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RESULTS ON CONVERGENCE AND STABILITY OF THE MODIFIED JUNGCK-MULTISTEP ITERATION SCHEME

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ABSTRACT

The aim of this paper is to examine some convergence as well as some stability results for a pair of nonself mappings using a newly introduced Jungck–multistep iteration and given contractive condition. The results are generalization, improvements and extensions of the works of Olaleru and Akewe [16] Olatinwo [20, 21], Bosede[4],Singh et al. [28], Rhoades[24] as well as of some other analogous ones in the literature.

1. INTRODUCTION AND PRELIMINARIES

Many authors have worked on multistep iteration schemes to approximate fixed points for a pair of quasicontractive maps in Banach spaces. First of all, Jungck introduced an iteration for a pair of contractive maps[8]. One of the most general contractive like operators which have been studied by several authors is the Zamfirescu operators[31] .Rhoades [24] used Zamfirescu operators to obtain some convergence results for Mann and Ishikawa iteration processes in a uniformly convex Banach space. Osilike [22] generalized and extended some of the results of Rhoades [24] by using a more general contractive definition than those of Rhoades: there exist $a \in [0, 1)$, $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \;\; \forall \; x, y \in X \tag{1.1}$$

In 2003, Imoru and Olatinwo [18] proved the stability of the Picard and the Mann iteration processes for the following operator which is more general than the one introduced by Osilike [22].The operator satisfies the following contractive definition: there exist $a \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$, such that

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + ad(x, y) \;\; \forall \; x, y \in X \tag{1.2}$$

Olatinwo *et al.* [19] also considered the stability of the Ishikawa and Kirk iteration process when the operator satisfies (1.2).

Let $(X, \|\cdot\|)$ be a normed linear space and $S, T: Y \to X$ are nonself operators with $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of X such that for each pair of points x, y in X at least one of the following is true:

(i) $d(Tx, Ty) \leq ad(Sx, Sy)$
(ii) $d(Tx, Ty) \leq b\,[d(Sx, Tx) + d(Sy, Ty)]$
(iii) $d(Tx, Ty) \leq c\,[d(Sx,Ty) + d(Sy, Tx)]$, (1.3)

Maps satisfying (1.3) are called generalized Zamfirescu operators.

Olatinwo and Imoru [17] proved some convergence results for the Jungck–Mann and Jungck–Ishikawa iteration process in the class of generalized Zamfirescu operator.

Singh *et al.* [28] established some stability results for Jungck and Jungck-Mann iteration processes by employing two contractive definitions:

$$\|Tx - Ty\| \leq \phi(\|Sx - Tx\|) + L\,\|Sx-Sy\|,\; L \geq 0,$$

$$\|Tx - Ty\| \leq \phi(\|Sx - Tx\|) + \delta\|Sx-Sy\|,\; \delta\in[0, 1), \tag{1.4}$$

both of which generalize those of Osilike [19] .

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Noor [12] introduced a three step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. It has been shown in [6] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Thereafter, Suantai [29] defined the new three-step iterations which are extensions of Noor iterations and gave some weak and strong convergence theorems of the modified Noor iterations for asymptotically nonexpansive mappings in Banach space.

Results obtained in this paper can be considered as a refinement and improvement of the previously known results:

$$x_{n+1} = (1-\alpha_n-\beta_n-\gamma_n)\,x_n + \alpha_n\,Ty_n + \beta_n\,Tz_n + \gamma_n\,Tx_n,$$
$$y_n = (1-b_n-c_n)x_n + b_n Tz_n + c_n Tx_n,$$
$$z_n = (1-a_n)\,x_n + a_n Tx_n,\ n = 0, 1, 2,\ldots, \tag{1.5}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{b_n + c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\alpha_n+\beta_n+\gamma_n\}$ are sequences in [0, 1] satisfying certain conditions.

