Between $\alpha$-closed sets and $g_\alpha$-closed sets

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ABSTRACT

In this paper we introduce and study a new class of generalized closed sets called $\psi \alpha$-closed sets in topological spaces. We analyze the relations between $\psi \alpha$-closed sets with already existing closed sets. We discuss some basic properties of $\psi \alpha$-closed sets. The class of $\psi \alpha$-closed sets is properly placed between the class of $\alpha$-closed sets and the class of $g_\alpha$-closed sets. We prove that the class of $\psi \alpha$-closed sets forms a topology.

Keywords: $\alpha$-closed sets, $\psi$-closed sets, $\psi g$-closed sets and $\psi \alpha$-closed sets

1. INTRODUCTION

Njastad [18] introduced the concept of an $\alpha$-open sets. Levine [13] introduced the notion of g-closed sets in topological spaces and studied their basic properties. Veerakumar [22] introduced and studied $\psi$-closed sets in topological spaces. Ramya and Parvathi [20] introduced a new concept of generalized closed sets called $\psi g\alpha$-closed sets and $\psi g\alpha$-closed sets in topological spaces. Jafari et al. [10] introduced the class of $g\alpha\alpha$-closed sets. In this paper we introduce a new class of generalized closed sets called $\psi \alpha$-closed sets in topological spaces. This class is obtained by generalizing $\alpha$-closed sets via $\psi g$-open sets.

2. PRELIMINARIES

Throughout this paper $(X, \tau)$ represents non-empty topological space on which no separation axioms are defined, unless otherwise mentioned. The interior, closure and complement of a subset $A$ of a space $(X, \tau)$ are denoted by $\text{int}(A)$, $\text{cl}(A)$ and $A'$ respectively.

Definition 2.1: A subset $A$ of a topological space $(X, \tau)$ is called
(i) Semi-open set [12] if $A \subseteq \text{cl}(\text{int}(A))$
(ii) $\alpha$-open set [18] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$
(iii) Pre-open set [17] if $A \subseteq \text{int}(\text{cl}(A))$
(iv) semi pre-open set [3] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$

The complements of the above mentioned sets are called semi-closed, $\alpha$-closed, pre-closed and semi pre-closed sets respectively.

The intersection of all semi-closed (resp. $\alpha$-closed, pre-closed and semi pre-closed) subsets of $(X, \tau)$ containing $A$ is called the semi-closure (resp. $\alpha$-closure, pre-closure and semi pre-closure) of $A$ and is denoted by $\text{scl}(A)$ (resp. $\alpha\text{cl}(A)$, $\text{pcl}(A)$ and $\text{spcl}(A)$). A subset $A$ of $(X, \tau)$ is called nowhere dense if $\text{int}(\text{cl}(A))=\emptyset$. A subset $A$ of a topological space $(X, \tau)$ is called semi-closed (resp. $\alpha$-closed) if and only if $\text{scl}(A)=\emptyset$ (resp. $\alpha\text{cl}(A)=A$).

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Definition 3.2: A subset $A$ of a topological space $(X, \tau)$ is called
(a) generalized closed set (briefly g-closed) [13] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(b) generalized semi-closed set (briefly gs-closed) [4] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(c) semi-generalized closed set (briefly sg-closed) [5] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
(d) generalised $\alpha$-closed set(briefly $g\alpha$-closed) [14] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.
(e) $\alpha$-generalized closed set (briefly $ag\alpha$-closed) [15] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(f) generalized semi-pre-closed set(briefly gsp-closed) [7] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(g) $g\star$ -closed set [24] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
(h) $g$ -closed set [23] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\star$-open in $(X, \tau)$.
(i) $g$ -closed set [30] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $(X, \tau)$.
(j) $g$ -closed set [16] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(k) $g$ -closed set [25] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\star$-open in $(X, \tau)$.
(l) $g$ -closed set [1] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\star$-closed in $(X, \tau)$.
(m) $gs$ -closed set[19] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
(n)$g\alpha$-closed set [26] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(o)$gs$-closed set [29] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(p)$g$-closed set [9] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(q)$gs$-closed set [10] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(r)$g$-semi-closed set [21] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(s)$pre$-closed set [8] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(t)$g\alpha$-closed set[27] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(u)$g\star$ -closed set[2] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(v)$\psi$-closed set[22] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(w)$\psi$-closed set[20] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(x)$\psi$ -closed set[28] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(y)$\psi$ -closed set[20] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $(X, \tau)$.
(z)$\psi$ -closed set[6] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.

The complements of the above mentioned sets are called their respective open-sets.

