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TENSOR AND DUAL REPRESENTATIONS FOR SU(2) BY THE MATRIX LIE ALGEBRAS su(2) AND sl(2)

SAAD OWAID*, ZAINAB SUBHI

Department of Mathematics,
College of Science, University of Al-Mustansiryah, Baghdad- Iraq.

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ABSTRACT

*In this work, we continue our study started in [6] on representations of the matrix lie group **SU(2)** resulting by conjugation action on the matrix lie algebras **su(2)** and **sl(2)**. We calculate the tensor and dual representations for the obtained adjoint representations Ad_1 and Ad_2.*

INTRODUCTION

In 1888, during his work at certain transformation groups, a Norwegian mathematician Sophus Lie initiated Lie theory. Later his researches led to a fundamental concept, namely, Lie algebras. Nowadays this theory becomes an indispensible for various branches in both mathematics and theoretical physics, for example, see [2] and [5] .One of the most fruitful approaches in representation theory is; choosing a group action on a vector space over a specific field; such procedure leads to a huge amount of research efforts in representation theory[1].

In [3] Helmer Aslaksen find certain summands in tensor products of Lie algebra representations. Mahmoud and his colleagues [4], constructed new representation of SU(4) in terms of Pauli matrices.

Follow the procedure that we used in [6], that is exploiting the generators of the matrix lie group SU(2) and the basis of the matrix lie algebras su(2) and sl(2), we construct tensor$\mathrm{Ad}_1 \otimes \mathrm{Ad}_2$ and dual$\mathrm{Ad}_1^{\hat{}}$, $\mathrm{Ad}_2^{\hat{}}$ representations .

1. TENSOR PRODUCT OF REPRESENTATIONS

Recall that if $\mathbb{U}$, $\mathbb{V}$are two vector spaces over a field F of dimensions n, m, and basis $\{\ell_i\}_{i=1}^{n}$, $\{h_j\}_{j=1}^{m}$ respectively, then the set $\{\ell_i \otimes h_j \,|\, 1 \le i \le n, 1 \le j \le m\}$ form a basis for the tensor product $\mathbb{U} \otimes \mathbb{V}$ such that
$\dim(\mathbb{U} \otimes \mathbb{V}) = \dim(\mathbb{U}).\dim(\mathbb{V}) = n.m$.

Definition 1.1: [2] Let G be a matrix Lie group and Let Π_1 be a representation of G acting on a space $\mathbb{U}$ and let Π_2 be a representation of G acting on a space $\mathbb{V}$. Then the tensor product of Π_1 and Π_2 is a representation $\Pi_1 \otimes \Pi_2$ of G acting on $\mathbb{U} \otimes \mathbb{V}$ defined by: $\Pi_1 \otimes \Pi_2(\mathrm{A}) = \Pi_1(\mathrm{A}) \otimes \Pi_2(\mathrm{A})$, for all A $\in$ G.

MAIN THEOREM

Theorem 1.2: Let G be a matrix Lie group and for each $\mathrm{i} \in [1, \dots, \mathrm{n}]$, $\mathrm{V_i}$ are complex vector spaces over a field F, Π_i are finite dimensional representations of G on$\mathrm{V_i}$ then the tensor product representation $\otimes_{\mathrm{i}=1}^{\mathrm{n}} \Pi_\mathrm{i} : \mathrm{G} \longrightarrow \mathrm{GL}(\otimes_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{V_i})$is completely determine by generators of G and basis of $\mathrm{V_i}$.

Proof: Let $\mathrm{s}_1, \mathrm{s}_2, \dots, \mathrm{s}_\mathrm{r}$ be generators of G and $\{\mathrm{V}_{1\mathrm{j}}\}_{\mathrm{j}=1}^{\mathrm{t}_1}, \dots, \{\mathrm{V}_{\mathrm{nj}}\}_{\mathrm{j}=1}^{\mathrm{t}_\mathrm{n}}$ be a basis of $\mathrm{V_i}$ where $\dim(\mathrm{V_i}) = \mathrm{t_i}$, $\mathrm{i} \in [1, \dots, \mathrm{n}]$.