The aim of this paper is to introduce and employ the newly iterative scheme, i.e., modified Jungck-multistep iteration process defined iteratively by the sequence $\{Sx_n\}_{n=0}^{\infty}$ as follows:

$$x_0\in Y,\quad Sx_{n+1} = (1-\alpha_n-\beta_n-\gamma_n)Sx_n + \alpha_n\,Ty^1_n + \beta_n\,Ty^{k-1}_n + \gamma_n\,Tx_n,$$
$$S\,y^i_n = (1-b^i_n-c^i_n)Sx_n + b^i_n Ty^{i+1}_n + c^i_n Tx_n,\ i = 1, 2,\ldots\ k-2, \tag{1.6}$$
$$S\,y^{k-1}_n = (1-b^{k-1}_n)Sx_n + b^{k-1}_n Tx_n,\ n = 0, 1, 2,\ldots,\ k \ge 2,$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{b^i_n\}_{n=1}^{\infty}$, $\{c^i_n\}_{n=1}^{\infty}$, i = 1, 2,…..k–1 are real sequences in [0,1) such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

Putting $\gamma_n = \beta_n = 0$ in (1.6), we obtain the Jungck-multistep iteration defined by J.O. Olaleru and H. Akewe [16];

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n\,Ty^1_n,$$
$$S\,y^i_n = (1-b^i_n)Sx_n + b^i_n Ty^{i+1}_n,\ i = 1, 2,\ldots\ k-2, \tag{1.7}$$
$$S\,y^{k-1}_n = (1-b^{k-1}_n)Sx_n + b^{k-1}_n Tx_n,\ n = 0, 1, 2,\ldots,\ k \ge 2,$$

where $\{\alpha_n\}_{n=0}^{\infty}$, , $\{b^i_n\}_{n=1}^{\infty}$, i = 1, 2,…..k–1 are real sequences in [0,1) such that $\sum_{n=0}^{\infty}\alpha_n = \infty$.

If $\gamma_n = \beta_n = 0$, k=3 in (1.6), we have the Jungck-Noor iteration [21]:

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n\,Ty_n,$$
$$S\,y_n = (1-b_n)Sx_n + b_n Tz_n \tag{1.8}$$
$$S\,z_n = (1-b^1_n)Sx_n + b^1_n Tx_n,\ n = 0, 1, 2,\ldots$$

If $\gamma_n = \beta_n = 0$, k=2 in (1.6), we have the Jungck-Ishikawa iteration [17]

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n\,Ty_n,$$
$$S\,y_n = (1-b_n)Sx_n + b_n Tx_n,\quad n = 0, 1, 2,\ldots \tag{1.9}$$

If $\gamma_n = \beta_n = 0$, k=1 in (1.6), we have the Jungck-Mann iteration[17]:

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n\,Tx_n,\quad n = 0, 1, 2,\ldots \tag{1.10}$$

In addition to multistep iteration process (1.6), the following contractive definition is used:

$$\|Tx - Ty\| \le e^{L\|Sx-Tx\|}\{\phi(\|Sx - Tx\|) + \delta\|Sx-Sy\|\}, \tag{1.11}$$

where ϕ is monotonic increasing function with $\phi(0) = 0$, $\delta\in[0, 1)$ and $L \ge 0$.

Remark 1.1: Contractive condition (1.11) Is more general than (1.4), as by putting L= 0 in (1.11), we get (1.4)

Definition 1.1: [9] A point $x\in X$ is called a coincidence point of a pair of self maps S, T if there exist a point w (called point of coincidence) in X such that w = Tx = Sx. Let C(S, T) = ($x\in X$ such that Sx = Tx). Self maps S and T are said to be occasionally weakly compatible if they commute at some of their coincidence points, that is, STx = TSx for some $x\in$ C(S, T)

Remark 1.2: Weakly compatible mappings are Occasionally weakly compatible but converse is not true.

Definition 1.2: [28] Let S, T: $Y \rightarrow X$ be nonself operators for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and z a coincidence point of S and T such that Tz = Sz = p. Let $\{Sx_n\}_{n=0}^{\infty} \subset X$, be the sequence generated by an iterative procedure

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1 \ldots\ldots\ldots \tag{1.12}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence in X and set $\varepsilon_n = d(y_{n+1}, f(T, y_n))$, n = 0,1,…Then, the iterative procedure (1.12) is said to be (S, T)-stable or stable if and only if $\lim_{n\to\infty} \in_n = 0$ implies $\lim_{n\to\infty} Sy_n = p$.