3. $\psi\alpha$ -CLOSED SETS

Definition 3.1: A subset $A$ of a topological space $(X, \tau)$ is said to be $\psi\alpha$ -closed set if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi\alpha$-open in $(X, \tau)$.

The class of all $\psi\alpha$ -closed sets of $(X, \tau)$ is denoted by $\psi\alpha C(X, \tau)$.

Proposition 3.2: Every closed set in $(X, \tau)$ is $\psi\alpha$ -closed but not conversely.

Proof: Let $A$ be a closed set and $U$ be any $\psi\alpha$ -open set containing $A$ in $X$. Since every closed set is $\alpha$-closed,
$ acl(A) \subseteq cl(A) = A \subseteq U$. Therefore $A$ is $\psi\alpha$ -closed.

Example 3.3: Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{a\}, \{a, b\}, X\}$. Then the subset $\{b\}$ is $\psi\alpha$ -closed but not closed in $(X, \tau)$.

Proposition 3.4: Every $\alpha$ -closed set in $(X, \tau)$ is $\psi\alpha$ -closed but not conversely.

Proof: Let $A$ be an $\alpha$-closed set and $U$ be any $\psi\alpha$ -open set containing $A$ in $X$. Since $A$ is $\alpha$-closed,
$ acl(A) = A \subseteq U$. Therefore $A$ is $\psi\alpha$ -closed.

Example 3.5: Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{a, b\}, X\}$. Then the subset $\{a, c\}$ is $\psi\alpha$ -closed but not $\alpha$ -closed in $(X, \tau)$.

Lemma 3.6: Every $\psi$ -open set in $(X, \tau)$ is $\psi\alpha$ -closed but not conversely.

Proof: Let $A$ be a $\psi$ -open set and $U$ be any $\psi$ -open set containing $A$ in $X$. Since every open set is $\psi\alpha$ -open,
and $A$ is $\psi\alpha$ -closed, $acl(A) \subseteq U$. Therefore $A$ is $\psi\alpha$ -closed.

Example 3.7: Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{a\}, X\}$. Then the subset $\{a, b\}$ is $\psi\alpha$ -closed but not $\psi$ -open in $(X, \tau)$.

Proposition 3.8: Every $\psi\alpha$ -closed set in $(X, \tau)$ is $\psi\alpha$ -closed but not conversely.

Proof: Let $A$ be a $\psi\alpha$ -closed set and $U$ be any $\psi$ -open set containing $A$ in $X$. Since every $\psi$ -open set is $\psi\alpha$ -open and $A$ is $\psi\alpha$ -closed, $acl(A) \subseteq U$. Therefore $A$ is $\psi\alpha$ -closed.
Example 3.9: Let $X=\{a, b, c, d\}$, $\tau =\{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}, X\}$. Then the subset $\{b, c, d\}$ is $\bar{g}_s$-closed but not $\psi \bar{\alpha}$ -closed in $(X, \tau)$.

Proposition 3.10: Every $\psi \bar{\alpha}$ -closed set in $(X, \tau)$ is $g\alpha$ (resp. $\alpha g$, $g_s$, $\bar{g}_s$)-closed but not conversely.

Proof: By [10], every $\bar{g}_s$ - closed set is $g\alpha$ ( resp. $\alpha g$, $g_s$, $\bar{g}_s$)-closed set. Hence it holds.

Example 3.11: Let $X=\{a, b, c, d\}$, $\tau =\{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}, X\}$. Then the subset $\{a, c, d\}$ is $g\alpha$-closed $\alpha g$-closed, $gs$-closed, $gs$-closed and $\bar{g}_s$-closed but not $\psi \bar{\alpha}$ -closed in $(X, \tau)$.

Proposition 3.12: Every $\psi \bar{\alpha}$ -closed set in $(X, \tau)$ is $\bar{g}$-pre closed but not conversely.

Proof: Follows from the fact that every $\bar{g}_s$ - closed is $\bar{g}$-pre closed.

Example 3.13: Let $X=\{a, b, c\}$, $\tau =\{\emptyset, \{a, b\}, X\}$. Then the subset $\{a\}$ is $\bar{g}$-pre closed but not $\psi \bar{\alpha}$ -closed in $(X, \tau)$.

Lemma 3.14: Every semi -closed set in $(X, \tau)$ is $\psi g$ - closed but not conversely.

Proof: Let $A$ be a semi - closed set and $U$ be any open set containing $A$ in $X$. Since $A$ is semi - closed, $scl(A)=A$. For every subset $A$ of $X$, $pcl(A)\subseteq scl(A)$ and so we have $\psi cl(A)\subseteq U$. Hence $A$ is $\psi g$ -closed.

