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IDEALS IN ALMOST SEMILATTICE

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ABSTRACT

The concept of ideal in an Almost Semilattice (ASL) is introduced and the smallest ideal containing a given nonempty subset of an $ASL\ L$ is described. Also, several properties of ideals are derived. Proved that the set $\mathscr{I}(L)$, of all ideals in an ASL L, is a distributive lattice and also, the set $P\mathscr{I}(L)$, of all principal ideals form semilattice is established. Derived set of equivalent identities for the intersection of any family of ideals is again an ideal and a 1-1 correspondence between ideals (prime) of L and ideals (prime) $P\mathscr{I}(L)$ is established. Finally obtained, every amicable set in L is embedded in a semilattice $P\mathscr{I}(L)$.

Key Words: *Ideal, Principal Ideal, Prime Ideal, Distributive Lattice, Complete Lattice, Amicable Set.*

AMS Subject classification (2000): *06A99, 06D10..*

1.INTRODUCTION

Ideals were first studied by Dedekind, who defined the concept for rings of algebraic integers. Later the concept of ideal was extended to rings in general. M. H. Stone investigated ideals in Boolean rings, which are lattice of a special kind. There is already a well-developed theory of ideals in lattice. We wish to show that it is useful to extend the notion of ideal to the more general systems called Almost Semilattice.

There are only one reasonable way of defining what is to be meant by an ideal in a lattice. Recall that Dedekind's definition of an ideal in a ring R is that it is a collection J of elements of R which (1) contains the difference $a-b$, and hence the sum $a+b$, of any two of its elements a and b for all $a,b\in J$, and (2) contains all multiples such as ax or ya of any of $x,y\in R$ and $a\in J$, By analogy, a collection J of elements of a lattice L is called an ideal if (1) it contains the lattice sum $a\cup b$ of any two of its elements a and b, and (2) it contains all multiples $a\cap x$ of any $x\in L$ and $a\in J$. The analogy is that the greatest lower bound , or lattice meet $a\cap b$ corresponds to product in a ring, and the least upper bound, or lattice join $a\cup b$ corresponds to the sum of two elements in a ring.

In this paper, we introduce the concept of an ideal and smallest ideal containing a given nonempty set in an $ASLL$ with binary operation $\circ$ and prove that the set $\mathscr{I}$(L) of all ideals of L forms a distributive lattice, the set P$\mathscr{I}$(L), of all principal ideals of an $ASL\ L$ is a semilattice. Also, given an equivalent conditions for the intersections of arbitrary family of ideals in an $ASL\ L$ is again an ideal. We establish a one-to-one correspondence between ideals of L and ideals of P$\mathscr{I}$(L), in particular a one-to-one correspondence between prime ideal of L and prime ideal of P$\mathscr{I}$(L). In this paper, we prove that if I is an ideal of L and K be a nonempty subset of L which is closed under the operation $\circ$ and $I\cap K=\emptyset$, then there exists a prime ideal P of L such that $I\subseteq P$ and $P\cap K=\emptyset$. Finally, we obtain every amicable set is embedded in a semilattice P$\mathscr{I}$(L).

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In section 2, we collect a few important definitions and results which are already known and which will be used more frequently in the paper. In section 3, we introduce the concept of ideal in an ASL L and prove that the set $\mathcal{I}$(L) of all ideals of L forms a distributive lattice for which the set P$\mathcal{I}$(L) of all principal ideals of L is a semilattice. In this section, we derive set of identities for the intersection of arbitrary family of ideals in L is again an ideal and hence the set $\mathcal{I}$(L) is a complete lattice. In section 4, we define a prime ideal in an almost semilattice and prove that a one-to-one correspondence between $\mathcal{I}$(L) and the set of all ideals in P$\mathcal{I}$(L) also, prove that a one-to-one correspondence between the set of all prime ideals in L and the set of all prime ideals in P$\mathcal{I}$(L). Finally, we prove that every amicable set in L is embedded in the semilattice P$\mathcal{I}$(L).

2.PRELIMINARY

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1: [2] A semilattice is an algebra $(S,*)$ satisfying where S is nonempty set and $*$ is a binary operation on S satisfying:

1. $x*(y*z)=(x*y)*z$
2. $x*y=y*x$
3. $x*x=x,$ for all $x,y,z\in S$.

Definition 2.2: An ideal in a semilattice L is a nonempty subset which closed under initial segments.

Definition 2.3: A proper ideal P of a semilattice $(L,\circ)$ is said to be prime ideal if for any $a,b\in L, a\circ b\in P$, then either $a\in P$ or $b\in P$.

In other words, a semilattice is an idempotent commutative semigroup. The symbol $*$ can be replaced by any binary operation symbol, and in fact we use one of the symbols of $\wedge, \vee, +$ or $.$, depending on the setting. The most natural example of a semilattice is $(\mathsf{P}(X)\ ,\ \cap)$, or more generally any collection of subsets of X closed under intersection. A sub semilattice of a semilattice $(S\ ,\ *)$ is a subset of S which is closed under the operation $*$. A homomorphism between two semilattices $(S,*)$ and $(T,*)$ is a map $h:S\to T$ with the property that $h(x*y)=h(x)\ *h(y)$ far all $x,y\in S$. An isomorphism is a homomorphism that is 1-1 and onto. It is worth nothing that, because the operation is determined by the order and vice versa.Also, it can be easily observed that two semilattices are isomorphic if and only if they are isomorphic as ordered sets.