Suppose $\mathrm{A} \in \mathrm{G}$ then $\mathrm{A} = \mathrm{s}_1^{\mathrm{n}_1} * \dots * \mathrm{s}_\mathrm{r}^{\mathrm{n}_\mathrm{r}}$ for some $\mathrm{n}_1, \dots, \mathrm{n}_\mathrm{r} \in \mathbb{Z}$.

$$\Pi_\mathrm{i}(\mathrm{A}) = \Pi_\mathrm{i}(\mathrm{s}_1^{\mathrm{n}_1} * \dots * \mathrm{s}_\mathrm{r}^{\mathrm{n}_\mathrm{r}}).$$

*Corresponding Author: Saad Owaid**
Department of Mathematics, College of Science, University of Al-Mustansiryah. Baghdad- Iraq.

If $X \in \otimes_{i=1}^{n} V_i$, X can be written as ;
$X = \sum_{j=1}^{t_1} C_{1j} V_{1j} \otimes \ldots \otimes \sum_{j=1}^{t_n} C_{nj} V_{nj}$ $(C_{i,j} \in F, 1 \le i \le n, 1 \le j \le \max(t_i))$.

$$\otimes_{i=1}^{n} \Pi_{i(A)}(X) = \otimes_{i=1}^{n} \Pi_{is_1^{n_1} * \ldots * s_r^{n_r}}(X) = \otimes_{i=1}^{n} \Pi_{is_1^{n_1} * \ldots * s_r^{n_r}} \left(\sum_{j=1}^{t_1} C_{1j} V_{1j} \otimes \ldots \otimes \sum_{j=1}^{t_n} C_{nj} V_{nj} \right)$$

By definition 1.1 above we have:

$= \Pi_{1s_1^{n_1} * \ldots * S_r^{n_r}} \left(\sum_{j=1}^{t_1} C_{1j} V_{1j} \right) \otimes \ldots \otimes \Pi_{ns_1^{n_1} * \ldots * S_r^{n_r}} \left(\sum_{j=1}^{t_n} C_{nj} V_{nj} \right)$

$= \left(\sum_{j=1}^{t_1} C_{1j} \Pi_{1s_1^{n_1} * \ldots * S_r^{n_r}}(V_{1j}) \right) \otimes \ldots \otimes \left(\sum_{j=1}^{t_n} C_{nj} \Pi_{ns_1^{n_1} * \ldots * S_r^{n_r}}(V_{nj}) \right)$

$= \left[C_{11} \Pi_{1s_1^{n_1} * \ldots * S_r^{n_r}}(V_{1j}) + \cdots + C_{nt_1} \Pi_{1s_1^{n_1} * \ldots * S_r^{n_r}}(V_{1t_1}) \right] \otimes \ldots \otimes \left[C_{n1} \Pi_{ns_1^{n_1} * \ldots * S_r^{n_r}} (V_{n1}) + \cdots + C_{nt_n} \Pi_{ns_1^{n_1} * \ldots * S_r^{n_r}} (V_{nt_n}) \right]$.

$= \left[C_{11}(\Pi_{1s_1^{n_1}}(V_{1j}) * \ldots * \Pi_{1S_r^{n_r}}(V_{1j})) + \cdots + C_{nt_1} (\Pi_{1s_1^{n_1}}(V_{1t_1}) * \ldots * \Pi_{1\, S_r^{n_r}}(V_{1t_1})) \right] \otimes \ldots \otimes \left[C_{n1}(\Pi_{ns_1^{n_1}} (V_{n1}) * \ldots * \right.$
ΠnSrnrVn1)+…+Cntn(Πns1n1Vntn∗…∗ΠnSrnrVntn*),* Since G acts on Vi for each i, we are done.

We knew that the set of matricesF_i, H_i *and* $X_i (1 \le i \le 3)$ "*listed below*", are generators of matrix Lie group SU(2) and basis for the matrix lie algebras **su(2)**and **sl(2)** respectively.In [6]we have computed the adjoint representations resulting from the conjugation action of this group on those algebraswhere:$Ad_1: SU(2) \to GL(\mathbf{su(2)})$,$Ad_2: SU(2) \to GL(\mathbf{sl(2)})$

$F_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, F_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$

$H_1 = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \mathbf{H_3} = \begin{pmatrix} \mathbf{0} & \mathbf{i/2} \\ \mathbf{i/2} & \mathbf{0} \end{pmatrix}$ and

$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Corollary 1.3: $Ad_1 \otimes Ad_2$ can be completely determined by generators of SU(2) and basis of **su(2)** and **sl(2)**.