Example 3.15: Let $X=\{a, b\}$, $\tau =\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the subset $\{a\}$ is $\psi g$ -closed but not semi -closed in $(X, \tau)$.

Proposition 3.16: Every $\psi \bar{\alpha}$ -closed set in $(X, \tau)$ is $\alpha gs$-closed but not conversely.

Proof: Let $A$ be a $\psi \bar{\alpha}$ -closed set and $U$ be any semi-open set containing $A$ in $X$. Since every semi-open set is $\psi g$-open and $A$ is $\psi \bar{\alpha}$ -closed, $acl(A) \subseteq U$. Hence $A$ is $\alpha gs$ -closed.

Example 3.17: Let $X=\{a, b, c\}$, $\tau =\{\emptyset, \{a\}, \{b, c\}, X\}$. Then the subset $\{a, c\}$ is $\alpha gs$-closed but not $\psi \bar{\alpha}$ -closed in $(X, \tau)$.

Lemma 3.18: Every $g$-closed set in $(X, \tau)$ is $\psi g$-closed but not conversely.

Proof: Let $A$ be a $g$-closed set and $U$ be any open set containing $A$ in $X$. Since $A$ is $g$-closed, $cl(A) \subseteq U$. For every subset $A$ of $X$, $pcl(A)\subseteq cl(A)$ and so we have $\psi cl(A)\subseteq U$. Hence $A$ is $\psi g$-closed.

Example 3.19: Let $X=\{a, b, c\}$, $\tau =\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the subset $\{a\}$ is $\psi g$-closed but not $g$-closed in $(X, \tau)$.

Proposition 3.20: Every $\psi \bar{\alpha}$ -closed set in $(X, \tau)$ is $gp$-closed ( $g^*p$-closed) but not conversely.

Proof: Let $A$ be a $\psi \bar{\alpha}$ -closed set and $U$ be any open ( $g$-open) set containing $A$ in $X$. Since every open ( $g$-open) set is $\psi g$-open and $A$ is $\psi \bar{\alpha}$ -closed, $acl(A) \subseteq U$. For every subset $A$ of $X$, $pcl(A)\subseteq acl(A)$ and so we have $pcl(A)\subseteq U$. Hence $A$ is $gp$-closed ( $g^*p$-closed).

Example 3.21: Let $X=\{a, b, c\}$, $\tau =\{\emptyset, \{a, b\}, X\}$. Then the subset $\{a\}$ is $gp$-closed ( $g^*p$-closed) but not $\psi \bar{\alpha}$ -closed in $(X, \tau)$.

Lemma 3.22: Every $sg$-closed set in $(X, \tau)$ is $\psi g$-closed but not conversely.

Proof: Let $A$ be a $sg$-closed set and $U$ be any open set containing $A$ in $X$. Since every open set is semi-open and $A$ is $sg$-closed, $scl(A)\subseteq U$. For every subset $A$ of $X$, $pcl(A)\subseteq scl(A)$ and so we have $scl(A)\subseteq U$. Hence $A$ is $\psi g$-closed.

Example 3.23: Let $X=\{a, b, c\}$, $\tau =\{\emptyset, \{a\}, \{a, b\}, X\}$. Then the subset $\{a, c\}$ is $\psi g$-closed but not $sg$-closed in $(X, \tau)$.

Proposition 3.24: Every $\psi \bar{\alpha}$ -closed set in $(X, \tau)$ is $\psi$-closed but not conversely.

Proof: Let $A$ be a $\psi \bar{\alpha}$ -closed set and $U$ be any open set containing $A$ in $X$. Since every open set is $\psi g$-open and $A$ is $\psi \bar{\alpha}$ -closed, $acl(A) \subseteq U$. For every subset $A$ of $X$, $scl(A)\subseteq acl(A)$ and so we have $scl(A)\subseteq U$. Hence $A$ is $\psi$-closed.

Example 3.25: Let $X=\{a, b, c\}$, $\tau =\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the subset $\{a\}$ is $\psi$-closed but not $\psi \bar{\alpha}$ -closed in $(X, \tau)$.
Proposition 3.36: Every $\psi\alpha$ -closed set in $(X, \tau)$ is $\psi\phi$ (resp. $\psi\alpha$, $\psi\alpha\alpha$)-closed but not conversely.

Proof: By [20], every $\psi$-closed set is $\psi\phi$ (resp. $\psi\alpha$, $\psi\alpha\alpha$)-closed. Therefore it holds

Example 3.37: Let $X=\{a, b, c\}, \tau=\{\phi,\{a\},\{b\},\{a, b\},X\}$. Then the subset $\{a\}$ is $\psi\phi$ -closed, $\psi\alpha$-closed, and $\psi\alpha\alpha$-closed but not $\psi\alpha$ -closed in $(X, \tau)$.