Lemma 1.1 [3]: If δ is a real number such that 0 ≤ δ < 1, and $\{\in_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \in_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \le \delta u_n + \in_n$, n = 0, 1, 2…..,
We have $\lim_{n\to\infty} u_n = 0$

2. MAIN RESULTS

Theorem 2.1: Let (X, ||·||) be an arbitrary Banach space, S, T: $Y \rightarrow X$ are nonself operator for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and that (1.11) holds. Suppose z is a coincidence point of S and T such that Tz = Sz = p. Then the multistep iteration $\{Sx_n\}$ given by (1.6) converges strongly to p. In addition, if Y=X and S, T are occasionally weakly compatible mappings, then p is unique common fixed point of S and T.

Proof: Since Tz = Sz = p, using (1) and (2), we have

$$\begin{aligned}
\|Sx_{n+1} - p\| &\le (1-\alpha_n-\beta_n-\gamma_n)\|Sx_n-p\| + \alpha_n\|Ty_n^1 - p\| + \beta_n\|Ty_n^{k-1} - p\| + \gamma_n\|Tx_n-p\| \\
&\le (1-\alpha_n-\beta_n-\gamma_n)\|Sx_n-p\| + \alpha_n e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sy_n^1 - Sz\|\} \\
&\quad + \beta_n e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sy_n^{k-1} - Sz\|\} + \gamma_n e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sx_n-Sz\|\} \\
&= (1-\alpha_n-\beta_n-\gamma_n)\|Sx_n-p\| + \alpha_n\delta\|Sy_n^1 - Sz\| + \beta_n\delta\|Sy_n^{k-1} - Sz\| + \gamma\delta\|Sx_n-Sz\| \\
&= (1-\alpha_n-\beta_n-\gamma_n+\gamma_n\delta)\|Sx_n-p\| + \alpha_n\delta\|Sy_n^1 - p\| + \beta_n\delta\|Sy_n^{k-1} - p\|
\end{aligned} \tag{2.1}$$

An application of (1.6) and (1.11) yields

$$\begin{aligned}
\|Sy_n^1 - p\| &\le (1-b_n^1-c_n^1)\|Sx_n-p\| + b_n^1\|Ty_n^2 - p\| + c_n^1\|Tx_n-p\| \\
&\le (1-b_n^1-c_n^1)\|Sx_n-p\| + b_n^1 e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sy_n^2 - Sz\|\} + c_n^1 e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sx_n-Sz\|\} \\
&= (1-b_n^1-c_n^1)\|Sx_n-p\| + \delta b_n^1\|Sy_n^2 - p\| + c_n^1\delta\|Sx_n-p\| \\
&= (1-b_n^1-c_n^1+c_n^1\delta)\|Sx_n-p\| + \delta b_n^1\|Sy_n^2 - p\|
\end{aligned} \tag{2.2}$$

Again using (1.6) and (1.11), we have

$$\begin{aligned}
\|Sy_n^{k-1} - p\| &\le (1-b^{k-1})\|Sx_n-p\| + b^{k-1}\|Tx_n - p\| \\
&\le (1-b^{k-1})\|Sx_n-p\| + b^{k-1} e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sx_n-Sz\|\} \\
&= (1-b^{k-1})\|Sx_n-p\| + \delta b^{k-1}\|Sx_n-p\| \\
&= (1-b^{k-1}+\delta b^{k-1})\|Sx_n-p\|
\end{aligned} \tag{2.3}$$

Substituting (2.2) and (2.3) in (2.1) and rearranging the terms, we have

$$\|Sx_{n+1} - p\| \le \{1-\alpha_n(1-\delta) - \beta_n(1-\delta) - \gamma_n(1-\delta) - \delta\alpha_n c^1(1-\delta) - \delta\beta_n b^{p-1}(1-\delta) - \delta\alpha_n b_n^1\}\|Sx_n-p\| + \delta^2\alpha_n b_n^1\|Sy_n^2 - p\| \tag{2.4}$$

Similar to (2.2), an application of (1.6) and (1.11) gives

$$\|Sy_n^2 - p\| \le (1-b_n^2-c_n^2+c_n^2\delta)\|Sx_n-p\| + \delta b_n^2\|Sy_n^3 - p\| \tag{2.5}$$

