BETWEEN REGULAR OPEN SETS AND OPEN SETS

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(Received On: 09-05-16; Revised & Accepted On: 31-05-16)

ABSTRACT

In this paper, we introduce a new class of sets called regular*-open sets, this class of sets lies between the classes of regular open and open sets. We also study its fundamental properties and compare it with some other types of sets and we investigate further topological properties of sets.

Mathematical Subject Classification: 540A5.

Keywords and Phrases: regular open set, generalized closure, regular*-open set, regular*-interior, regular*-closed set, regular*-closure.

1. INTRODUCTION

In 1970 Levine [4] introduced generalized closed sets. In 1937, regular open sets were introduced and used to define the semi-regularization space of a topological space. Dunham [2] introduced the concept of generalized closure using Levine’s generalized closed sets and defined new topology \( \tau^* \) and studied some of their properties.

In this paper, we define a new class of sets, namely regular*-open sets, using the generalized closure operator \( \text{Cl}^* \) due to Dunham. We investigate fundamental properties of regular*-open sets. We also define regular*-interior point and regular*-interior of a subset. We also introduce the concept of regular*-closed sets and investigate many fundamental properties of regular*-closed sets. We also define the regular*-closure of a subset and study their properties.

2. PRELIMINARIES

Throughout this paper, \((X, \tau)\) will always denote topological space on which no separation axioms are assumed, unless explicitly stated. If \(A\) is a subset of the space \((X, \tau)\), \(\text{Cl}(A)\) and \(\text{Int}(A)\) denote the closure and interior of \(A\) respectively.

Definition 2.1: A subset \(A\) of a topological space \((X, \tau)\) is said to be

(i) **regular open** [11] if \(A = \text{Int}(\text{Cl}(A))\).

(ii) **\(\alpha\)-open** [6] if \(A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))\).

(iii) **semi-open** [3] if there exits an open set \(G\) such that \(G \subseteq A \subseteq \text{Cl}(G)\) equivalently \(A \subseteq \text{Cl}(\text{Int}(A))\).

(iv) **pre-open** [5] if \(A \subseteq \text{Int}(\text{Cl}(A))\).

Definition 2.2: A subset \(A\) of a topological space \((X, \tau)\) is said to be

(i) **regular closed** if \(X \setminus A\) is regular open in \(X\) (i.e.) \(A = \text{Cl}(\text{Int}(A))\).

(ii) **\(\alpha\)-closed** if \(\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A\).

(iii) **semi-closed** if \(\text{Int}(\text{Cl}(A)) \subseteq A\).

(iv) **pre-closed** if \(\text{Cl}(\text{Int}(A)) \subseteq A\).
Remark 2.3:
(i) The class of all regular open sets in \((X, \tau)\) is denoted by RO \((X, \tau)\).
(ii) The class of all regular closed sets in \((X, \tau)\) is denoted by RC \((X, \tau)\).

Definition 2.4:
(i) The class of all regular open sets in \((X, \tau)\) is denoted by RO \((X, \tau)\).
(ii) The class of all regular closed sets in \((X, \tau)\) is denoted by RC \((X, \tau)\).

Definition 2.5:
(i) A subset \(A\) of a topological space \((X, \tau)\) is said to be generalized closed \([4]\) (briefly g-closed) if \(Cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
(ii) A subset \(A\) of a topological space \((X, \tau)\) is said to be generalized open \([4]\) (briefly g-open) if \(X \setminus A\) is g-closed in \(X\).

Definition 2.6:
(i) The finite intersection of regular open sets is regular open.
(ii) The finite union of regular closed sets is regular closed.

Remark 2.7:
(i) The union of two regular open sets need not be regular open.
(ii) Every regular open set is open.
(iii) Every clopen set is regular open.
(iv) The intersection of two regular closed sets need not be regular closed.

Theorem 2.8:
(i) \(\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} Int(A_i)\).
(ii) \(\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} Cl(A_i)\).

Theorem 2.9:
(i) \(Int(X \setminus A) = X \setminus Cl(A)\).
(ii) \(Cl(X \setminus A) = X \setminus Int(A)\).
(iii) \(X \setminus (X \setminus A) = A\).