According to definition 1.1 and corollary 1.3 we have:

Tensor representation $\mathbf{Ad_1 \otimes Ad_2}$

The tensor product of the representations Ad_1 and Ad_2, with the rule

$\Pi_1 \otimes \Pi_2(A, B) = \Pi_1(A) \otimes \Pi_2(B)$, for all $A \in SU(2)$, is given by formula;

$Ad_1 \otimes dA_2(F_i) = Ad_1(F_i) \otimes dA_2(F_i)$, $\quad 1 \le i \le 3$.

i-$\mathbf{Ad_1 \otimes dA_2(F_1) = Ad_1(F_1) \otimes dA_2(F_1)}$

1- $Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_1) = \frac{-1}{4} H_1 \otimes \frac{-1}{4} X_1$.
2- $Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_2) = \frac{-1}{4} H_1 \otimes \frac{1}{4} X_3$.
3- 3-$Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_3) = \frac{-1}{4} H_1 \otimes \frac{1}{4} X_2$.
4- $Ad_{1F_1}(H_2) \otimes dA_{2F_1}(X_1) = \frac{-1}{4} H_2 \otimes \frac{1}{4} X_1$.
5- $Ad_{1F_1}(H_2) \otimes dA_{2F_1}(X_2) = \frac{-1}{4} H_2 \otimes \frac{1}{4} X_3$.
6- $Ad_{1F_1}(H_2) \otimes dA_{2F_1}(X_3) = \frac{-1}{4} H_2 \otimes \frac{1}{4} X_2$.
7- $Ad_{1F_1}(H_3) \otimes dA_{2F_1}(X_1) = \frac{1}{4} H_3 \otimes \frac{-1}{4} X_1$.
8- $Ad_{1F_1}(H_3) \otimes dA_{2F_1}(X_2) = \frac{1}{4} H_3 \otimes \frac{1}{4} X_3$.
9- $Ad_{1F_1}(H_3) \otimes dA_{2F_1}(X_3) = \frac{1}{4} H_3 \otimes \frac{1}{4} X_2$.

ii-$\mathbf{Ad_1 \otimes dA_2(F_2) = Ad_1(F_2) \otimes dA_2(F_2)}$

1- $Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_1) = \frac{-1}{4} H_1 \otimes \frac{-1}{4} X_1$.
2- $Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_2) = \frac{-1}{4} H_1 \otimes \frac{-1}{4} X_3$.
3- 3-$Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_3) = \frac{-1}{4} H_1 \otimes \frac{1}{4} X_2$.
4- $Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_1) = \frac{1}{4} H_2 \otimes \frac{-1}{4} X_1$.
5- $Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_2) = \frac{1}{4} H_2 \otimes \frac{-1}{4} X_3$.

6- $Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_3) = \frac{1}{4}H_2 \otimes \frac{1}{4}X_3$.
7- $Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_1) = \frac{-1}{4}H_3 \otimes \frac{-1}{4}X_1$.
8- $Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_2) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_3$.
9- $Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_3) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_3$.

iii-$\mathbf{Ad_1 \otimes dA_2(F_3) = Ad_1(F_3) \otimes dA_2(F_3)}$
1- $Ad_{1F_3}(H_1) \otimes dA_{2F_3}(X_1) = \frac{1}{4}H_1 \otimes \frac{1}{4}X_1$.
2- $Ad_{1F_3}(H_1) \otimes dA_{2F_3}(X_2) = \frac{1}{4}H_1 \otimes \frac{1}{4}X_2$.
3- 3-$Ad_{1F_3}(H_1) \otimes dA_{2F_3}(X_3) = \frac{1}{4}H_1 \otimes \frac{1}{4}X_3$.
4- $Ad_{1F_3}(H_2) \otimes dA_{2F_3}(X_1) = \frac{-1}{4}H_2 \otimes \frac{1}{4}X_1$.
5- $Ad_{1F_3}(H_2) \otimes dA_{2F_3}(X_2) = \frac{-1}{4}H_2 \otimes \frac{1}{4}X_2$.
6- $Ad_{1F_3}(H_2) \otimes dA_{2F_3}(X_3) = \frac{-1}{4}H_2 \otimes \frac{1}{4}X_3$.
7- $Ad_{1F_3}(H_3) \otimes dA_{2F_3}(X_1) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_1$.
8- $Ad_{1F_3}(H_3) \otimes dA_{2F_3}(X_2) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_2$.
9- $Ad_{1F_3}(H_3) \otimes dA_{2F_3}(X_3) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_3$.