Substituting (2.5) in (2.4) and rearranging the terms, we have

$$\begin{aligned}
\|Sx_{n+1} - p\| &\le \{1-\alpha_n(1-\delta) - \beta_n(1-\delta) - \gamma_n(1-\delta) - \delta\alpha_n c_n^1(1-\delta) - \delta\beta_n b^{k-1}(1-\delta) \\
&\quad - \delta\alpha_n b_n^1(1-\delta) - \delta^2\alpha_n b_n^1 c_n^2(1-\delta) - \delta^2\alpha_n b_n^1 b_n^2\}\|Sx_n-p\| + \delta^3\alpha_n b_n^1 b_n^2\|Sy_n^3 - p\|
\end{aligned} \tag{2.6}$$

Similar to (2.5), an application of (1.6) and (1.11) gives

$$\|Sy_n^3 - p\| \le (1 - b_n^3 - c_n^3 + c_n^3\delta)\ \|Sx_n - p\| + \delta\, b_n^3 \|Sy_n^4 - p\| \tag{2.7}$$

Substituting (2.7) in (2.6) and rearranging the terms, we have

$$\|Sx_{n+1} - p\| \le \{1 - \alpha_n(1-\delta) - \beta_n(1-\delta) - \gamma_n(1-\delta) - \delta\alpha_n b_n^1 (1-\delta) - \delta\alpha_n c_n^1 (1-\delta)$$

$$- \delta\beta\, b^{p-1}(1-\delta) - \delta^2 \alpha_n b_n^1 c_n^2 (1-\delta) - \delta^2 \alpha_n b_n^1 b_n^2 (1-\delta) - \delta^3 \alpha_n b_n^1 b_n^2 c_n^3 (1-\delta)$$

$$- \delta^3 \alpha_n b_n^1 b_n^2 b_n^3\} \|Sx_n - p\| + \delta^4 \alpha_n b_n^1 b_n^2 b_n^3 \|Sy_n^4 - p\|$$

$$\le \{1 - \alpha_n(1-\delta) - \delta^3 \alpha_n b_n^1 b_n^2 b_n^3\} \|Sx_n - p\| + \delta^4 \alpha_n b_n^1 b_n^2 b_n^3 \|Sy_n^4 - p\| \tag{2.8}$$

Continuing the above process, we have

$$\|Sx_{n+1} - p\| \le \{1 - \alpha_n(1-\delta) - \delta^{k-2} \alpha_n b_n^1 b_n^2 b_n^3 \ldots\ldots b_n^{k-2}\} \|Sx_n - p\| + \delta^{k-1} \alpha_n b_n^1 b_n^2 b_n^3 \ldots\ldots b_n^{k-2} \|Sy_n^{k-1} - p\|$$

$$\le \{1 - \alpha_n(1-\delta) - \delta^{k-2} \alpha_n b_n^1 b_n^2 b_n^3 \ldots b_n^{k-2}\} \|Sx_n - p\|$$

$$+ \delta^{k-1} \alpha_n b_n^1 b_n^2 b_n^3 \ldots\ldots b_n^{k-2} \{(1 - b^{k-1}) \|Sx_n - p\| + b^{k-1} \|Tx_n - p\|\}$$

$$\le \{1 - (1-\delta)\alpha_n - \delta^{k-2} \alpha_n b_n^1 b_n^2 b_n^3 \ldots\ldots b_n^{k-2}\} \|Sx_n - p\|$$

$$+ \delta^{k-1} \alpha_n b_n^1 b_n^2 b_n^3 + \ldots b_n^{k-2} [(1 - b^{k-1}) \|Sx_n - p\| + b_n^{k-1} \mathbf{e^{L\|Sz - Tz\|}\{\phi(\|Sz - Tz\|) + \delta\|Sx_n - Sz\|\}}]$$

$$= \{1 - (1-\delta)\alpha_n - \delta^{k-2} \alpha_n b_n^1 b_n^2 \ldots\ldots b_n^{k-2}\} \|Sx_n - p\| + \delta^{k-1} \alpha_n b_n^1 b_n^2 b_n^3 \ldots\ldots b_n^{k-2} \{1 - b_n^{k-1} + \delta\, b_n^{k-1}\} \|Sx_n - p\|$$

$$\le 1 - (1-\delta)\alpha_n - \delta^{k-2} \alpha_n b_n^1 b_n^2 \ldots\ldots b_n^{k-2} (1-\delta)\} \|Sx_n - p\|$$