Definition 2.10:
A subset \(A\) of a topological space \((X, \tau)\) is called clopen if it is both open and closed in \((X, \tau)\).

3. REGULAR STAR OPEN SETS

Definition 3.1:
A subset \(A\) of a topological space \((X, \tau)\) is said to be regular*-open if \(A = Int(Cl^*(A))\).

Notation 3.2:
The set of all regular*-open sets (r*-open sets) in \((X, \tau)\) is denoted by \(R^*O(X, \tau)\) (or) \(R^*O(X)\).

Remark 3.3:
In any space \((X, \tau)\), \(\emptyset\) and \(X\) are regular*-open sets.

Example 3.4:
Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}\). In this space, regular open sets are \(\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\}\) and regular*-open sets are \(\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}\).

Example 3.5:
Let \(X = \{a, b, c, d, e\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d, e\}, X\}\). In this space, regular open sets are \(\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d, e\}, X\}\) and regular*-open sets are \(\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d, e\}, X\}\).

Example 3.6:
Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c, d\}, X\}\). In this space, regular open sets are \(\{\emptyset, \{a\}, \{b, c, d\}, X\}\) and regular*-open sets are \(\{\emptyset, \{a\}, \{b, c, d\}, X\}\).

Remark 3.8:
The union of two regular*-open sets need not be regular*-open, as seen from the following example.
Example 3.9: Consider the space as in Example: 3.4. In this space (X, τ), the subsets A={a, b} and B={b, c} are regular*-open sets, but A∪B = {a, b, c} is not regular*-open.

Theorem 3.10: Intersection of any two regular*-open sets is regular*-open.

Proof: Let A and B be regular*-open sets then \( A = \text{Int} (\text{Cl}^*(A)) \), \( B = \text{Int} (\text{Cl}^*(B)) \). Consider,

\[
\text{Int}(\text{Cl}^*(A \cap B)) = \text{Int}(\text{Cl}(X \setminus (A \cap B))) = \text{Int}(X \setminus \text{Int}((X \setminus A) \cup (X \setminus B))) = \text{Int}(\text{Cl}^*(X \setminus (X \setminus A) \cup (X \setminus B)))
\]

\[
= \text{Int}(\text{Cl}^*(X \setminus A) \cap \text{Cl}^*(X \setminus B)) = \text{Int}(\text{Cl}^*(A) \cap \text{Cl}^*(B)) = \text{Int}(\text{Cl}^*(A)) \cap \text{Int}(\text{Cl}^*(B)) = A \cap B.
\]

Hence A ∩ B is regular*-open set.

Theorem 3.11: \( R^*(X, \tau) \) forms a topology on X if and only if it is closed under arbitrary union.

Proof: Follows from Remark 3.3 and Theorem 3.10

Theorem 3.12: Every regular open set is regular*-open.

Proof: Let A be a regular open set then \( A = \text{Int} (\text{Cl}(A)) \). Since A is regular open, it is clopen (i.e) A is closed and every closed set is generalized closed, hence \( \text{Cl}(A) = \text{Cl}^*(A) \Rightarrow \text{Int}(\text{Cl}(A)) = \text{Int}(\text{Cl}^*(A)) \). Hence A is regular*-open.

Remark 3.13: Converse of the above theorem 3.12 need not be true, as seen from the following example.

Example 3.14: Consider the space as in Example 3.5, in this space (X, τ), the set \{a, b\} is regular*-open but not regular open.

Theorem 3.15: Every regular*-open set is open.

Proof: Let A be a regular*-open set then \( A = \text{Int} (\text{Cl}^*(A)) \). Now \( \text{Int}(A) = \text{Int}(\text{Int}(\text{Cl}^*(A))) = \text{Int}(\text{Cl}^*(A)) = A \). (i.e) \( \text{Int}(A) = A \).

Hence A is open.

Remark 3.16: The converse of the above theorem 3.15 need not be true, as seen from the following example.

Example 3.17: Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \). In this space, \( R^*(X) = \{\phi, \{a\}, \{b\}, \{a, b, c\}, X\} \). Here, the set \{a, b, c\} is open but not regular*-open.