Definition 2.4: [3] An algebra $(L,\circ)$ of type (2) is called an Almost Semilattice if it satisfies the following axioms:

(AS_1)	$(x\circ y)\circ z=x\circ(y\circ z)$	(Associative Law)
(AS_2)	$(x\circ y)\circ z=(y\circ x)\circ z$	(Almost Commutative Law)
(AS_3)	$x\circ x=x$	(Idempotent Law)

Definition 2.5: [3] Let L be a nonempty set. Define a binary operation $\circ$ on L by: $x\circ y=y$, for all $x,y\in L$. Then $(L,\circ)$ is called discrete ASL.

Definition 2.6: [3] For any $a,b\in L$ where L is an ASL, we say that a is less or equal to b and write $a\le b$, if $a\circ b=a$.

Definition 2.7: [3] Let L be an ASL. Then for any $a,b\in L$, we say that a is $compatible$ with b and write $a\ \sim b$ if and only if $a\circ b=b\circ a$. A subset S of L is said to be $compatible\ set$ if $a\ \sim b$, for all $a,b\in S$

Definition 2.8: [3] Let L be an ASL. Then a maximal compatible set in L is called a $maximal\ \ set$.

Definition 2.9: [3] Let M be a maximal set in L. Then an element $x\in L$ is said to be $M-\ amicable$ if there exists $a\in M$ such that $a\circ x=x$.

Lemma 2.10: [3] Let L be an ASL. Then for any $a, b \in L, a \circ b = b \circ a$ whenever $a \leq b$.

Theorem 2.11: [3] Let M be a maximal set in L and $a \in M$. Then for any $x \in L, x \circ a \in M$.

Corollary 2.12: [3] If M is a maximal set and $x \in L$ is M-amicable, then there is a smallest element $a \in M$ with the property $a \circ x = x$. We denote this element a of L by x^M.

Corollary 2.13: [3] Let M be a maximal set and $x \in L$ be M-amicable. Then x^M is the unique element of M such that $x^M \circ x = x$ and $x \circ x^M = x^M$.

Definition 2.14:. [3] If M is a maximal set in L, then we denote the set of all M-amicable elements of L by $A_M(L)$.

Theorem 2.15: [3] Let M be a maximal set. Then $(A_M(L), \circ)$ is an ASL. Moreover, for any $x, y \in A_M(L)$, we have $(x \circ y)^M = x^M \circ y^M$.

Definition 2.16: [3] A maximal set M in L is said to be *amicable* if $A_M(L) = L$. That is, every element in L is M-amicable.

Definition 2.17: [3] An element $m \in L$ is said to be *unimaximal* if $m \circ x = x$ for all $x \in L$.

Definition 2.18: If $(P, \leq)$ is a poset which is bounded above in which every nonempty subset of P has glb, then every nonempty subset of P has lub and hence is a complete lattice.

3.IDEALS

In this section, we introduce the concept of an ideal in an $ASL\ L$ and describe the smallest ideal containing a given nonempty subset of L. We further prove that the set $\mathcal{I}(L)$ of all ideals of L forms a distributive lattice for which the set $P\mathcal{I}(L)$ of all principal ideals of L is a semilattice.

Throughout the remaining of this section, by L we mean an ASL $(L, \circ)$ unless otherwise specified. In the following, we give the definition of an ideal in an $ASL\ L$.

Definition 3.1: A nonempty subset I of an $ASL\ L$ is said to be an *ideal* if $x \in I$ *and* $a \in L$, then $x \circ a \in I$.

From the definition of ideal in $ASL\ L$, it can be easily seen that every ideal is closed under the operation $\circ$ and hence every ideal is a sub ASL of L. But, any subset of L which is closed under the operation $\circ$ need not be an ideal. For, consider the following example:

Example 3.1: Let $L = \{a, b, c\}$. Define a binary operation $\circ$ on L as below:

$\circ$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

In this ASL, the set $\{a, b\}$ is closed under the operation $\circ$, but not an ideal, since $b \circ c = c \notin \{a, b\}$.

In the following theorem, we describe the ideal generated by a given nonempty subset S of L; that is, the smallest ideal of L containing S.

Theorem 3.2: Let S be a nonempty subset of L. Then $(S] = \{(\circ_{i=1}^{n} s_i) \circ x \mid x \in L,\ s_i \in S \quad where\ 1 \leq i \leq n\ and\ n\ is\ a\ positive\ integer\}$ is the smallest ideal of L containing S.

Proof: Suppose S is a nonempty subset of L. Then for any $s \in S$, we have $s = s \circ s$ and hence $S \subseteq (S]$. Thus $(S]$ is nonempty. We shall prove that $(S]$ is an ideal. Let $a \in (S]$ and $t \in L$. Then $a = (\circ_{i=1}^{n} s_i) \circ x$ for some $x \in L$ and $s_i \in S$ for all $1 \leq i \leq n$. Now, $a \circ t = \left((\circ_{i=1}^{n} s_i) \circ x\right) \circ t = (\circ_{i=1}^{n} s_i) \circ \left(x \circ t\right) = (\circ_{i=1}^{n} s_i) \circ y$ where $y = x \circ t \in L$. Thus $a \circ t \in (S]$. Hence $(S]$ is an ideal of L. Now, it remains to prove that $(S]$ is the smallest ideal of L containing S. Suppose J is an ideal of L such that $S \subseteq J$. Then for any $t \in (S]$, we have $t = (\circ_{i=1}^{n} s_i) \circ x$ for some $x \in L$ and $s_i \in S$ where $1 \leq i \leq n$. This implies that $t = (\circ_{i=1}^{n} s_i) \circ x$, $x \in L$ and $s_i \in S \subseteq J$ for all $1 \leq i \leq n$. Thus $t \in J$. Hence $(S] \subseteq J$. Therefore $(S]$ is the smallest ideal containing S.