$$\le \{1 - (1-\delta)\alpha_n\} \|Sx_n - p\|$$

$$\le \prod_{k=0}^{n} \{1 - (1-\delta)\alpha_k\} \|Sx_0 - p\|$$

$$\le \prod_{k=0}^{n} e^{-(1-\delta)\alpha_k} \|Sx_0 - p\|$$

$$= e^{-(1\delta)\sum_{k=0}^{n} \alpha_k} \|Sx_0 - p\| \tag{2.9}$$

Since $0 \le \delta < 1$, $\alpha_k \in [0, 1)$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\lim_{n\to\infty} e^{-(1-\delta)\sum_{k=0}^{n}\alpha_k} = 0$

It follows from (2.9) that $\lim_{n\to\infty} \|Sx_{n+1} - p\| = 0$. Therefore $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Now we show that p is unique. If possible, let p^* be another point of coincidence then $Tz^* = Sz^* = p^*$ for some $z^* \in X$.

Hence, from (1.7) we have

$\|Tz - Tz^*\| = \|p - p^*\| \le e^{L\|Sz - Tz\|} \{\phi(\|Sz - Tz\|) + \delta\|Sz - Sz^*\|\} = \delta\|p - p^*\|$.

Since $0 \le \delta < 1$, then $p = p^*$ and so p is unique.

Since S, T are occasionally weakly compatible, then Sp = STz =TSz = Tp and hence p is a coincidence point of S and T and since the coincidence point is unique, then $p = p^*$ and hence Sp= Tp = p and therefore p is unique common fixed point of S and T.

Corollary 2.2: [Theorem 3.2 [16]] Let (X, ||·||) be an arbitrary Banach space, S, T: $Y \to X$ are nonself operator for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and that (1.4) holds. Suppose z is a coincidense point of S and T such that Tz = Sz = p. Then the multistep iteration $\{Sx_n\}$ given by (1.7) converges strongly to p. In addition, if Y = X and S, T are weakly compatible mappings, then p is unique common fixed point of S and T.

Corollary 2.3: Let (X, ||·||) be an arbitrary Banach space, S, T: $Y \to X$ are nonself operator for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and that (1.11) holds. Suppose z is a coincidence point of S and T such that Tz = Sz = p. Then the Jungck- Noor iteration (1.8) converges strongly to p. In addition, if Y=X and S, T are occasionally weakly compatible mappings, then p is unique common fixed point of S and T.

Remarks:

2.1 If L= 0 in (1.10), then weaker version of corollary 2.3 gives the results of [20] where the convergence is to the coincidence point of S and T and S is assumed injective.
2.2 We know that Jungck-Mann and Jungck Ishikawa iterations are special cases of Jungck-Noor iteration, so if L= 0, then weaker version of corollary 2.3 gives results of [20]. where the convergence is to the coincidence point of S and T and S is assumed injective. Also weaker version of corollary 2.2 gives results of [4].
2.3 Since the contractive condition(1.11) is more general than (1.4) and(1.3) so convergence theorems for operators satisfying(1.4) and (1.3) using multistep iterations (1.6), (1.7) and Jungck –Noor iteration (1.8) are obtained in theorem 2.1, corollary 2.2 and corollary 2.3 respectively.
2.4 If $S = I_d$ (identity map), corollary 2.3 gives the results of [2] with the help of remark 2.2.

Theorem 2.4: Let $(X, \|\cdot\|)$ be a normed linear space and S, T: $Y \to X$ are nonself operators such that (1.11) holds and $T(Y) \subseteq S(Y)$, S(Y) a complete subspace of X. Let z be a coincidence point of S and T such that Tz = Sz = p. For any $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-multistep iteration defined by (1.6) with $0 \le \alpha \le \alpha_n$, $0 \le \beta \le \beta_n$, $0 \le \gamma \le \gamma_n$, $0 \le b_n^i \le b$, for all i =1, 2, 3,…k-1 and for all n, converges to p. Then the Jungck-Multistep iteration is (S, T) stable.

