TENSOR AND DUAL REPRESENTATIONS
FOR SU(2) BY THE MATRIX LIE ALGEBRAS su(2) AND sl(2)

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ABSTRACT
In this work, we continue our study started in [6] on representations of the matrix lie group SU(2) resulting by conjugation action on the matrix lie algebras su(2) and sl(2). We calculate the tensor and dual representations for the obtained adjoint representations Ad_1 and Ad_2.

INTRODUCTION
In 1888, during his work at certain transformation groups, a Norwegian mathematician Sophus Lie initiated Lie theory. Later his researches led to a fundamental concept, namely, Lie algebras. Nowadays this theory becomes an indispensable for various branches in both mathematics and theoretical physics, for example, see [2] and [5]. One of the most fruitful approaches in representation theory is; choosing a group action on a vector space over a specific field; such procedure leads to a huge amount of research efforts in representation theory[1].

In [3] Helmer Aslaksen find certain summands in tensor products of Lie algebra representations. Mahmoud and his colleagues [4], constructed new representation of SU(4) in terms of Pauli matrices.

Follow the procedure that we used in [6], that is exploiting the generators of the matrix lie group SU(2) and the basis of the matrix lie algebras su(2) and sl(2), we construct tensor \( \text{Ad}_1 \otimes \text{Ad}_2 \) and dual \( \text{Ad}_1^* \otimes \text{Ad}_2^* \) representations.

1. TENSOR PRODUCT OF REPRESENTATIONS
Recall that if \( U, V \) are two vector spaces over a field \( F \) of dimensions \( n, m \), and basis \( \{ \ell_i \}_{i=1}^n, \{ h_j \}_{j=1}^m \) respectively, then the set \( \{ \ell_i \otimes h_j | 1 \leq i \leq n, 1 \leq j \leq m \} \) form a basis for the tensor product \( U \otimes V \) such that \( \dim(U \otimes V) = \dim(U) \cdot \dim(V) = n \cdot m \).

Definition 1.1: [2] Let \( G \) be a matrix Lie group and let \( \Pi_1 \) be a representation of \( G \) acting on a space \( U \) and let \( \Pi_2 \) be a representation of \( G \) acting on a space \( V \). Then the tensor product of \( \Pi_1 \) and \( \Pi_2 \) is a representation \( \Pi_1 \otimes \Pi_2 \) of \( G \) acting on \( U \otimes V \) defined by: \( \Pi_1 \otimes \Pi_2(A) = \Pi_1(A) \otimes \Pi_2(A) \), for all \( A \in G \).

MAIN THEOREM
Theorem 1.2: Let \( G \) be a matrix Lie group and for each \( i \in [1, ..., n] \), \( V_i \) are complex vector spaces over a field \( F \), \( \Pi_i \) are finite dimensional representations of \( G \) on \( V_i \). Then the tensor product representation \( \bigotimes_{i=1}^n \Pi_i : G \rightarrow \text{GL}(\bigotimes_{i=1}^n V_i) \) is completely determine by generators of \( G \) and basis of \( V_i \).

Proof: Let \( s_1, s_2, ..., s_r \) be generators of \( G \) and \( \{ V_{ij} \}_{i=1}^{t_i} \) be a basis of \( V_i \) where \( \dim(V_i) = t_i \), \( i \in [1, ..., n] \).

Suppose \( A \in G \) then \( A = s_1^{n_1} \ast \cdots \ast s_r^{n_r} \) for some \( n_1, ..., n_r \in \mathbb{Z} \).

\[ \Pi_i(A) = \Pi_i(s_1^{n_1} \ast \cdots \ast s_r^{n_r}) \]

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If \( X \in \bigotimes_{i=1}^{n} V_i \), \( X \) can be written as:
\[
X = \sum_{i=1}^{t_1} C_{1i} V_{1i} \otimes \cdots \otimes \sum_{j=1}^{t_n} C_{nj} V_{nj} \quad (C_{ij} \in F, 1 \leq i \leq n, 1 \leq j \leq \max(t_i)).
\]
\[
\bigotimes_{i=1}^{n} \Pi_i(X) = \bigotimes_{i=1}^{n} \Pi_i^{n_1^{t_1} \cdots n_j^{t_j}} \left( \sum_{j=1}^{t_1} C_{1j} V_{1j} \otimes \cdots \otimes \sum_{j=1}^{t_n} C_{nj} V_{nj} \right)
\]