If $S = \{a\}$, then we write $(a]$ instead of $(S]$ and is called the principal ideal of L generated by a. Now, we have the following.

Corollary 3.3: Let L be an ASL and $a \in L$. Then $(a] = \{a \circ x \mid x \in L\}$ is an ideal of L.

Corollary 3.4: For any $a, b \in L, a \in (b]$ if and only if $a = b \circ a$.

Proof: Suppose $a \in (b]$. Then $a = b \circ t$ for some $t \in L$. Now, $b \circ a = b \circ (b \circ t) = (b \circ b) \circ t = b \circ t = a$. Therefore $a = b \circ a$. Converse follows by the definition of $(a]$.

Corollary 3.5: Let I be an ideal of L. Then, for any $a, b \in L,\ \ a \circ b \in I$ if and only if $b \circ a \in I$.

Proof: Suppose I is an ideal of L and suppose that $a \circ b \in I$. Then $(a \circ b) \circ a \in I$. It follows that $b \circ a = b \circ (a \circ a) = (b \circ a) \circ a = (a \circ b) \circ a \in I$. Similarly, we can prove the converse.

Corollary 3.6: For any $a, b \in L, (a \circ b] = (b \circ a]$.

Proof: Since $(a \circ b) \circ t = (b \circ a) \circ t$ for all $t \in L$, it follows that $(a \circ b] = (b \circ a]$.

Recall that for any $a, b \in L$, with $a \leq b$, we have $a \circ b = b \circ a$. Now, we have the following.

Corollary 3.7: Let I be an ideal of L. Then, for any $x \in I$ and $a \in L$, $a \circ x \in I$ and hence I is an initial segment of L; that is, $x \in I$ and $a \in L$ such that $a \leq x$ imply that $a \in I$.

Proof: Suppose I is an ideal of L and suppose $x \in I$ and $a \in L$ such that $a \leq x$. Then $a = a \circ x = x \circ a$. It follows that $a \in I$, since $x \circ a \in I$.

It is clear that every initial segment I of L contains the zero element 0. Now, we have the following theorem.

Theorem 3.8: Let L be an ASL. Then the intersection of any class of an initial segments of L is also an initial segment of L.

Proof: Let $\{I_j\}_{j \in J}$ be a class of an initial segments of L. If the index set J is empty, then $\bigcap_{j \in J} I_j = L$, which is clearly an initial segment of L. Suppose J is nonempty. Since each initial segment contains 0, it follows that $\bigcap_{j \in J} I_j$ contains 0. Now, we shall prove that $\bigcap_{j \in J} I_j$ is an initial segment of L. Let $a \in L$ and $x \in \bigcap_{j \in J} I_j$ such that $a \leq x$. Then $x \in I_j$ for all $j \in J$ and $a \leq x$. Since I_j is an initial segment of L, $a \in I_j$ for all $j \in J$. Therefore $a \in \bigcap_{j \in J} I_j$. Thus $\bigcap_{j \in J} I_j$ is an initial segment of L.

Theorem 3.9: Let $\{I_j\}_{j\in J}$ be a nonempty class of an initials segment of an $ASL\ L$. Then $\bigcup_{j\in J} I_j$ is also an initial segment of L.

Proof: Put $I = \bigcup_{j\in J} I_j$. Since the index set is nonempty and each I_j is nonempty, it follows that I is nonempty. Suppose $a \in L$ and $x \in I$ with $a \le x$. Then $x \in I_j$ for some $j \in J$. Since I_j is an initial segment of L, $x \in I_j$ for some $j \in J$. Therefore $x \in \bigcup_{j\in J} I_j$. Thus $\bigcup_{j\in J} I_j$ is an initial segment of L.

Note that, $In(L)$ denote the set of all initial segment of an $ASL\ L$.

Theorem 3.10: $In(L)$ is a complete lattice, with respect to set inclusion, in which for any $\{I_j\}_{j\in J}$, glb $\{I_J\}_{j\in J} = \bigcap_{j\in J} I_j$ and lub $\{I_j\}_{j\in J} = \bigcup_{j\in J} I_j$.

In the following, we prove that, the set $\mathcal{I}$(L) of all ideals in an $ASLL$ is a distributive lattice. For this, first we need the following.

Definition 3.11: *Let* I *and* J *are ideals in* L. *Then* $I \cap J = \{x \in L \mid x \in I \ and \ x \in J\}$.

Lemma 3.12: *If* I *and* J *are ideals of an* $ASL\ L$, *then* $I \cap J$ *is an ideal of* L.