Theorem 3.18: In any topological space (X, τ), \( RO(X, \tau) \subseteq R^*(X, \tau) \subseteq \tau \). That is, the class of regular*-open sets is placed between the class of regular open sets and the class of open sets.

Proof: Follows from Theorem 3.12 and Theorem 3.15

Theorem 3.19: Every regular*-open sets are pre-open.

Proof: Let A be a regular*-open set, then \( A = \text{Int} (\text{Cl}^*(A)) \). Since \( \text{Cl}^*(A) \subseteq \text{Cl}(A) \) Then \( \text{Int}(\text{Cl}^*(A)) \subseteq \text{Int}(\text{Cl}(A)) \). Hence A is pre-open.

Remark 3.20: The converse of the above theorem 3.19 need not be true, as seen from the following example.

Example 3.21: Consider the space as in Example 3.7, in this space (X, τ), the subset \{a, b, c, d\} is pre-open but not regular*-open.

From the above discussions we have the following implication diagram

Diagram 3.22:
Definition 3.23: The regular*-interior of $A$ is defined as the union of all regular*-open sets of $X$ contained in $A$. It is denoted by $r^*\text{Int}(A)$.

Definition 3.24: Let $A$ be a subset of $X$. A point $x$ in $X$ is called a regular*-interior point of $A$ if $A$ contains a regular*-open set containing $x$.

Remark 3.25: If $A$ is any subset of $X$, $r^*\text{Int}(A)$ need not be regular*-open set, as seen from the following example.

Example 3.26: Consider the space as in Example 3.4, in this space $(X, \tau)$. Let $A = \{a, b, c\}$ then $r^*\text{Int}(A) = \{a, b, c\}$. But $\{a, b, c\}$ is not regular*-open.

Theorem 3.27: In any topological space $(X, \tau)$, if $A$ and $B$ are subsets of $X$ then the following hold:
(i) $r^*\text{Int}(\emptyset) = \emptyset$,
(ii) $r^*\text{Int}(X) = X$,
(iii) $r^*\text{Int}(A) \subseteq A$,
(iv) $A \subseteq B \Rightarrow r^*\text{Int}(A) \subseteq r^*\text{Int}(B)$,
(v) $\text{Int}(r^*\text{Int}(A)) \subseteq \text{Int}(A)$,
(vi) $r^*\text{Int}(A) \subseteq r^*\text{Int}(A) \subseteq \text{Int}(A) \subseteq A$,
(vii) $r^*\text{Int}(A \cup B) \supseteq r^*\text{Int}(A) \cup r^*\text{Int}(B)$,
(viii) $r^*\text{Int}(A \cap B) = r^*\text{Int}(A) \cap r^*\text{Int}(B)$.

Proof: (i), (ii), (iii) and (iv) follows from Definition 3.22 and (v) follows from Definition 3.22 and (iv) above. (vi) follows from Theorems 3.12 and 3.15. (vii) follows from (iv) above. (viii) follows from Theorem 2.8

Remark 3.28: In (vi) of Theorem 3.27, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 3.29: In the space $(X, \tau)$, where $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. Here $\text{RO}(X) = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\text{R*O}(X) = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Let $A = \{a, b, c\}$ then $\text{Int}(A) = r^*\text{Int}(A) = \{a, b, c\}$. Here $\text{Int}(A) = r^*\text{Int}(A) = \{a, b, c\} = A$. Let $B = \{a, b\}$ then $\text{Int}(B) = r^*\text{Int}(B) = \{a\}$. Here $\text{Int}(A) = r^*\text{Int}(A) \subseteq \text{Int}(B) = B$. Let $C = \{a, b, c, e\}$ then $\text{Int}(C) = r^*\text{Int}(C) = \{a, b, c\} = C$. Here $\text{Int}(A) = r^*\text{Int}(A) = \{a, b, c\} = C$.