By definition 1.1 above we have:
\[
= \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq t_i} s_{i_j}^* \left( \sum_{j=1}^{t_1} C_{1j} V_{1j} \otimes \cdots \otimes \sum_{j=1}^{t_n} C_{nj} V_{nj} \right)
\]

We knew that the set of matrices \( F_i \), \( H_i \) and \( X_i \) \((1 \leq i \leq 3)\) "listed below", are generators of Lie group SU(2) and basis for the matrix Lie algebras \( su(2) \) and \( sl(2) \) respectively. In [6] we have computed the adjoint representations resulting from the conjugation action of this group on those algebra, where: \( Ad_1 : SU(2) \rightarrow GL(su(2)) \), \( Ad_2 : SU(2) \rightarrow GL(sl(2)) \)

\[
F_1 = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}
\]

\[
H_1 = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}
\]

\[
X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

Corollary 1.3: \( Ad_1 \otimes Ad_2 \) can be completely determined by generators of SU(2) and basis of \( su(2) \) and \( sl(2) \).

According to definition 1.1 and corollary 1.3 we have:

**Tensor representation** \( Ad_1 \otimes Ad_2 \)

The tensor product of the representations \( Ad_1 \) and \( Ad_2 \), with the rule
\[
\Pi_1 \otimes \Pi_2(A, B) = \Pi_1(A) \otimes \Pi_2(B), \quad \text{for all } A \in SU(2),\text{ is given by formula;}
\]

\[
Ad_1 \otimes dA_2(F_i) = dA_1(F_i) \otimes dA_2(F_i), \quad 1 \leq i \leq 3.
\]

**i-Ad\(_1\) \otimes dA\(_2\)(F\(_i\)) = Ad\(_1\)(F\(_i\)) \otimes dA\(_2\)(F\(_i\))**

1- \( Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_1) = -\frac{1}{4} H_1 \otimes -\frac{1}{4} X_1. \)

2- \( Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_2) = -\frac{1}{4} H_1 \otimes \frac{1}{4} X_3. \)

3- \( 3Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_3) = -\frac{1}{4} H_1 \otimes \frac{3}{4} X_2. \)

4- \( Ad_{1F_1}(H_2) \otimes dA_{2F_1}(X_1) = -\frac{1}{4} H_2 \otimes -\frac{1}{4} X_1. \)

5- \( Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_2) = -\frac{1}{4} H_2 \otimes \frac{1}{4} X_3. \)

6- \( Ad_{1F_2}(H_2) \otimes dA_{2F_3}(X_3) = -\frac{1}{4} H_2 \otimes \frac{3}{4} X_2. \)

7- \( Ad_{1F_1}(H_3) \otimes dA_{2F_1}(X_1) = -\frac{1}{4} H_3 \otimes -\frac{1}{4} X_1. \)

8- \( Ad_{1F_1}(H_3) \otimes dA_{2F_2}(X_2) = -\frac{1}{4} H_3 \otimes \frac{1}{4} X_3. \)

9- \( Ad_{1F_1}(H_3) \otimes dA_{2F_3}(X_3) = -\frac{1}{4} H_3 \otimes \frac{3}{4} X_2. \)

**ii-Ad\(_1\) \otimes dA\(_2\)(F\(_2\)) = Ad\(_1\)(F\(_2\)) \otimes dA\(_2\)(F\(_2\))**

1- \( Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_1) = -\frac{1}{4} H_1 \otimes -\frac{1}{4} X_1. \)

2- \( Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_2) = -\frac{1}{4} H_1 \otimes \frac{1}{4} X_3. \)

3- \( 3Ad_{1F_1}(H_1) \otimes dA_{2F_1}(X_3) = -\frac{1}{4} H_1 \otimes \frac{3}{4} X_2. \)