Proof: Suppose I and J are an ideals of L. Then clearly I and J are nonempty subsets of L. Hence we can choose $x \in I$ and $y \in J$. Then we have $x \circ y \in I$ and hence $y \circ x \in I$. Also, we have $y \circ x \in J$. Therefore $y \circ x \in I \cap J$. Hence $I \cap J$ is nonempty. Clearly $I \cap J$ is an ideal of L.

It can be easily seen that $I \cap J = \{a \circ b \mid a \in I \ and \ b \in J\}$.

Definition 3.13: *Let* I *and* J *are ideals in* L. *Then* $I \cup J = \{x \in L \mid x \in I \ or \ x \in J\}$.

Lemma 3.14: *If* I *and* J *are ideals of* L, *then* $I \cup J$ *is an ideal of* L.

Proof: Suppose I and J are ideals of L. Then clearly I and J are nonempty subsets of L. Hence $I \cup J$ is nonempty. We shall prove that $I \cup J$ is an ideal of L. Let $x \in I \cup J$ and $a \in L$. Then either $x \in I$ or $x \in J$. If $x \in I$, then $x \circ a \in I \subseteq I \cup J$. Thus $I \cup J$ is an ideal of L.

It can be easily seen that the intersection (union) of any finite family of ideals of an $ASL\ L$ is again an ideal. Now, we have the following theorem whose proof is straightforward.

Theorem 3.15: The set $\mathcal{I}$(L) of all ideals of an $ASL\ L$ is a distributive lattice with respect to set inclusion.

Next, we prove that the set P$\mathcal{I}$(L), of all principal ideals in an $ASL\ L$ is a semilattice. For this, we need the following.

Lemma 3.16: For any $a, b \in L$, $b \in (a]$ if and only if $(b] \subseteq (a]$.

Proof: Suppose $b \in (a]$. Then $b = a \circ b$. Now, let $t \in (b]$. Then $t = b \circ t = (a \circ b) \circ t = a \circ (b \circ t) \in (a]$. Thus $(b] \subseteq (a]$. Converse is trivial, since $a \in (b]$.

Lemma 3.17: Let $a, b \in L$. Then $(a] \subseteq (b]$ whenever $a \le b$.

Proof: Suppose $a \le b$. Then $a = a \circ b$. Now, let $t \in (a]$. Then $t = a \circ t = (a \circ b) \circ t = (b \circ a) \circ t = b \circ (a \circ t) \in (b]$. Therefore $(a] \subseteq (b]$.

Lemma 3.18: For any $a,b \in L$, $(a \circ b] = (a] \circ (b] = (b \circ a]$.

Proof: Suppose $a,b \in L$ and suppose $t \in (a] \circ (b] = (a] \cap (b]$. Then $t \in (a]$ and $t \in (b]$. Thus we have $t = a \circ t$ and $t = b \circ t$. Now, $t = a \circ t = a \circ (b \circ t) = (a \circ b) \circ t \in (a \circ b]$. Hence $(a] \circ (b] \subseteq (a \circ b]$. Conversely, let $t \in (a \circ b]$. Then $t = (a \circ b) \circ t = a \circ (b \circ t) \in (a]$ and also, $t = (a \circ b) \circ t = (b \circ a) \circ t = b \circ (a \circ t) \in (b]$. Thus $t \in (a] \cap (b]$. Hence $(a \circ b] \subseteq (a] \cap (b] = (a] \circ (b]$. Therefore $(a] \circ (b] = (a \circ b]$. Hence $(a \circ b] = (a] \circ (b] = (a] \cap (b] = (b] \cap (a] = (b \circ a]$.

Now, we have the following theorem, whose proof follows by the above lemmas.

Theorem 3.19: Let L be an ASL. Then the set P$\mathscr{I}$(L) of all principal ideals of L is a semilattice.

It can be easily seen that the above semilattice P$\mathscr{I}$(L) is not a sub-lattice of the distributive lattice $\mathscr{I}$(L). Let us recall that an element a of L is said to be minimal if $x \in L$, $x \le a$ imply that $x = a$. Observe that, for any $a \in L$, a is minimal if and only if $b \circ a = a$. for all $b \in L$. Also, observe that L has 0 if and only if L has unique minimal element. Also seen that, if x is a minimal element in L and I is an ideal of L, then $x \in I$. We have observed that the intersection of a finite family of ideals is again an ideal. But, the intersection of an arbitrary family of ideals need not be an ideal again, in a general ASL. In the following theorem we give set of identities for the intersection of an arbitrary family of ideals is again an ideal. For this, first we need the following.

Lemma 3.20: If L has minimal element, then the set of all minimal elements of L forms an ideal.

Proof: Suppose L has a minimal element say a. Now, put $I = \{m \mid m \textit{ is a minimal element in } L\}$. Then clearly I is nonempty, since $a \in I$. Let $x \in I$ and $t \in L$. Then $s \circ x = x$ for all $s \in L$.

Now, $s \circ (x \circ t) = (s \circ x) \circ t = x \circ t$ for all $s \in L$. Thus $x \circ t$ is a minimal element of L. Hence $x \circ t \in I$. Therefore I is an ideal.

Theorem 3.21: The following conditions are equivalent in an ASL L.