Proof: Suppose that $\{Sy_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and define

$$\{\in_n\}_{n=0}^{\infty} \text{ by } \in_n = \|Sy_{n+1} - (1-\alpha_n-\beta_n-\gamma_n)Sy_n - \alpha_n Tp_n - \beta_n Tq_n - \gamma_n Ty_n\|,$$

where
$$Sp_n = (1-b_n^i-c_n^i)Sy_n + b_n^i Tq_n + c_n^i Ty_n,\ i = 1, 2, \ldots\ k-2,$$
$$Sq_n = (1-b_n^{k-1})Sy_n + b_n^{k-1}Ty_n,\ n = 0, 1, 2, \ldots,\ k \ge 2, \tag{2.10}$$

Let $\lim_{n\to\infty} \in_n = 0$, then by employing contractive condition (1.11) and the triangle inequality to establish that $\lim_{n\to\infty} Sy_n = p$:

$$\|Sy_{n+1}-p\| \le (1-\alpha_n-\beta_n-\gamma_n)\|Sy_n-p\| + \alpha_n\|Tp_n-p\| + \beta_n\|Tq_n-p\| + \gamma_n\|Ty_n-p\| + \in_n \tag{2.11}$$

An application of (1.11) and (2.10) gives

$$\|Tp_n - Tz\| \le e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sp_n - Sz\|\}$$
$$= \delta\|Sp_n - Sz\|$$
$$\le \delta\{(1-b_n^i-c_n^i)\|Sy_n-Sz\| + b_n^i\|Tq_n-Tz\| + c_n^i\|Ty_n-Tz\|\},\ i = 1, 2, ..\ k-2 \tag{2.12}$$

$$\|Tq_n - Tz\| \le e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sq_n - Sz\|\}$$
$$= \delta\|Sq_n - Sz\|$$
$$\le \delta\{(1-b_n^{k-1})\|Sy_n-Sz\| + b_n^{k-1}\|Ty_n-Tz\|\},\ i = 1, 2, ..\ k-2$$
$$\le \delta\ (1-b_n^{k-1})\|Sy_n-Sz\| + \delta b_n^{k-1}\{e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sy_n-Sz\|\}$$
$$\le \delta\ (1-b_n^{k-1}+a^2 b_n^{k-1})\|Sy_n-Sz\| \tag{2.13}$$

From (1.11) we have

$$\|Ty_n - Tz\| \le e^{L\|Sz-Tz\|}\{\phi(\|Sz-Tz\|) + \delta\|Sy_n - Sz\|\}$$
$$= \delta\|Sy_n - Sz\| \tag{2.14}$$

Substituting (2.13) and (2.14) in (2.12), we have

$$\|Tp_n - Tz\| \le (\delta - \delta b_n^i - \delta c_n^i + \delta^2 b_n^i - \delta^2 b_n^i b_n^{k-1} + \delta^3 b_n^i b_n^{k-1} + \delta^2 c_n^i)\|Sy_n - Sz\| \tag{2.15}$$

Substituting (2.13), (2.14) and (2.15) in (2.11) and rearranging the terms, we have

$$\|Sy_{n+1}-p\| \le [1-\alpha_n(1-\delta)-\beta_n(1-\delta)-\gamma_n(1-\delta) - \alpha_n\delta\, b_n^i(1-\delta) - \alpha_n\delta\, c_n^i(1-\delta)$$
$$-\beta_n\delta\, b_n^{k-1}(1-\delta) - \alpha_n\delta^2 b_n^i\, b_n^{k-1}(1-\delta)]\,\|Sy_n - Sz\| + \in_n$$
$$\le [1-\alpha(1-\delta)-\beta(1-\delta)-\gamma(1-\delta)-\alpha\delta b(1-\delta)-\alpha\delta c(1-\delta)-\beta\delta b(1-\delta)-\alpha\delta^2 b^2(1-\delta)]\,\|Sy_n - Sz\| + \in_n \tag{2.16}$$

Since $0 \le 1-\alpha(1-\delta)-\beta(1-\delta) -\gamma(1-\delta) -\alpha\delta b (1-\delta) -\alpha\delta c(1-\delta) -\beta\delta b(1-\delta) -\alpha \delta^2 b^2 (1-\delta) < 0$ and $\lim_{n\to\infty} \in_n = 0$, using Lemma (1.1) in (2.16) yield $\lim_{n\to\infty} Sy_n = p$.