Lemma 3.38: Every $\alpha g$ -closed set in $(X, \tau)$ is $\psi g$ - closed but not conversely.

Proof: Let $A$ be an $\alpha g$ -closed set and $U$ be any open set containing $A$ in $X$. Since every open set is $\alpha g$ -closed, $\alpha cl(A) \subseteq U$. For every subset $A$ of $X$, $\psi cl(A) \subseteq \alpha cl(A)$ and so $\psi cl(A) \subseteq U$. Hence $A$ is $\psi g$ -closed.

Example 3.39: Let $X=\{a, b, c\}, \tau=\{\phi,\{a\},\{b\},\{a, b\},X\}$. Then the subset $\{a\}$ is $\psi g$ -closed but not $\alpha g$ -closed in $(X, \tau)$.

Proposition 3.40: Every $\psi\alpha$ -closed set in $(X, \tau)$ is $\alpha g$ - closed but not conversely.

Proof: Let $A$ be a $\psi\alpha$ -closed set and $U$ be any $\alpha g$ -open set containing $A$ in $X$. Since every $\alpha g$-open set is $\psi g$-open and $A$ is $\psi\alpha$-closed, $\alpha cl(A) \subseteq U$. For every subset $A$ of $X$, $\psi cl(A) \subseteq \alpha cl(A)$ and so we have $\psi cl(A) \subseteq U$. Hence $A$ is $\alpha g$ -closed.

Example 3.41: Let $X=\{a, b, c\}, \tau=\{\phi,\{a\},\{a, b\},X\}$. Then the subset $\{a\}$ is $\alpha g$ -closed but not $\psi\alpha$ -closed in $(X, \tau)$.
Proposition 3.42: Every $\psi^*\alpha$ -closed set in $(X, \tau)$ is $g^*\psi$ -closed but not conversely.

Proof: Follows from the fact that every $\psi$-closed is $g^*\psi$ -closed

Example 3.43: Let $X=\{a, b, c, d\}, \tau=\{\phi, \{a\}, \{a, b\}, X\}$. Then the subset $\{a, c, d\}$ is $g^*\psi$ -closed but not $\psi^*\alpha$ -closed in $(X, \tau)$.

Proposition 3.44: Every $\psi^*\alpha$ -closed set in $(X, \tau)$ is $\alpha\psi$ -closed but not conversely.

Proof: Let $A$ be a $\psi^*\alpha$ -closed set and $U$ be any $\alpha$ -open set containing $A$ in $X$. Since every $\alpha$ -open set is $\psi g$ -open and $A$ is $\psi^*\alpha$ -closed, $\alpha cl(A) \subseteq U$. For every subset $A$ of $X$, $\psi cl(A) \subseteq \alpha cl(A)$. and so we have $\psi cl(A) \subseteq U$. Hence $A$ is $\alpha\psi$ -closed.

Example 3.45: Let $X=\{a, b, c\}, \tau=\{\phi, \{a\}, \{a, b\}, \{a, b\}, X\}$. Then the subset $\{a\}$ is $\alpha\psi$ -closed but not $\psi^*\alpha$ -closed in $(X, \tau)$.

Remark 3.46: The following example shows that $\psi^*\alpha$ -closedness is independent from $g$-closedness, $g^*\psi$ -closedness, $g^*\alpha$ -closedness and $g^*\alpha$ -closedness.

Example 3.47: Let $X=\{a, b, c\}, \tau=\{\phi, \{a\}, \{a, b\}, X\}$. In this topology the set $\{a, c\}$ is $g$-closed, $g^*\alpha$ -closed, $g$ - closed and $g^*\alpha$ -closed but not $\psi^*\alpha$ -closed. The set $\{b\}$ is $\psi^*\alpha$ -closed but not g-closed, $g^*\psi$ -closed, $g^*\alpha$ -closed and $g^*\alpha$ -closed.

Remark 3.48: The following examples show that $\psi^*\alpha$ -closedness is independent from semi-closedness.

Example 3.49: Let $X=\{a, b, c\}, \tau=\{\phi, \{a\}, \{a, b\}, X\}$. In this topology the set $\{b, c\}$ is $\psi^*\alpha$ -closed but not semi-closed.

Example 3.50: Let $X=\{a, b, c\}, \tau=\{\phi, \{a\}, \{a, b\}, X\}$. In this topology the set $\{b\}$ is semi-closed but not $\psi^*\alpha$ -closed.

Remark 3.51: The following examples show that $\psi^*\alpha$ -closedness is independent from $g^*\alpha$ -closedness, $g^*\alpha$ -closedness and $g^*\alpha$ -closedness.