(1) The intersections of any family of ideals is nonempty
(2) The intersections of any family of ideals is again an ideal
(3) The class $\mathscr{I}$(L) has least element
(4) The class $\mathscr{I}$(L) is complete
(5) The class P$\mathscr{I}$(L) has least element
(6) L has a minimal element

Proof: Suppose $\{I_\alpha\}_{\alpha \in \delta}$ be a family of ideals in L and suppose $I = \bigcap_{\alpha \in \delta} I_\alpha$ is nonempty. Then clearly I is nonempty and I is an ideal of L. This proves $(1) \Rightarrow (2)$. If we take $I = \bigcap_{I_\alpha \in \mathrm{I}(L)} I_\alpha$. Then by (2), I is an ideal of L, and clearly I is the least element of $\mathscr{I}$(L). This proves $(2) \Rightarrow (3)$. Since $\mathscr{I}$(L) is bounded above by L and any nonempty subset of $\mathscr{I}$(L) has glb, follows by $\mathscr{I}$(L) has least element. Hence $(3) \Rightarrow (4)$. Clearly $(4) \Rightarrow (5)$. $(5) \Rightarrow (6)$: Suppose P$\mathscr{I}$(L) has least element say $(a]$. Now, we shall prove that a is a minimal element in L. Suppose $x \in L$ such that $x \le a$. Then we have $(x] \subseteq (a]$. It follows that $(x] = (a]$, since $(a]$ is minimal. Hence $a \in (a] = (x]$. Therefore $a = x \circ a = a \circ x$, since $x \le a$. Hence $a \le x$ Therefore by antisymmetric, $x = a$. Thus a is minimal. $(6) \Rightarrow (1)$ follows by every ideal contains a minimal element.

4.THE SEMILATTICE P$\mathscr{I}$(L)

We have proved in the previous section that the class P$\mathscr{I}$(L) of all principal ideals of L forms a semilattice. In this section, we prove that a one-to-one correspondence between set of all ideals (prime ideals) in L and set of all ideals (prime ideals) in P$\mathscr{I}$(L).

Throughout the remaining of this section, by L we mean an ASL $(L,\circ)$ unless otherwise specified. In view of $corolary\ 3.5$, we give the following definition of a prime ideal in an ASL L which coincides with the well known concepts of prime ideal in semilattice.

Definition 4.1: A proper ideal P of L is said to be $a\ prime\ ideal$ if for any $x, y \in L$, $x \circ y \in P$ implies that $x \in P$ or $y \in P$.

Lemma 4.2: A proper ideal P of L is prime if and only if for any ideals I and J of L, $I \cap J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Proof: Suppose I and J are an ideals of L such that $I \cap J \subseteq P$. If $I \subseteq P$, then the $lemma$ holds true. Assume that $I \not\subset P$. Then there exists $x \in I$ such that $x \notin P$. Let $y \in J$. Then $y \circ x \in J$ and hence $x \circ y \in J$. Also, $x \circ y \in I$. Hence $x \circ y \in I \cap J \subseteq P$. Since P is prime ideal and $x \notin P$, $y \in P$. Thus $J \subseteq P$. Conversely, assume the condition. We shall prove that P is a prime ideal. Let $x, y \in L$ such that $x \circ y \in P$. Then $(x] \circ (y] = (x \circ y] \subseteq P$. Therefore $(x] \subseteq P$ or $(y] \subseteq P$. Hence $x \in P$ or $y \in P$. Therefore P is prime.

The following theorem establishes the relation between the ideals (prime ideals) of L and the ideals (prime ideals) of the semilattice P$\mathscr{I}$(L).

Theorem 4.3: *Let* L *be an* ASL. *Then we have the following:*

1. For any ideal I of L, $I^e := \{(a] \mid a \in I\}$ is an ideal of P$\mathscr{I}$(L). Moreover, I is prime if and only if I^e is prime.
2. For any ideal K of the semilattice P$\mathscr{I}$(L), $K^c = \{a \in L \mid (a] \in K\}$ is an ideal of L. Further, K is prime if and only if so is K^c.
3. For any ideals I_1 and I_2 of L, $I_1 \subseteq I_2$ if and only if $I_1^{\,e} \subseteq I_2^{\,e}$.
4. For any ideals K_1 and K_2 of P$\mathscr{I}$(L), $K_1 \subseteq K_2$ if and only if $K_1^{\,c} \subseteq K_2^{\,c}$.
5. $I^{e^c} = I$ for all ideals I of L.
6. $K^{c^e} = K$ for all ideals K of P$\mathscr{I}$(L).

Proof:

1. Suppose I is an ideal of L. Then $I^e = \{(a] \mid a \in I\}$. Now, we shall prove that I^e is an ideal of P$\mathscr{I}$(L). Since I is nonempty, it follows that I^e is nonempty. Let $(a] \in I^e$ and $(t] \in$ P$\mathscr{I}$(L). Then $a \in I$ and $t \in L$. Therefore $a \circ t \in I$. Hence $(a] \circ (t] = (a \circ t] \in I^e$. Thus I^e is an ideal of P$\mathscr{I}$(L). Suppose I is a prime ideal of L. We shall prove that I^e is a prime ideal of P$\mathscr{I}$(L). Let $(a], (b] \in$ P$\mathscr{I}$(L) such that $(a] \circ (b] \in I^e$. Then $(a \circ b] \in I^e$. Therefore $(a \circ b] = (t]$ for some $t \in I$. Since $a \circ b \in (a \circ b] = (t]$, $a \circ b = t \circ (a \circ b)$. Therefore $a \circ b \in I$. Since I is prime, either $a \in I$ or $b \in I$. It follows that $(a] \in I^e$ or $(b] \in I^e$. Thus I^e is a prime ideal of P$\mathscr{I}$(L). Conversely, suppose I^e is a prime ideal of P$\mathscr{I}$(L). Let $a, b \in L$ such that $a \circ b \in I$. Then $(a] \circ (b] = (a \circ b] \in I^e$. Therefore $(a] \in I^e$ or $(b] \in I^e$. Hence $(a] = (s]$ or $(b] = (t]$ for some $s, t \in I$. Therefore $a = s \circ a \in I$ or $b = t \circ b \in I$ and hence I is prime.
2. Suppose K is an ideal of P$\mathscr{I}$(L). Then $K^c = \{a \in L \mid (a] \in K\}$. We shall prove that K^c is an ideal of L. Since K is nonempty, K^c is nonempty. Let $a \in K^c$ and $t \in L$. Then $(a] \in K$ and $(t] \in$ P$\mathscr{I}$(L). Therefore $(a \circ t] = (a] \circ (t] \in K$. Hence $a \circ t \in K^c$. Thus K^c is an ideal of L. Now, suppose K is a prime ideal of P$\mathscr{I}$(L). We shall prove that K^c is a prime ideal of L. Let $a, b \in L$ such that $a \circ b \in K^c$. Then $(a] \circ (b] = (a \circ b] \in K$. Therefore either $(a] \in K$ or $(b] \in K$, since K is prime. It follows that

$a \in K^c$ or $b \in K^c$. Hence K^c is a prime ideal of L. Conversely, suppose K^c is a prime ideal of L. Now, let $(a],(b] \in$ P$\mathscr{I}$(L) such that $(a] \circ (b] \in K$. Then $(a \circ b] \in K$. It follows that $a \circ b \in K^c$. Since K^c is prime, either $a \in K^c$ or $b \in K^c$. Therefore $(a] \in K$ or $(b] \in K$. Hence K is a prime ideal of P$\mathscr{I}$(L).

3. Suppose I_1 and I_2 are ideals of L such that $I_1 \subseteq I_2$. Let $(a] \in I_1^{\,e}$. Then $a \in I_1$ and hence $a \in I_2$. Therefore $(a] \in I_2^{\,e}$. Thus $I_1^{\,e} \subseteq I_2^{\,e}$. Conversely, suppose $I_1^{\,e} \subseteq I_2^{\,e}$. Let $a \in I_1$. Then $(a] \in I_1^{\,e} \subseteq I_2^{\,e}$. Therefore $(a] = (t]$ for some $t \in I_2$. Hence $a = t \circ a \in I_2$. Thus $I_1 \subseteq I_2$.
4. Suppose K_1 and K_2 are an ideals of P$\mathscr{I}$(L) such that $K_1 \subseteq K_2$. Let $a \in K_1^{\,c}$. Then $(a] \in K_1$. Thus $(a] \in K_2$. Therefore $(a] = (t]$ for some $t \in K_2$. Hence $a = t \circ a \in K_2^{\,c}$. Thus $K_1^{\,c} \subseteq K_2^{\,c}$. Conversely, suppose $K_1^{\,c} \subseteq K_2^{\,c}$. Let $(a] \in K_1$. Then $a \in K_1^{\,c} \subseteq K_2^{\,c}$. Thus $a \in K_2^{\,c}$. Hence $(a] \in K_2$. Therefore $K_1 \subseteq K_2$.
5. Suppose $a \in I^{e^c}$. Then $(a] \in I^e$. Therefore $(a] = (t]$ for some $t \in I$. Hence $a = t \circ a \in I$. Therefore $I^{e^c} \subseteq I$. Clearly $I \subseteq I^{e^c}$. Thus $I = I^{e^c}$. Similarly, we can prove (6).

Lemma 4.4: *Let I and J be an ideals of L. Then $(I \cap J)^e = I^e \cap J^e$.*

Proof: Suppose I and J are an ideals of L. Then $I \cap J \subseteq I, J$. Therefore by *theorem* 4.3, we have $(I \cap J)^e \subseteq I^e, J^e$. Hence $(I \cap J)^e \subseteq I^e \cap J^e$. Conversely, suppose $(a] \in I^e \cap J^e$. Then $(a] \in I^e$ and $(a] \in J^e$. Hence $(a] = (t]$ for some $t \in I$ and $(a] = (s]$ for some $s \in J$. Therefore $a \in (a] = (t]$ and hence $a = t \circ a$. Similarly we get $a = s \circ a$. Since $t \in I$, $a = t \circ a \in I$. Similarly we get $a \in J$. Hence $a \in I \cap J$. It follows that $(a] \in (I \cap J)^e$. Therefore $I^e \cap J^e \subseteq (I \cap J)^e$ and hence $(I \cap J)^e = I^e \cap J^e$.
Thus we have the following theorem, whose proof follows by *theorem* 4.3 and *lemma*4.4.

Theorem 4.5: The mapping $I \mapsto I^e$ is a one-to-one correspondence of $\mathscr{I}$(L) onto $\mathscr{I}$(P$\mathscr{I}$(L)). Moreover, this correspondence gives one- to - one correspondence between the prime ideals of L and those of P$\mathscr{I}$(L).

