 11) + 4 - 8s - 2t\}$$

Example: The molecular graph representing the chemical compound in figure below is 1, 2-Butadiene isomorphic to $S_{3,1}$. Where $S_{3,1}$ is the carbon skeleton of 1, 2-Butadiene.



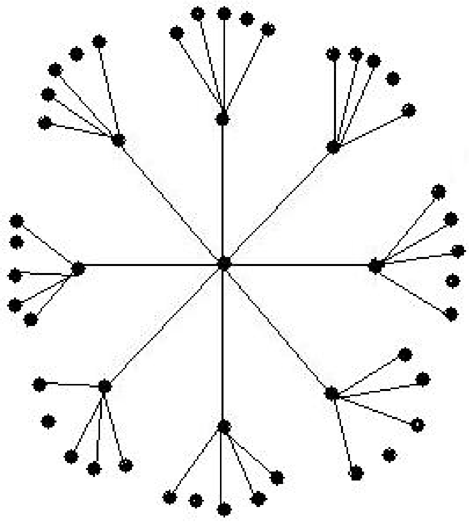
$t = 3, s = 1, W(G) = 10$, and $WW(G) = 15$.

Theorem 2: (Dendrimers) Let H be the star graph on t vertices. The graph G obtained by attaching s-number of pendent vertices to each pendent vertex of H with common vertex then its Wiener and hyper-Wiener index given by

$$W(G) = \frac{1}{2}\{t - 1 + 2p + (t-1)(2t + 3p - 7s - 3 + 3st + 4sp - 2s^2)\}$$

$$WW(G) = \frac{1}{2}\{3p + t - 1 + (t-1)(3t + 6p - 16s - 5 + 6st + 10sp - 7s^2\}$$

Proof: The Schematic representation of G is shown as in below,



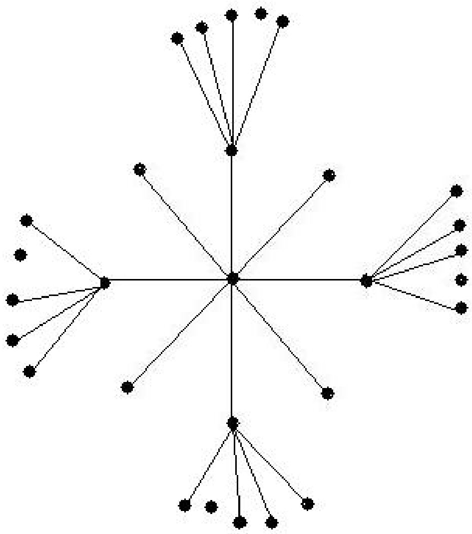
The Proof is similar to above theorem.

Theorem 3: (Dendrimers) Let H be the star graph on t vertices ($t-1$ is even). The graph G obtained by attaching s-number of pendent vertices to alternative pendent vertex of graph H with common vertex then its Wiener and hyper-Wiener index given by

$$W(G) = \frac{1}{2}\{\frac{t-1}{2}(4t+6p-3s-5)+4p^2+3pt-2ps-3p+t-1\}$$

$$WW(G) = \frac{1}{2}\{\frac{t-1}{2}(6t+12p-6s-9)+10p^2+6pt-7ps-8p+t-1\}$$

Proof: The Schematic representation of G is shown as in below,



The Proof is similar to above theorem 1.

2.2 Bounds for Wiener and hyper-Wiener indices of acyclic molecular graphs:

Lemma 4: For acyclic molecular graphs following inequalities holds good,

1. (a). $W(T) \geq \frac{1}{2}\{t(2t+3s-3)+S(3t+2s-5)+2-3s-t\}$
   (b). $WW(T) \geq \frac{1}{2}\{t(3t+6s-5)+s(6t+3s-11)+4-8s-2t\}$
2. (a). $W(G) \geq \frac{1}{2}\{t-1+2p+(t-1)(2t+3p-7s-3+3st+4sp-2s^2)\}$
   (b). $WW(G) \geq \frac{1}{2}\{3p+t-1+(t-1)(3t+6p-16s-5+6st+10sp-7s^2\}$
3. (a). $W(G) \geq \frac{1}{2}\{\frac{t-1}{2}(4t+6p-3s-5)+4p^2+3pt-2ps-3p+t-1\}$
   (b). $WW(G) \geq \frac{1}{2}\{\frac{t-1}{2}(6t+12p-6s-9)+10p^2+6pt-7ps-8p+t-1\}$

Proof: To prove, assertion 1. a. We know that among all acyclic graphs, the star $K_{1,(t-1)}$ has a minimum Wiener index and path graph P_n has a maximum Wiener index. So, we can get extreme class of graphs in acyclic molecular trees. Therefore symbolically represented as follows,

$$W(K_{1,(t-1)}) \leq W(G) \leq W(P_n)$$

In the assertion a. Graph under consideration is a star, so above inequality holds good.