Example 3.30: In the space $(X, \tau)$, where $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{b, c, d\}, \{a, b, c\}, \{a, b, c, d, e\}, X\}$. Here $\text{RO}(X) = \{\emptyset, \{a\}, \{b, c, d, e\}, X\}$ and $\text{R*O}(X) = \{\emptyset, \{a\}, \{b, c, d, e\}, \{a, b, c\}, \{a, b, c, d, e\}, X\}$. Let $A = \{a, b\}$ then $\text{Int}(A) = r^*\text{Int}(A) = \{a\}$ and $\text{Int}(A) = r^*\text{Int}(A) = \{a\}$. Here $\text{Int}(A) = r^*\text{Int}(A) = \{a\} \subseteq \text{Int}(A) = A$.

Remark 3.31: The inclusion in (vii) of Theorem 3.27 may be strict and equality may also hold. This can be seen from the following examples.

Example 3.32: Consider the space $(X, \tau)$ as in Example 3.29. Let $A = \{a\}$ and $B = \{b\}$ then $A \cup B = \{a, b\}$ and then $r^*\text{Int}(A) = \{a\}$, $r^*\text{Int}(B) = \{b\}$, $r^*\text{Int}(A \cup B) = \{a, b\}$. Here $r^*\text{Int}(A) = r^*\text{Int}(A) \cup r^*\text{Int}(B)$. Let $C = \{a, b, c\}$ and $D = \{c, e\}$ then $C \cup D = \{a, b, c, e\}$, $r^*\text{Int}(C) = \{a\}$, $r^*\text{Int}(D) = \emptyset$, $r^*\text{Int}(C \cup D) = \{a, b, c\}$. Here $r^*\text{Int}(C \cup D) \supseteq r^*\text{Int}(C) \cup r^*\text{Int}(D)$.

4. REGULAR* CLOSED SET

Definition 4.1: The complement of regular*-open set is called regular*-closed set (i.e) $A = \text{Cl}(\text{Int}^*(A))$

Notation 4.2: The set of all regular*-closed sets in $(X, \tau)$ is denoted by $\text{R*C}(X, \tau)$ (or) $\text{R*C}(X)$.

Remark 3.32: The intersection of two regular*-closed sets need not be regular*-closed, as seen from the following example.
Example 4.6: Consider \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\} \). In this space, \( R^*C(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\} \). Here \( A = \{a, d\} \) and \( B = \{b, d\} \) are regular*-closed but \( A \cap B = \{d\} \) is not regular*-closed.

Theorem 4.7: Every regular*-closed set is closed.

Proof: Let \( A \) be regular*-closed, then \( X \setminus A \) is regular*-open. By theorem 3.15, \( X \setminus A \) is open \( \Rightarrow \) \( A \) is closed.

Remark 4.8: The converse of the above theorem 4.7 need not be true, as seen from the following example.

Example 4.9: Consider \( X = \{a, b, c, d\} \), \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \). In this space, \( R^*C(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\} \). Here \( \{d\} \) is closed but not regular*-closed.

Theorem 4.10: Every regular closed set is regular*-closed.

Proof: Let \( A \) be a regular closed set, then \( X \setminus A \) is regular open. By theorem 3.12, \( X \setminus A \) is regular*-open \( \Rightarrow \) \( A \) is regular*-closed.

Remark 4.11: The converse of the above theorem 4.10 need not be true, as seen from the following example.

Example 4.12: Consider \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, X\} \). In this space, \( RC(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, X\} \) and \( R^*C(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, X\} \). Here \( \{c, d\} \), \( \{a, c, d\} \), \( \{a, b, c, d\} \) are regular*-closed but not regular closed. From the above discussions we have the following implication diagram

Diagram 4.13:

\[ \text{Semi-closed} \quad \uparrow \quad \text{Regular closed} \quad \longrightarrow \quad \text{Regular*-closed} \quad \longrightarrow \quad \text{Closed} \quad \longrightarrow \quad \alpha\text{-closed} \quad \downarrow \quad \text{Pre-closed} \]

Definition 4.14: The regular*closure of \( A \) is defined as the intersection of all regular*closed sets of \( X \) containing \( A \). It is denoted by \( r^*Cl(A) \).