4- \( Ad_{1F_1}(H_2) \otimes dA_{2F_2}(X_1) = -\frac{1}{4} H_2 \otimes -\frac{1}{4} X_1. \)

5- \( Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_2) = -\frac{1}{4} H_2 \otimes \frac{1}{4} X_3. \)
6. Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_5) = \frac{1}{4}H_2 \otimes \frac{1}{4}X_3.
7. Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_1) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_1.
8. Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_2) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_3.
9. Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_5) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_3.

iii-Ad_1 \otimes dA_2(F_3) = Ad_1(F_3) \otimes dA_2(F_1)
1. Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_1) = \frac{1}{4}H_1 \otimes \frac{1}{4}X_1.
2. Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_2) = \frac{1}{4}H_1 \otimes \frac{1}{4}X_2.
3. Ad_{1F_2}(H_1) \otimes dA_{2F_2}(X_3) = \frac{1}{4}H_1 \otimes \frac{1}{4}X_3.
4. Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_1) = \frac{-1}{4}H_2 \otimes \frac{1}{4}X_1.
5. Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_2) = \frac{-1}{4}H_2 \otimes \frac{1}{4}X_2.
6. Ad_{1F_2}(H_2) \otimes dA_{2F_2}(X_3) = \frac{-1}{4}H_2 \otimes \frac{1}{4}X_3.
7. Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_1) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_1.
8. Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_2) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_2.
9. Ad_{1F_2}(H_3) \otimes dA_{2F_2}(X_3) = \frac{-1}{4}H_3 \otimes \frac{1}{4}X_3.

We can display the resulting calculations by the following table (1)

<table>
<thead>
<tr>
<th>Basis Of SU(2) \otimes sl(2)</th>
<th>F_1</th>
<th>F_2</th>
<th>F_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H_1, X_1)</td>
<td>\frac{-1}{4}H_1 \otimes \frac{1}{4}X_1</td>
<td>\frac{-1}{4}H_1 \otimes \frac{1}{4}X_1</td>
<td>\frac{1}{4}H_1 \otimes \frac{1}{4}X_1</td>
</tr>
<tr>
<td>(H_1, X_2)</td>
<td>\frac{-1}{4}H_1 \otimes \frac{1}{4}X_2</td>
<td>\frac{-1}{4}H_1 \otimes \frac{1}{4}X_3</td>
<td>\frac{1}{4}H_1 \otimes \frac{1}{4}X_2</td>
</tr>
<tr>
<td>(H_1, X_3)</td>
<td>\frac{-1}{4}H_1 \otimes \frac{1}{4}X_3</td>
<td>\frac{-1}{4}H_1 \otimes \frac{1}{4}X_2</td>
<td>\frac{1}{4}H_1 \otimes \frac{1}{4}X_3</td>
</tr>
<tr>
<td>(H_2, X_1)</td>
<td>\frac{-1}{4}H_2 \otimes \frac{1}{4}X_1</td>
<td>\frac{1}{4}H_2 \otimes \frac{1}{4}X_1</td>
<td>\frac{-1}{4}H_2 \otimes \frac{1}{4}X_1</td>
</tr>
<tr>
<td>(H_2, X_2)</td>
<td>\frac{-1}{4}H_2 \otimes \frac{1}{4}X_2</td>
<td>\frac{1}{4}H_2 \otimes \frac{1}{4}X_3</td>
<td>\frac{-1}{4}H_2 \otimes \frac{1}{4}X_2</td>
</tr>
<tr>
<td>(H_2, X_3)</td>
<td>\frac{-1}{4}H_2 \otimes \frac{1}{4}X_3</td>
<td>\frac{1}{4}H_2 \otimes \frac{1}{4}X_2</td>
<td>\frac{-1}{4}H_2 \otimes \frac{1}{4}X_3</td>
</tr>
<tr>
<td>(H_3, X_1)</td>
<td>\frac{1}{4}H_3 \otimes \frac{1}{4}X_1</td>
<td>\frac{-1}{4}H_3 \otimes \frac{1}{4}X_1</td>
<td>\frac{1}{4}H_3 \otimes \frac{1}{4}X_1</td>
</tr>
<tr>
<td>(H_3, X_2)</td>
<td>\frac{1}{4}H_3 \otimes \frac{1}{4}X_2</td>
<td>\frac{-1}{4}H_3 \otimes \frac{1}{4}X_3</td>
<td>\frac{1}{4}H_3 \otimes \frac{1}{4}X_2</td>
</tr>
<tr>
<td>(H_3, X_3)</td>
<td>\frac{1}{4}H_3 \otimes \frac{1}{4}X_3</td>
<td>\frac{-1}{4}H_3 \otimes \frac{1}{4}X_2</td>
<td>\frac{1}{4}H_3 \otimes \frac{1}{4}X_3</td>
</tr>
</tbody>
</table>