Example 3.52: Let $X=\{a, b, c\}$ with $\tau=\{\phi, \{a\}, \{a, b\}, X\}$. In this topology the set $\{b\}$ is $\psi^*\alpha$ -closed but not $g^*\psi$ -closed, $g^*\alpha$ -closed and $g^*\alpha$ -closed.

Example 3.53: Let $X=\{a, b, c, d\}$ with $\tau=\{\phi, \{d\}, \{a, b\}, \{a, b, d\}, X\}$. In this topology the set $\{a, c, d\}$ is $g\alpha$ -closed, $g^*\psi$ -closed and $g^*\psi$ -closed but not $\psi^*\alpha$ -closed.

Remark 3.54: The following diagram has shown the relationship of $\psi^*\alpha$ -closed sets with already existing various closed sets. where $A \rightarrow B$ represents $A$ implies $B$ but not conversely. where $A \leftrightarrow B$ represents $A$ and $B$ are independent of each other.

\[ \begin{align*} 
1. & \psi^*\alpha \text{-closed} & 2. & \text{closed} & 3. & \alpha\text{-closed} & 4. & \tilde{g}_a\text{-closed} & 5. & \text{ga-closed} & 6. & \alpha g\text{-closed} & 7. & \text{sg-closed} & 8. & \text{gs-closed} & 9. & \text{g-semi-closed} & 10. & \text{g-pre-closed} & 11. & \text{ags-closed} & 12. & \text{gp-closed} & 13. & \text{g}^*\tilde{g}\text{-closed} & 14. & \psi\text{-closed} & 15. & \text{g}^*\psi\text{-closed} & 16. & \text{g}\tilde{g}\text{-closed} & 17. & \text{gsp-closed} & 18. & \text{g}^*\text{s-closed} & 19. & \text{g}^*\text{-closed} & 20. & \text{gs-closed} & 21. & \text{g}^*\psi\text{-closed} & 22. & \alpha\psi\text{-closed} & 23. & \text{g}\text{-closed} & 24. & \text{g}^*\text{-closed} & 25. & \text{g}\text{-closed} & 26. & \text{g}^*\tilde{g}\text{-closed} & 27. & \text{semi-closed} & 28. & \tilde{g}\text{-closed} & 29. & \text{g}^\alpha\text{-closed} & 30. & \tilde{g}\text{-closed} \end{align*} \]
Definition 3.55: A subset $A$ of a topological space $(X, \tau)$ is said to be $\psi^* \alpha$-open if its complement $A^c$ is $\psi^* \alpha$-closed.

The class of all $\psi^* \alpha$-open sets in $(X, \tau)$ is denoted by $\psi^* \alpha O(X, \tau)$.

Proposition 3.56: Every open (respectively $\alpha$-open) set is $\psi^* \alpha$-open.

Proposition 3.57: Every $\psi^* \alpha$-open set is $\tilde{\alpha}_a$-open (respectively $g\alpha$-open, $og$-open, sg-open, gs-open, $\tilde{g}$-semi-open, $\tilde{g}$-pre-open, $args$-open, gp-open, $\tilde{g}$-p-open, $\psi g\tilde{g}$-open, $\psi g$-open, $gsp$-open, $gs$-open, $\tilde{g}gs$-open, $g\psi\psi$-open and $\alpha\psi\psi$-open).

4. PROPERTIES OF $\psi^* \alpha$-CLOSED SETS AND $\psi^* \alpha$-OPEN SETS

Theorem 4.1: If $A$ and $B$ are $\psi^* \alpha$-closed sets in a topological space $(X, \tau)$, then $A \cup B$ is $\psi^* \alpha$-closed set in $(X, \tau)$.

Proof: Let $A$ and $B$ be any two $\psi^* \alpha$-closed sets in $(X, \tau)$ and $U$ be any $\psi g$-open set containing $A$ and $B$. We have $\alpha cl(A) \subseteq U$ and $\alpha cl(B) \subseteq U$. Always $\alpha cl(A \cup B) = \alpha cl(A) \cup \alpha cl(B) \subseteq U$. Hence $A \cup B$ is $\psi^* \alpha$-closed in $(X, \tau)$.

Theorem 4.2: Let $A$ be a $\psi^* \alpha$-closed set in $(X, \tau)$. Then $\alpha cl(A)$-A contains no non-empty closed set in $(X, \tau)$.