Definition 4.15: Let \( A \subseteq X \). An element \( x \in X \) is called regular*-adherent point of \( A \) if every regular*-open set in \( X \) containing \( x \) intersects \( A \).

Definition 4.16: Let \( A \subseteq X \). An element \( x \in X \) is called a regular*-limit point of \( A \) if every regular*-open set in \( X \) containing \( x \) intersects \( A \) in a point different from \( x \).

Definition 4.17: The set of all regular*-limit points of \( A \) is called the regular*-Derived set of \( A \). It is denoted by \( D_{r^*}(A) \).

Remark 4.18: If \( A \) is any subset of \( X \), \( r^*Cl(A) \) need not be a regular*closed set, as seen from the following example.

Example 4.19: In the space \( (X, \tau) \), \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, X\} \). Here \( R^*C(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, X\} \). Let \( A = \{d\} \) then \( r^*Cl(A) = \{d\} \) but \( \{d\} \) is not regular*-closed.

Theorem 4.20: In any topological space \( (X, \tau) \), the following result hold:

1. \( r^*Cl(\phi) = \phi \).
2. \( r^*Cl(X) = X \).
3. \( A \subseteq r^*Cl(A) \).
4. \( A \subseteq B \Rightarrow r^*Cl(A) \subseteq r^*Cl(B) \), if \( A \) and \( B \) are subsets of \( X \).
5. \( A \subseteq Cl(A) \subseteq r^*Cl(A) \).
6. \( r^*Cl(A \cup B) = r^*Cl(A) \cup r^*Cl(B) \), if \( A \) and \( B \) are subsets of \( X \).
7. \( r^*Cl(A \cap B) \subseteq r^*Cl(A) \cap r^*Cl(B) \), if \( A \) and \( B \) are subsets of \( X \).
8. \( Cl(A) \subseteq Cl(r^*Cl(A)) \).
Proof: (i), (ii), (iii) and (iv) follow from Definition 4.14. (v) follows from Theorem 4.10 and Theorem 4.7. (vi) follows from Theorem 2.8. (vii) follow from (iv) above. From (iii) above, we have $A \subseteq r^*\text{Cl}(A)$ and hence $\text{Cl}(A) \subseteq \text{Cl}(r^*\text{Cl}(A))$.

Remark 4.21: In (v) of Theorem 4.20, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 4.22: In the space $(X, \tau)$, $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. Here $\text{RC}(X) = \{\emptyset, \{a, d, e\}, \{b, c, d, e\}, X\}$ and $\text{R}*\text{C}(X) = \{\emptyset, \{b, d, e\}, \{a, d, e\}, \{b, c, d, e\}, X\}$. Let $A = \{a, d, e\}$ then $\text{Cl}(A) = \{a, d, e\}, r\text{Cl}(A) = \{a, d, e\}$ and $r^*\text{Cl}(A) = \{a, d, e\}$. Here $A = \text{Cl}(A) = r^*\text{Cl}(A) = r\text{Cl}(A)$.

Remark 4.23: The inclusion in (vii) of Theorem 4.20 may be strict and equality may also hold. This can be seen from the following examples.

Example 4.24: In the space $(X, \tau)$, $X = \{a, b, c, d, e, f, g\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{b, f, g\}, \{a, b, c, d, e\}, \{a, b, c, d, f, g\}, X\}$. Here $\text{R}*\text{C}(X) = \{\emptyset, \{c, d\}, \{f, g\}, \{a, c, d, e\}, \{b, e, f, g\}, \{c, d, e, f, g\}, \{b, c, d, e, f, g\}, X\}$. Let $A = \{a, d, e\}, B = \{a, d, f\}$ then $A \cap B = \{a, d\}$, $r^*\text{Cl}(A) = \{a, c, d, e\}, r^*\text{Cl}(B) = \{a, c, d, e, f, g\}$ and $r^*\text{Cl}(A \cap B) = \{a, c, d, e\}$. Here $r^*\text{Cl}(A \cap B) = r^*\text{Cl}(A) \cap r^*\text{Cl}(B)$.

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Source of support: Nil, Conflict of interest: None Declared

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