Table 1: (Ad_1 \otimes Ad_2)

2. DUAL REPRESENTATIONS

**Definition 2.1:** [2] Suppose G is a Lie group and \( \Pi \) is representation of G acting on a finite dimensional vector space V. Then the dual representation \( \Pi^* \) to \( \Pi \) is the representation of G acting on \( V^* \) given by \( \Pi^*(A) = \Pi(A^{-1}))^\Pi, \forall A \in G \). The dual representation is also called contragredient representation.

**Remark 2.2:** We can extend our result of theorem 1.2 above to include dual representations which proved in similar procedure, and have the following result:

**Proposition 2.2:** Let G be a matrix Lie group, V complex vector space over a field F, \( \Pi: G \rightarrow GL(V) \) be a representation of G on V, then the dual representation \( \Pi^* \) can be completely determined by generators of G and basis of V.
In particular, proposition 2.2 applies to the dual representations $\text{Ad}_1$ and $\text{Ad}_2$ which we compute separately as follows:

### Dual representation $\text{Ad}_1$

Combining definition 2.1 and proposition 2.2 we have:

1- $\text{Ad}_1'(F_1) = [\text{Ad}_1(F_1)]^{-1} = [\text{Ad}_1(F_1)']^{-1} = [\text{Ad}_1(F_1)][\text{Ad}_1(F_1)] = 1$

i-$[\text{Ad}_1(F_1)] = [F_1^*H_1(F_1^*)] = [F_1^*H_1F_1] = -\frac{1}{4}H_1$.

ii-$[\text{Ad}_1(F_2)] = [F_2^*H_1(F_2^*)] = [F_2^*H_2F_1] = \frac{3}{4}H_2$.

iii-$[\text{Ad}_1(F_3)] = [F_3^*H_3(F_1^*)] = [F_3^*H_3F_1] = \frac{1}{4}H_3$.

2- $\text{Ad}_1'(F_2) = [\text{Ad}_1(F_2)] = [\text{Ad}_1(F_2)'] = [\text{Ad}_1(F_2)][\text{Ad}_1(F_2)] = 1$

i-$[\text{Ad}_1(F_1)] = [F_1^*H_1(F_2^*)] = [F_2^*H_1F_2] = -\frac{1}{4}H_1$.

ii-$[\text{Ad}_1(F_2)] = [F_2^*H_2(F_2^*)] = [F_2^*H_2F_2] = \frac{1}{4}H_2$.

iii-$[\text{Ad}_1(F_3)] = [F_3^*H_3(F_2^*)] = [F_3^*H_3F_2] = -\frac{1}{4}H_3$.

3- $\text{Ad}_1'(F_3) = [\text{Ad}_1(F_3)] = [\text{Ad}_1(F_3)'] = [\text{Ad}_1(F_3)][\text{Ad}_1(F_3)] = 1$

i-$[\text{Ad}_1(F_1)] = [F_1^*H_1(F_3^*)] = [F_1^*H_1F_3] = \frac{1}{4}H_1$.

ii-$[\text{Ad}_1(F_2)] = [F_2^*H_2(F_3^*)] = [F_2^*H_2F_3] = -\frac{1}{4}H_2$.

iii-$[\text{Ad}_1(F_3)] = [F_3^*H_3(F_3^*)] = [F_3^*H_3F_3] = \frac{1}{4}H_3$.

We can display the resulting calculations as table 2 below.