Proof: Suppose that $A$ is $\psi^* \alpha$-closed. Let $F$ be a closed subset of $\alpha cl(A)$-A. Then $F^c$ is open and hence $\psi g$-open such that $A^c \subseteq F^c$. Since $A$ is a $\psi^* \alpha$-closed set, $\alpha cl(A) \subseteq F^c$. Thus $F \subseteq (\alpha cl(A))^c$. Since every closed set is $\alpha$-closed, $F$ is $\alpha$-closed. Hence $F \subseteq \alpha cl(A)$. Therefore $F \subseteq \alpha cl(A) \cap (\alpha cl(A))^c = \phi$. Hence $F = \phi$.

Remark 4.3: The converse of the above theorem is not true as seen from the following example.

Example 4.4: Let $X=\{a, b, c\}$, $\tau=\{\varnothing, \{a\}, \{b, c\}, X\}$. If $A= \{b\}$ then $\alpha cl(A)-A=\{b, c\}-\{b\} = \{c\}$ does not contain non-empty closed set. However $A$ is not a $\psi^* \alpha$-closed subset of $(X, \tau)$.

Theorem 4.5: A set $A$ is $\psi^* \alpha$-closed in $(X, \tau)$ if and only if $\alpha cl(A)$-A contains no non-empty $\psi g$-closed set in $(X, \tau)$.

Proof: (Necessity): Suppose that $A$ is $\psi^* \alpha$-closed. Let $F$ be a $\psi g$-closed set contained in $\alpha cl(A)$-A. Now $F^c$ is a $\psi g$-open set in such that $A^c \subseteq F^c$. Since $A$ is a $\psi^* \alpha$-closed set in $X$, $\alpha cl(A) \subseteq F^c$. Thus $F \subseteq (\alpha cl(A))^c$. Also $F \subseteq \alpha cl(A)$-A. Therefore $F \subseteq \alpha cl(A) \cap (\alpha cl(A))^c = \phi$. Hence $F = \phi$.

Sufficiency: Suppose that $\alpha cl(A)$-A contains no non empty $\psi g$-closed set. Let $A \subseteq G$ and $G$ be $\psi g$-open. If $\alpha cl(A)$ is not a subset of $G$ then $\alpha cl(A) \cap G^c$ is a non-empty $\psi g$-closed subset of $\alpha cl(A)$-A, which is a contradiction. Therefore $\alpha cl(A) \subseteq G$ and hence $A$ is $\psi^* \alpha$-closed.

Proposition 4.6: If $A$ is $\psi g$-open and $\psi^* \alpha$-closed subset of $(X, \tau)$. Then $A$ is an $\alpha$-closed set of $(X, \tau)$.

Proof: Since $A$ is $\psi g$-open and $\psi^* \alpha$-closed, $\alpha cl(A) \subseteq A$. Hence $A$ is $\alpha$-closed.

Theorem 4.7: If a set $A$ is $\psi^* \alpha$-closed and $\psi g$-open and $F$ is $\alpha$-closed in $(X, \tau)$, then $A \cap F$ is $\alpha$-closed.

Proof: Since $A$ is $\psi^* \alpha$-closed and $\psi g$-open, $A$ is $\alpha$-closed by Proposition 4.6 Since $F$ is $\alpha$-closed in $X$, $A \cap F$ is $\alpha$-closed in $X$.

Theorem 4.8: If $A$ is a $\psi^* \alpha$-closed set in $(X, \tau)$ and $A \subseteq B \subseteq \alpha cl(A)$ . Then $B$ is also a $\psi^* \alpha$-closed set in $(X, \tau)$.

Proof: Let $U$ be a $\psi g$-open set of $(X, \tau)$ such that $B \subseteq U$. Then $A \subseteq U$. Since $A$ is a $\psi^* \alpha$-closed set, $\alpha cl(A) \subseteq U$. Also since $B \subseteq \alpha cl(A)$, $\alpha cl(B) \subseteq \alpha cl(\alpha cl(A)) = \alpha cl(A)$. Hence $\alpha cl(B) \subseteq U$. Therefore $B$ is also a $\psi^* \alpha$-closed set in $(X, \tau)$.

Theorem 4.9: Let $A$ be a $\psi^* \alpha$-closed set of $(X, \tau)$. Then $A$ is $\alpha$-closed if and only if $\alpha cl(A)$-A is $\psi g$-closed.

Proof: (Necessity): Let $A$ be an $\alpha$-closed subset of $(X, \tau)$. Then $\alpha cl(A)=A$ and therefore $\alpha cl(A)-A=\phi$ which is $\psi g$-closed in $(X, \tau)$.