Now, we prove the following theorem.

Theorem 4.6: Let L be an ASL with a minimal element and let M_o denote the least element of $\mathscr{I}$(L). Then M_o contains precisely the minimal elements of L.

Proof: Suppose $x \in M_o$. We shall prove that x is minimal. Suppose $a \in L$ such that $a \le x$. Then by *lemma* 3.17 $(a] \subseteq (x]$ and $(x] \subseteq M_o$. On the other hand, M_o is the least element of $\mathscr{I}$(L), $M_o \subseteq (a] \subseteq (x]$. It follows that $M_o \subseteq (a] \subseteq (x] \subseteq M_o$. Hence $M_o = (a] = (x]$. Therefore $x \in (x] = (a]$, and hence $x = a \circ x = a$, since $a \le x$. Thus x is minimal. Now, suppose $x \in L$ such that x is minimal. Since M_o is an ideal, we can choose $a \in M_o$. Therefore $x = a \circ x$ since x is minimal. Thus $x \in M_o$, since $a \in M_o$. Thus M_o contains precisely all minimal elements in L.

Corollary 4.7: Let L be an ASL with a minimal element. Then, for any $x, y \in L$, $x \circ y$ is minimal if and only if $y \circ x$ is minimal.

Proof: We have, for any ideal I of L, $x \circ y \in I$ if and only if $y \circ x \in I$. It follows that $x \circ y$ is minimal if and only if $y \circ x$ is minimal.
Already, we have observed that a nonempty subset of an ASL L which is closed under the binary operation $\circ$ need not be an ideal. Now, we have the following theorem.

Theorem 4.8: Let I be an ideal of L and K be a nonempty subset of L which is closed under the operation $\circ$ with $I \cap K = \emptyset$. Then there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap K = \emptyset$.

Proof: Write $T = \{J \in \mathcal{I}(L) \mid I \subseteq J \text{ and } J \cap K = \emptyset\}$. Then $T \neq \emptyset$, since $I \in T$. Clearly T is a poset under set inclusion and it can be easily verified that T satisfies the hypothesis of Zorn's lemma. Therefore, by Zorn's lemma, T has maximal element say P. We shall prove that P is prime. Let $x, y \in L$ such that $x \notin P$ and $y \notin P$. Then $P \subset (x] \cup P$ and $P \subset (y] \cup P$. Therefore $(x] \cup P, (y] \cup P \notin T$, since P is the maximal element in T. Hence $((x] \cup P) \cap K \neq \emptyset$ and $((y] \cup P) \cap K \neq \emptyset$. Choose $t_1, t_2 \in L$ such that $t_1 \in ((x] \cup P) \cap K$ and $t_2 \in ((y] \cup P) \cap K$. Then we have $t_1, t_2 \in K$ and hence $t_1 \circ t_2 \in K$. Also, $t_1 \in (x] \cup P$ and hence $t_1 \circ t_2 \in (x] \cup P$. Similarly, $t_1 \circ t_2 \in (y] \cup P$. It follows that $t_1 \circ t_2 \in ((x] \cup P) \cap ((y] \cup P)$. Now, $((x] \cup P) \cap ((y] \cup P) = ((x] \cap (y]) \cup P = ((x] \circ (y]) \cup P = (x \circ y] \cup P$. If $x \circ y \in P$, then $(x \circ y] \subseteq P$. Hence $(x \circ y] \cup P = P$. Thus $t_1 \circ t_2 \in P$. Therefore $t_1 \circ t_2 \in P \cap K$ which is a contradiction to $P \cap K = \emptyset$. Hence $x \circ y \notin P$. Thus P is a prime ideal of L. Therefore there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap K = \emptyset$.

Corollary 4.9: Let I be an ideal of L and $a \notin I$. Then there exists a prime ideal P of L such that $I \subseteq P$ and $a \notin P$

Corollary 4.10: If $0 \neq a \in L$, then there exist a prime ideal P of L such that $0 \in P$ and $a \notin P$.

Corollary 4.11: Let I be a proper ideal of L. Then the intersection of all prime ideals of L containing I is I itself.

Proof: Suppose I is a proper ideal of L and write $T = \{P \mid P \text{ is a prime ideal of } L \text{ and } I \subseteq P\}$. Put $J = \bigcap_{P \in T} P$. We shall prove that $I = J$. Since $I \subseteq P$ for all $P \in T$, $I \subseteq J$. Conversely, suppose $J \not\subset I$. Then there exist $x \in J$ such that $x \notin I$. By Corollary 4.9, there exists a prime ideal P of L such that $I \subseteq P$ and $x \notin P$. Therefore $x \notin J$ which is a contradiction to $x \in J$. Thus $J \subseteq I$. Hence $I = J$.

Corollary 4.12: Let $a, b \in L$ and $(a] \neq (b]$. Then there exists a prime ideal P of L containing a and not containing b or vice versa.

Proof: Suppose $a, b \in L$ such that $(a] \neq (b]$. Then either $(a] \not\subset (b]$ or $(b] \not\subset (a]$. Without loss of generality, assume that $(a] \not\subset (b]$. Then $a \notin (b]$. Therefore $\{a\} \cap (b] = \emptyset$. Now, by corollary 4.8, there exists a prime ideal P of L such that $(b] \subseteq P$ and $\{a\} \cap P = \emptyset$. Thus $b \in P$ and $a \notin P$. Similarly, we can prove that $a \in P$ and $b \notin P$.