We can display the resulting calculations by the following table (1)

Generators Basis of SU(2) / Basis Of $su(2) \otimes sl(2)$	F_1	F_2	F_3
(H_1, X_1)	$\frac{-1}{4}H_1 \otimes \frac{-1}{4}X_1$	$\frac{-1}{4}H_1 \otimes \frac{-1}{4}X_1$	$\frac{1}{4}H_1 \otimes \frac{1}{4}X_1$
(H_1, X_2)	$\frac{-1}{4}H_1 \otimes \frac{1}{4}X_3$	$\frac{-1}{4}H_1 \otimes \frac{-1}{4}X_3$	$\frac{1}{4}H_1 \otimes \frac{1}{4}X_2$
(H_1, X_3)	$\frac{-1}{4}H_1 \otimes \frac{1}{4}X_2$	$\frac{-1}{4}H_1 \otimes \frac{1}{4}X_2$	$\frac{1}{4}H_1 \otimes \frac{1}{4}X_3$
(H_2, X_1)	$\frac{-1}{4}H_2 \otimes \frac{1}{4}X_1$	$\frac{1}{4}H_2 \otimes \frac{-1}{4}X_1$	$\frac{-1}{4}H_2 \otimes \frac{1}{4}X_1$
(H_2, X_2)	$\frac{-1}{4}H_2 \otimes \frac{1}{4}X_3$	$\frac{1}{4}H_2 \otimes \frac{-1}{4}X_3$	$\frac{-1}{4}H_2 \otimes \frac{1}{4}X_2$
(H_2, X_3)	$\frac{-1}{4}H_2 \otimes \frac{1}{4}X_2$	$\frac{1}{4}H_2 \otimes \frac{1}{4}X_3$	$\frac{-1}{4}H_2 \otimes \frac{1}{4}X_3$
(H_3, X_1)	$\frac{1}{4}H_3 \otimes \frac{-1}{4}X_1$	$\frac{-1}{4}H_3 \otimes \frac{-1}{4}X_1$	$\frac{-1}{4}H_3 \otimes \frac{1}{4}X_1$
(H_3, X_2)	$\frac{1}{4}H_3 \otimes \frac{1}{4}X_3$	$\frac{-1}{4}H_3 \otimes \frac{1}{4}X_3$	$\frac{-1}{4}H_3 \otimes \frac{1}{4}X_2$
(H_3, X_3)	$\frac{1}{4}H_3 \otimes \frac{1}{4}X_2$	$\frac{-1}{4}H_3 \otimes \frac{1}{4}X_3$	$\frac{-1}{4}H_3 \otimes \frac{1}{4}X_3$

Table-1: $(Ad_1 \otimes Ad_2)$

2. DUAL REPRESENTATIONS

Definition 2.1: [2] Suppose G is a Lie group and Π is representation of G acting on a finite dimensional vector space V. Then the dual representation$\Pi^{\wedge}$ to Π is the representation of G acting on $V^{\wedge}$ given by $\Pi^{\wedge}(A) = [\Pi(A^{-1})]^{tr}$, $\forall A \in G$. The dual representation is also called contragredient representation.

Remark 2.2: We can extend our result of theorem 1.2 above to include dual representations which proved in similar procedure, and have the following result:

Proposition 2.2: Let G be a matrix Lie group, V complex vector space over a field F, $\Pi: G \longrightarrow GL(V)$ be a representation of G on V, then the dual representation $\Pi^{\wedge}$ can be completely determined by generators of G and basis of V.