$$W(T) \geq \frac{1}{2}\{t(2t+3s-3)+S(3t+2s-5)+2-3s-t\}$$

To prove, assertion 1. b. We know that among all acyclic graphs, the star $K_{1,(t-1)}$ has a minimum hyper-Wiener index and path graph P_n has a maximum hyper-Wiener index. So, we can get extreme class of graphs in acyclic molecular trees. Therefore symbolically represented as follows,

$$WW(K_{1,(t-1)}) \leq WW(G) \leq WW(P_n)$$

In the assertion b. Graph under consideration is a star, so above inequality holds good.

$$WW(T) \geq \frac{1}{2}\{t(3t+6s-5)+s(6t+3s-11)+4-8s-2t\}$$

Proofs for results **2** and **3** are similar to **1. a.** and **1. b.**

We end the paper with the following simple but elegant lemma.

Lemma 5: Let T_1, T_2 and T_3 are the thorny-star graphs then following inequalities holds good.

a. $W(T_1) \leq W(T_3) \leq W(T_2)$
b. $WW(T_1) \leq WW(T_3) \leq WW(T_2)$
c. $W(T_i) < WW(T_i)$, Where $i = 1,2,3$.

Where

$$W(T_1) = \frac{1}{2}\{t(2t+3s-3) + S(3t+2s-5) + 2 - 3s - t\}$$

$$WW(T_1) = \frac{1}{2}\{t(3t+6s-5) + s(6t+3s-11) + 4 - 8s - 2t\}$$

$$W(T_2) = \frac{1}{2}\{t - 1 + 2p + (t-1)(2t+3p-7s-3+3st+4sp-2s^2)\}$$

$$WW(T_2) = \frac{1}{2}\{3p + t - 1 + (t-1)(3t+6p-16s-5+6st+10sp-7s^2\}$$

$$W(T_3) = \frac{1}{2}\{\frac{t-1}{2}(4t+6p-3s-5) + 4p^2 + 3pt - 2ps - 3p + t - 1\}$$

$$WW(T_3) = \frac{1}{2}\{\frac{t-1}{2}(6t+12p-6s-9) + 10p^2 + 6pt - 7ps - 8p + t - 1\}$$

The proof is straightforward and omitted.

REFERENCES

1. D. Bonchev and D. H. Rouvary, Chemical Graph Theory, introduction and Fundamentals, 1991.
2. D. J. Klein, I. Lukovits, I. Gutman, On the definition of the hyper-Wiener index for cycle-containing structures, J. Chem. Inf. Comput. Sci. 35 (1995).
3. F. Buckley, F. Harary, Distances in Graphs, Addison-Wesley, Redwood, 1990.
4. G. C. Cash, Polynomial expressions for the hyper-Wiener index of extended hydrocarbon networks, Comput. Chem. 25 (2001) 577-582.
5. G. C. Cash, Relationship between the Hosaya Polynomial and the hyper-Wiener index, Appl. Math. Lett. 15 (2002) 893-895.
6. H. B. Walikar, H. S. Ramane, V. S. Shigehalli, Wiener number of Dendrimers, In: Proc. National Conf. on Mathematical and Computational Models, (Eds. R. Nadarajan and G. Arulmozhi), Applied Publishers, New Delhi, 2003, 361-368.
7. H. B. Walikar, V. S. Shigehalli, H. S. Ramane, Bounds on the Wiener number of a graph, MATCH comm. Math. Comp. Chem., 50 (2004), 117-132.
8. H. Wiener, Structural determination of paraffin boiling points, J. Amer. Chem. Soc., 69 (1947), 17-20.
9. I. Gutman, property of the Wiener number and its modifications, Indian J. Chem. 36A (1997) 128-132.
10. I. Gutman, Relation between hyper-Wiener and Wiener index, Chem. Phys. Lett. 364 (2002) 352-356.
11. Randic, M., Novel molecular description for structure-property studies, Chem. Phys. Lett., 211 (1993), 478-483.
12. Shigehalli V. S. and Shanmukh Kuchabal, “On the invariance of topological indices”, International Journal of Mathematical Archive-5(12), 2014, 122-125.
13. Shigehalli V. S. and Shanmukh Kuchabal, On the hyper-Wiener index of thorny-cocktail party graphs, International Journal of Mathematical Archive-6(3), 2015,1-6.
14. Shigehalli V. S. and Shanmukh Kuchabal, On the hyper-Wiener index of thorny-complete graph”, Journal of Global Research in Mathematical Archives, Volume 2, No. 6, June 2014 Page No. 55-61, 2014.
15. Shigehalli V. S. and Shanmukh Kuchabal, On the hyper-Wiener index of thorny-wheel graphs”, Bulletin of Mathematics and Statistics Research, Vol. 3. Issue. 1. Page No. 25-33 2015.

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