<table>
<thead>
<tr>
<th>Basis of $\text{su}(2)$</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basis of SU(2)</td>
<td>$F_1$</td>
<td>$F_2$</td>
<td>$F_3$</td>
</tr>
<tr>
<td>$F_1$</td>
<td>$-\frac{1}{4}H_1$</td>
<td>$\frac{1}{4}H_2$</td>
<td>$\frac{1}{4}H_3$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$-\frac{1}{4}H_1$</td>
<td>$-\frac{1}{4}H_2$</td>
<td>$-\frac{1}{4}H_3$</td>
</tr>
<tr>
<td>$F_3$</td>
<td>$\frac{1}{4}H_1$</td>
<td>$\frac{1}{4}H_2$</td>
<td>$-\frac{1}{4}H_3$</td>
</tr>
</tbody>
</table>

**Table-2:** The dual representation $\text{Ad}_1$

### Dual representation $\text{Ad}_2$

Using the same manner in the case of $\text{Ad}_1$ above we have:

1- $\text{Ad}_2'(F_1) = [\text{Ad}_2(F_1)]^{-1} = [\text{Ad}_2(F_1)']^{-1} = 1$

i-$[\text{Ad}_2(F_1)] = [F_1^*X_1(F_1^*)] = [F_1^*X_1F_1] = -\frac{1}{4}X_1$.

ii-$[\text{Ad}_2(F_2)] = [F_2^*X_1(F_2^*)] = [F_2^*X_1F_1] = \frac{1}{4}X_2$.

iii-$[\text{Ad}_2(F_3)] = [F_3^*X_1(F_3^*)] = [F_3^*X_1F_1] = \frac{1}{4}X_3$.

2- $\text{Ad}_2'(F_2) = [\text{Ad}_2(F_2)] = [\text{Ad}_2(F_2)'] = 1$

i-$[\text{Ad}_2(F_1)] = [F_1^*X_2(F_2^*)] = [F_2^*X_2F_1] = -\frac{1}{4}X_1$.

ii-$[\text{Ad}_2(F_2)] = [F_2^*X_2(F_2^*)] = [F_2^*X_2F_2] = -\frac{1}{4}X_2$.

iii-$[\text{Ad}_2(F_3)] = [F_3^*X_2(F_2^*)] = [F_3^*X_2F_2] = -\frac{1}{4}X_3$.

3- $\text{Ad}_2'(F_3) = [\text{Ad}_2(F_3)] = [\text{Ad}_2(F_3)'] = 1$

i-$[\text{Ad}_2(F_1)] = [F_1^*X_3(F_3^*)] = [F_1^*X_3F_3] = \frac{1}{4}X_1$.

ii-$[\text{Ad}_2(F_2)] = [F_2^*X_3(F_3^*)] = [F_2^*X_3F_3] = \frac{1}{4}X_3$.

iii-$[\text{Ad}_2(F_3)] = [F_3^*X_3(F_3^*)] = [F_3^*X_3F_3] = \frac{1}{4}X_2$. 

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We can display the resulting calculations as in table 3 below.

<table>
<thead>
<tr>
<th>Generators</th>
<th>Basis of $\text{SU}(2)$</th>
<th>Basis of $\text{sl}(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>$\frac{1}{4}X_1$</td>
<td>$\frac{1}{4}X_1$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}X_2$</td>
<td>$\frac{1}{4}X_2$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}X_3$</td>
<td>$\frac{1}{4}X_3$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$\frac{1}{4}X_1$</td>
<td>$\frac{1}{4}X_1$</td>
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<tr>
<td></td>
<td>$\frac{1}{4}X_2$</td>
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<td>$F_3$</td>
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<td>$\frac{1}{4}X_2$</td>
<td>$\frac{1}{4}X_2$</td>
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<tr>
<td></td>
<td>$\frac{1}{4}X_3$</td>
<td>$\frac{1}{4}X_3$</td>
</tr>
</tbody>
</table>

Table-3: The dual representation $\text{Ad}_2^\pi$

REFERENCES

7. Saad Owaid and Zainab Subhi, "Representations of SU(2) by the matrix Lie algebras su(2) and sl(2)", to appear, Al-Mustansiriya J. Sci. 2016.

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