In particular, proposition 2.2 applies to the dual representations $\mathrm{Ad}_1^{\wedge}$ and $\mathrm{Ad}_2^{\wedge}$ which we compute separately as follows:

Dual representation $\mathbf{Ad}_1^{\wedge}$

Combining definition 2.1 and proposition 2.2 we have:

1-$Ad_1^{\wedge}(\mathrm{F}_1) = [\mathrm{Ad}_1(\mathrm{F}_1)^{-1}]^{\mathrm{tr}} = [\mathrm{Ad}_1(\mathrm{F}_1)^*]^{\mathrm{tr}} =$
i-$\left[\mathrm{Ad}_{1\mathrm{F}_1{}^*}(\mathrm{H}_1)\right]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{H}_1(\mathrm{F}_1^*)^*]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{H}_1\mathrm{F}_1]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{H}_1.$
ii-$\left[\mathrm{Ad}_{1\mathrm{F}_1{}^*}(\mathrm{H}_2)\right]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{H}_2(\mathrm{F}_1^*)^*]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{H}_2\mathrm{F}_1]^{\mathrm{tr}} = \frac{1}{4}\mathrm{H}_2.$
iii-$\left[\mathrm{Ad}_{1\mathrm{F}_1{}^*}(\mathrm{H}_3)\right]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{H}_3(\mathrm{F}_1^*)^*]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{H}_3\mathrm{F}_1]^{\mathrm{tr}} = \frac{1}{4}\mathrm{H}_3.$

2- $Ad_1^{\wedge}(\mathrm{F}_2) = [\mathrm{Ad}_1(\mathrm{F}_2)^{-1}]^{\mathrm{tr}} = [\mathrm{Ad}_1(\mathrm{F}_2)^*]^{\mathrm{tr}} =$
i-$\left[\mathrm{Ad}_{1\mathrm{F}_2{}^*}(\mathrm{H}_1)\right]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{H}_1(\mathrm{F}_2^*)^*]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{H}_1\mathrm{F}_2]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{H}_1.$
ii-$\left[\mathrm{Ad}_{1\mathrm{F}_2{}^*}(\mathrm{H}_2)\right]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{H}_2(\mathrm{F}_2^*)^*]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{H}_2\mathrm{F}_2]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{H}_2.$
iii-$\left[\mathrm{Ad}_{1\mathrm{F}_2{}^*}(\mathrm{H}_3)\right]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{H}_3(\mathrm{F}_2^*)^*]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{H}_3\mathrm{F}_2]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{H}_3.$

3- $Ad_1^{\wedge}(\mathrm{F}_3) = [\mathrm{Ad}_1(\mathrm{F}_3)^{-1}]^{\mathrm{tr}} = [\mathrm{Ad}_1(\mathrm{F}_3)^*]^{\mathrm{tr}} =$
i-$\left[\mathrm{Ad}_{1\mathrm{F}_3{}^*}(\mathrm{H}_1)\right]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{H}_1(\mathrm{F}_3^*)^*]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{H}_1\mathrm{F}_3]^{\mathrm{tr}} = \frac{1}{4}\mathrm{H}_1.$
ii-$\left[\mathrm{Ad}_{1\mathrm{F}_3{}^*}(\mathrm{H}_2)\right]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{H}_2(\mathrm{F}_3^*)^*]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{H}_2\mathrm{F}_3]^{\mathrm{tr}} = \frac{1}{4}\mathrm{H}_2.$
iii-$\left[\mathrm{Ad}_{1\mathrm{F}_3{}^*}(\mathrm{H}_3)\right]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{H}_3(\mathrm{F}_3^*)^*]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{H}_3\mathrm{F}_3]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{H}_3.$

We can display the resulting calculations as table 2 below.

Basis of **su(2)** / enerators Basis of SU(2)	H_1	H_2	H_3
F_1	$-\frac{1}{4}H_1$	$\frac{1}{4}H_2$	$\frac{1}{4}H_3$
F_2	$-\frac{1}{4}H_1$	$-\frac{1}{4}H_2$	$-\frac{1}{4}H_3$
F_3	$\frac{1}{4}H_1$	$\frac{1}{4}H_2$	$-\frac{1}{4}H_3$

Table-2: The dual representation $\mathrm{Ad}_1^{\wedge}$

Dual representation $\mathrm{Ad}_2^{\wedge}$

Using the same manner in the case of $\mathbf{Ad}_1^{\wedge}$ above we have:

1- $\mathrm{Ad}_2^{\wedge}(\mathrm{F}_1) = [\mathrm{Ad}_2(\mathrm{F}_1)^{-1}]^{\mathrm{tr}} = [\mathrm{Ad}_2(\mathrm{F}_1)^*]^{\mathrm{tr}} =$
i-$\left[\mathrm{Ad}_{2\mathrm{F}_1{}^*}(\mathrm{X}_1)\right]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{X}_1(\mathrm{F}_1^*)^*]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{X}_1\mathrm{F}_1]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{X}_1.$
ii-$\left[\mathrm{Ad}_{2\mathrm{F}_1{}^*}(\mathrm{X}_2)\right]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{X}_2(\mathrm{F}_1^*)^*]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{X}_2\mathrm{F}_1]^{\mathrm{tr}} = \frac{1}{4}\mathrm{X}_2.$
iii-$\left[\mathrm{Ad}_{2\mathrm{F}_1{}^*}(\mathrm{X}_3)\right]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{X}_3(\mathrm{F}_1^*)^*]^{\mathrm{tr}} = [\mathrm{F}_1^*\mathrm{X}_3\mathrm{F}_1]^{\mathrm{tr}} = \frac{1}{4}\mathrm{X}_3.$

2- $\mathrm{Ad}_2^{\wedge}(\mathrm{F}_2) = [\mathrm{Ad}_2(\mathrm{F}_2)^{-1}]^{\mathrm{tr}} = [\mathrm{Ad}_2(\mathrm{F}_2)^*]^{\mathrm{tr}} =$
i-$\left[\mathrm{Ad}_{2\mathrm{F}_2{}^*}(\mathrm{X}_1)\right]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{X}_1(\mathrm{F}_2^*)^*]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{X}_1\mathrm{F}_2]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{X}_1.$
ii-$\left[\mathrm{Ad}_{2\mathrm{F}_2{}^*}(\mathrm{X}_2)\right]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{X}_2(\mathrm{F}_2^*)^*]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{X}_2\mathrm{F}_2]^{\mathrm{tr}} = \frac{-1}{4}\mathrm{X}_2.$
iii-$\left[\mathrm{Ad}_{2\mathrm{F}_2{}^*}(\mathrm{X}_3)\right]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{X}_3(\mathrm{F}_2^*)^*]^{\mathrm{tr}} = [\mathrm{F}_2^*\mathrm{X}_3\mathrm{F}_2]^{\mathrm{tr}} = \frac{1}{4}\mathrm{X}_2.$

3-$\mathrm{Ad}_2^{\wedge}(\mathrm{F}_3) = [\mathrm{Ad}_2(\mathrm{F}_3)^{-1}]^{\mathrm{tr}} = [\mathrm{Ad}_2(\mathrm{F}_3)^*]^{\mathrm{tr}} =$
i-$\left[\mathrm{Ad}_{2\mathrm{F}_3{}^*}(\mathrm{X}_1)\right]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{X}_1(\mathrm{F}_3^*)^*]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{X}_1\mathrm{F}_3]^{\mathrm{tr}} = \frac{1}{4}\mathrm{X}_1.$
ii-$\left[\mathrm{Ad}_{2\mathrm{F}_3{}^*}(\mathrm{X}_2)\right]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{X}_2(\mathrm{F}_3^*)^*]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{X}_2\mathrm{F}_3]^{\mathrm{tr}} = \frac{1}{4}\mathrm{X}_3.$
iii-$\left[\mathrm{Ad}_{2\mathrm{F}_3{}^*}(\mathrm{X}_3)\right]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{X}_3(\mathrm{F}_3^*)^*]^{\mathrm{tr}} = [\mathrm{F}_3^*\mathrm{X}_3\mathrm{F}_3]^{\mathrm{tr}} = \frac{1}{4}\mathrm{X}_2.$

We can display the resulting calculations as in table 3 below.

Basis of **sl(2)** / Generators Basis of SU(2)	X_1	X_2	X_3
F_1	$\frac{-1}{4}X_1$	$\frac{1}{4}X_2$	$\frac{1}{4}X_3$
F_2	$\frac{-1}{4}X_1$	$\frac{-1}{4}X_2$	$\frac{1}{4}X_2$
F_3	$\frac{1}{4}X_1$	$\frac{1}{4}X_3$	$\frac{1}{4}X_2$

Table-3: The dual representation $Ad_2^{\wedge}$

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