Sufficiency: Let $\alpha cl(A)$-A be a $\psi g$-closed set. Since $A$ is $\psi^* \alpha$-closed by Theorem 4.5, $\alpha cl(A)$-A contains no non-empty $\psi g$-closed set which implies $\alpha cl(A)=A$. That is $\alpha cl(A)=A$. Hence $A$ is $\alpha$-closed.
Definition 4.10: Let \((X,\tau)\) be a topological space and let \(B \subseteq A \subseteq X\) then \(B\) is \(\psi \alpha\) -closed relative to \(A\) if \((\text{cl}(A))\cup B \subseteq U\), whenever \(B \cup U\), \(U\) is \(\psi g\)-open in \(A\).

Theorem 4.11: Let \(B \subseteq A \subseteq X\) and suppose that \(B\) is \(\psi \alpha\) -closed in \((X, \tau)\), then \(B\) is \(\psi \alpha\) -closed relative to \(A\). The converse is true if \(A\) is \(\alpha\) -open and \(\psi \alpha\) -closed in \((X, \tau)\).

Proof: Suppose that \(B\) is a \(\psi \alpha\) -closed in \((X, \tau)\). Let \(B \subseteq U\), \(U\) is \(\psi g\) - open in \(A\). Since \(U\) is \(\psi g\) - open set in \(A\), \(U \cap V \subseteq A\), where \(V\) is \(\psi g\) - open in \(X\). Hence \(B \subseteq U \subseteq V\). Suppose that \(B\) is \(\psi \alpha\) -closed in \((X, \tau)\). Hence \((\text{cl}(B))\cup A \subseteq U\). Since \(A\) is \(\alpha\) -open, it is \(\psi g\) - open in \(X\). Since \(A \subseteq U\) and \(B\) is \(\psi \alpha\) -closed relative to \(A\), \(B\) is a \(\psi \alpha\) -closed relative to \(A\). Let \(B \subseteq U\) and \(U\) be \(\psi g\) - open in \(A\). Then \(A \cap U\) is \(\psi g\) - open in \(A\). Since \(B \subseteq A\) and \(B \subseteq U\), \(B \subseteq U\subseteq V\). Since \(B\) is a \(\psi \alpha\) -closed relative to \(A\), \((\text{cl}(B))\cup A \subseteq U\). Therefore \(B\) is \(\psi \alpha\) -closed relative to \(A\).

Now, to prove the converse, assume that \(B \subseteq A \subseteq X\) where \(A\) is \(\alpha\) -open and \(\psi \alpha\) -closed in \(X\) and \(B\) is a \(\psi \alpha\) -closed relative to \(A\). Let \(B \subseteq U\) and \(U\) be \(\psi g\) - open in \(X\). Then \(A \cap U\) is \(\psi g\) - open in \(A\). Since \(B \subseteq A\) and \(B \subseteq U\), \(B \subseteq U\subseteq V\). Since \(B\) is \(\psi \alpha\) -closed relative to \(A\), \((\text{cl}(B))\cup A \subseteq U\). Therefore \(B\) is \(\psi \alpha\) -closed relative to \(A\).

Theorem 4.12: In a topological space \((X, \tau)\), for each \(x \in X\), either \(\{x\}\) is \(\psi g\)-closed or \(X\) -\(\{x\}\) is \(\psi \alpha\) closed set in \((X, \tau)\).

Proof: Suppose that \(\{x\}\) is not \(\psi g\)-closed in \(X\). Then \(\{x\}\) is not \(\psi g\)-open in \(X\). Hence \(X\) is the only \(\psi g\)-open set containing \(X\) -\(\{x\}\). That is \((X\) -\(\{x\}\)) \(\subseteq X\). Therefore \(\text{cl}(X\) -\(\{x\}\)) \(\subseteq X\), which implies that \(X\) -\(\{x\}\) is \(\psi \alpha\) -closed in \((X, \tau)\).

Definition 4.13: The intersection of all \(\psi g\)-open subsets of \((X, \tau)\) containing \(A\) is called \(\psi g\)- kernel of \(A\) and is denoted by \(\psi g\text{-ker}(A)\).

i.e \(\psi g\text{-ker}(A) = \cap\{U \subseteq X: U\) is \(\psi g\)-open in \((X, \tau)\) and \(A \subseteq U\}\) \(\subseteq \) \(\psi g\text{-ker}(A)\).

Theorem 4.14: A subset \(A\) of a topological space \((X, \tau)\) is \(\psi \alpha\) -closed in \((X, \tau)\) if and only if \(\text{cl}(A) \subseteq \psi g\text{-ker}(A)\).