Conversely, let $\lim_{n\to\infty} Sy_n = p$. Then an application of contractive condition (1.11) and the triangle inequality gives

$$\in_n \le \| Sy_{n+1} - p \| + (1-\alpha_n-\beta_n-\gamma_n) \|Sy_n - p\| + \alpha_n \| Tp_n - p \| + \beta_n \| Tq_n - p \| + \gamma_n \|Ty_n - p\| \quad (2.17)$$

Substituting (2.13), (2.14) and (2.15) in (2.17), we have

$$\begin{aligned} \in_n &\le \| Sy_{n+1} - Sz \| + [1-\alpha_n (1-\delta)-\beta_n(1-\delta)-\gamma_n (1-\delta) - \alpha_n\delta\, b_n^i (1-\delta) \\ &\quad -\alpha_n\delta\, c_n^i (1-\delta) -\beta_n \delta\, b_n^{k-1} (1-\delta)-\alpha_n\delta^2 b_n^i\, b_n^{k-1} (1-\delta)] \| Sy_n - Sz \| \\ &\le \| Sy_{n+1} - Sz \| + [1-\alpha(1-\delta)-\beta(1-\delta) -\gamma(1-\delta)-\alpha\delta b(1-\delta) -\alpha\delta c(1-\delta) -\beta \delta b(1-\delta)-\alpha\delta^2b^2(1-\delta)] \| Sy_n - Sz \quad (2.18) \end{aligned}$$

Since
$0 \le 1-\alpha(1-\delta)-\beta(1-\delta) -\gamma(1-\delta)-\alpha\delta b (1-\delta)-\alpha\delta c(1-\delta) -\beta\delta b(1-\delta) -\alpha\delta^2b^2 (1-\delta) < 0$ and $\lim_{n\to\infty} Sy_n = p$, from (2.18) we have $\lim_{n\to\infty} \in_n = 0$

Corollary 2.5: Let (X, ||·||) be a normed linear space and S, T: Y→X are nonself operators such that (1.11) holds and $T(Y) \subseteq S(Y)$, S(Y) a complete subspace of X. Let z be a coincidence point of S and T such that Tz = Sz = p. For any $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-multistep iteration defined by (1.7) with $0\le \alpha\le \alpha_n$, $0\le b_n^i \le b$, for all i =1,2,3,…k-1 and for all n, converges to p. Then the Jungck- multistep iteration (1.7) is (S, T) stable.

Corollary 2.6: Let (X, ||·||) be a normed linear space and S, T: Y→X are nonself operators such that (1.11) holds and $T(Y) \subseteq S(Y)$, S(Y) a complete subspace of X. Let z be a coincidence point of S and T such that Tz = Sz = p. For any $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Noor iteration defined by (1.8) with $0\le \alpha\le \alpha_n$, $0\le b_n^1$, $b_n \le b$ for all n, converges to p. Then the Jungck-Noor iteration (1.8) is (S, T) stable.

Corollary 2.7: Let (X, ||·||) be a normed linear space and S, T: Y→X are nonself operators such that (1.9) holds and $T(Y) \subseteq S(Y)$, S(Y) a complete subspace of X. Let z be a coincidence point of S and T such that Tz = Sz = p. For any $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Ishikawa iteration defined by (1.10) with $0\le \alpha\le \alpha_n$, $0\le b_n^1 \le b$ for all n, converges to p. Then the Jungck-Ishikawa iteration is (S, T) stable.

Remark 2.5: If L=0, then corollary 2.2 gives theorem 3.2 in [19]

Corollary 2.8: Let (X, ||·||) be a normed linear space and S, T: Y→X are nonself operators such that (1.10) holds and $T(Y) \subseteq S(Y)$, S(Y) a complete subspace of X. Let z be a coincidence point of S and T such that Tz = Sz = p. For any $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Mann iteration defined by (1.10) with $0\le \alpha\le \alpha_n$ for all n, converges to p. Then the Jungck-Mann iteration is (S, T) stable.

Remark 2.6: As contractive condition (1.11) more general than [1.4] corollary 2.8 gives results in [28]

Remark 2.7: As given condition 1.11 is more general than generalized Zamfirescu operators, the above results [2.5-2.8] also hold for generalized Zamfirescu operators (1.3).

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