Corollary 4.13: If $a \in L$ is not minimal, then there exists a prime ideal of L not containing a.

Proof: Suppose a is not a minimal element of L. Then there exist $x \in L$ such that $x \circ a \neq a$. Thus $(x \circ a] \neq (a]$. Suppose $(x \circ a] = (a]$. Then $a \in (a] = (x \circ a]$. Hence $a = (x \circ a) \circ a = x \circ (a \circ a) = x \circ a$ which is a contradiction to $a \neq x \circ a$. Thus by Corollary 4.12, there exist a prime ideal of L not containing a.

Theorem 4.14. The following are equivalent, in L.

1. The intersection of all prime ideals of L is nonempty
2. The intersection of all prime ideals of L is again an ideal
3. L has a minimal element

In the following theorem we can see that, if an ideal I of L contains a unimaximal element, then an ideal I and an ASL L are equal.

Theorem 4.15: Let L be an ASL and $m \in L$ be unimaximal. If I is an ideal in L such that $m \in I$, then $I = L$.

Proof: Suppose I is an ideal of L and suppose $m \in I$ is a unimaximal. We shall prove that $I = L$. Now, let $x \in L$. Then $x = m \circ x$. Therefore $x \in I$. Hence $L \subseteq I$. But, $I \subseteq L$. Thus $I = L$.

Finally, we characterize amicable subsets in L. More precisely, we prove that if L has amicable set, then every amicable set is isomorphic to the semilattice P$\mathcal{I}$(L). First we need the following.

Lemma 4.16L: Let M be a maximal set in L. Then $A_M(L)$, the set of all M-amicable elements of L is an ideal of L

Proof: Suppose M is a maximal set in L. Then clearly $A_M(L)$ is nonempty, since every element in M is M-amicable. Let $a \in A_M(L)$ and $t \in L$. Then there exists $m \in M$ such that $m \circ a = a.$ Now, $a \circ t = (m \circ a) \circ t = m \circ (a \circ t)$. Thus $a \circ t \in A_M(L)$. Therefore $A_M(L)$ is an ideal of L.

Recall that if M is a maximal set and $x \in L$ is M-amicable, then there exists a unique element x^M in M with the property $x^M \circ x = x$ and $x \circ x^M = x^M$. Now, we have the following whose proof is straight forward.

Lemma 4.17: Let M be an amicable set in L. Then for any $x \in L, (x] = (x^M]$.

Theorem 4.18: Let M be an amicable set in L. Then for any $x, y \in L$, the following are equivalent:

1. $(x] = (y]$
2. $(x^M] = (y^M]$
3. $x^M = y^M$

Proof: Suppose M is an amicable set in L. Then $A_M(L) = L$. Now, let $x, y \in L = A_M(L)$. Then there exists a unique element $x^M, y^M \in M$ such that $x \circ x^M = x^M$ and $x^M \circ x = x$ and also $y \circ y^M = y^M$ and $y^M \circ y = y$. Thus by *corollary* 3.4 and *lemma* 3.16, we get $(x] = (x^M]$ and $(y] = (y^M]$. Therefore $(1) \Rightarrow (2)$ is clear. Assume (2). Then $x^M \in (x^M] = (y^M]$. Thus $x^M = y^M \circ x^M = x^M \circ y^M$, since $x^M, y^M \in M$. Hence $x^M \leq y^M$. Similarly we get $y^M \leq x^M$. Therefore $x^M = y^M$. Assume (3). We need to show that $(x] = (y]$. Let $t \in (x]$.

Then $t = x \circ t = (x^M \circ x) \circ t = (x \circ x^M) \circ t = x^M \circ t = y^M \circ t = (y \circ y^M) \circ t = (y^M \circ y) \circ t = y \circ t \in (y]$.

Hence $(x] \subseteq (y]$. Similarly we can prove that $(y] \subseteq (x]$. Therefore $(x] = (y]$.

Corollary 4.19: Let M be a maximal set in L. Then for any $x, y \in M$, the following are equivalent.

1. $x = y$
2. $(x] = (y]$

Recall that if M is a maximal set in L and $a \in M$, then for any $x \in L$, $x \circ a \in M$. Finally we prove the following theorem.

Theorem 4.20: Let M be an amicable set in L. Then the mapping $x \mapsto (x]$ is an isomorphism of M onto a semilattice P$\mathcal{I}$(L).

Proof: Let M be an amicable set in L. Define, $f : M \to$ P$\mathcal{I}$(L) by $f(x) = (x]$ for all $x \in M$. Then f is both well defined and 1-1. Also, let $(x] \in$ $\mathcal{I}$(L). Then $x \in L = A_M(L)$. Therefore there exists $a \in M$ such that $a \circ x = x$. Since $x \circ a \in M$, we get $f(x \circ a) = (x \circ a] = (a \circ x] = (x]$. Hence f is onto. Now, it remains to show that f is a homomorphism. Suppose $x, y \in M$. Then $f(x \circ y) = (x \circ y] = (x] \circ (y] = f(x) \circ f(y)$.

Therefore f is a homomorphism. Thus f is an isomorphism.

From the above theorem, every amicable set is embedded in the semilattice $P\mathscr{I}(L)$.

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