Proof: (Necessity): Suppose that \(A\) is \(\psi \alpha\) -closed set in \((X, \tau)\) and \(x \in \text{cl}(A)\). If \(x \notin \psi g\text{-ker}(A)\), then there exists a \(\psi g\) - open set \(U\) in \((X, \tau)\) such that \(A \subseteq U\) and \(x \in U\). Since \(U\) is \(\psi g\) - open set containing \(A\) and \(A\) is \(\psi \alpha\) -closed, we have \(\text{cl}(A) \subseteq U\), which is a contradiction to \(x \in \text{cl}(A)\) and \(x \notin U\).

Sufficiency: Suppose that \(\text{cl}(A) \subseteq \psi g\text{-ker}(A)\). If \(U\) is any \(\psi g\)-open set containing \(A\), then \(\psi g\text{-ker}(A) \subseteq U\) so we have \(\text{cl}(A) \subseteq U\). Hence \(A\) is \(\psi \alpha\) -closed.

Remark 4.15: Jankovic and Reilly [11] stated that “If \(x\) is any point in a topological space \((X, \tau)\), then every singleton \(\{x\}\) is either nowhere dense or preopen in \((X, \tau)\).” Also this provides another decomposition namely \(X = X_1 \cup X_2\) where \(X_1 = \{x \in X: \{x\}\) is nowhere dense\} and \(X_2 = \{x \in X: \{x\}\) is preopen\}.

Proposition 4.16: For any subset \(A\) of a topological space \((X, \tau)\), \(X_1 \cap \text{cl}(A) \subseteq \psi g\text{-ker}(A)\).

Proof: Let \(x \in X_1 \cap \text{cl}(A)\) and if \(x \notin \psi g\text{-ker}(A)\). Then there is a \(\psi g\)-open set \(U\) containing \(A\) such that \(x \notin U\). Then \(U\) is \(\psi g\)-closed set containing \(x\). Since \(x \in \text{cl}(A)\), \(\text{cl}(\{x\}) \subseteq \text{cl}(A)\). Since \(x \in X_1\), \(\text{int}(\text{cl}(\{x\})) \subseteq \text{int}(\text{cl}(\{x\})) \neq \phi\). Also \(x \in \text{cl}(A)\), so \(A \cap \text{int}(\text{cl}(\{x\})) \neq \phi\). Hence there is some point \(y \in A \cap \text{int}(\text{cl}(\{x\}))\) and therefore \(y \in A \cap U\), which is a contradiction.

Theorem 4.17: A subset \(A\) of a topological space \((X, \tau)\) is \(\psi \alpha\) -closed in \((X, \tau)\) if and only if \(X_1 \cap \text{cl}(A) \subseteq A\).

Proof: (Necessity): Suppose that \(A\) is \(\psi \alpha\) -closed in \((X, \tau)\) and \(x \in X_1 \cap \text{cl}(A)\) but \(x \notin A\). Since \(x \in X_1\), \(\text{int}(\text{cl}(\{x\})) = \phi\) so we have \(\text{int}(\text{cl}(\{x\})) = \phi \subseteq \{x\}\). Therefore \(\{x\}\) is semi-closed. Since every semi-closed set is \(\psi g\) -closed, \(\{x\}\) is \(\psi g\) -closed and hence \(U = X\) -\(\{x\}\) is \(\psi g\) -open set containing \(A\) and so \(\text{cl}(A) \subseteq U\). Since \(x \in \text{cl}(A)\) so we have \(x \in U\), which is a contradiction.

Sufficiency: Suppose that \(X_1 \cap \text{cl}(A) \subseteq A\). Since \(A \subseteq \psi g\text{-ker}(A)\), \(X_1 \cap \text{cl}(A) \subseteq \psi g\text{-ker}(A)\). Therefore \(\text{cl}(A) = X \cap \text{cl}(A) = (X_1 \cup X_2) \cap \text{cl}(A) = (X_1 \cap \text{cl}(A)) \cup (X_2 \cap \text{cl}(A))\). By hypothesis \(X_1 \cap \text{cl}(A) \subseteq \psi g\text{-ker}(A)\) and by

Proposition 4.16: \(X_2 \cap \text{cl}(A) \subseteq \psi g\text{-ker}(A)\). Hence \(\text{cl}(A) \subseteq \psi g\text{-ker}(A)\). Therefore by Theorem 4.14 \(A\) is \(\psi \alpha\) -closed.
Theorem 4.18: Arbitrary intersection of \( \psi^* \alpha \)-closed sets in a topological space \( (X, \tau) \) is \( \psi^* \alpha \)-closed in \( (X, \tau) \).

Proof: Let \( F=\{A_i : i \in \Lambda \} \) be a family of \( \psi^* \alpha \)-closed sets and \( \Lambda=\bigcap_{i \in \Lambda} A_i \) Since \( A \subseteq \Lambda \), \( X_1 \cap \text{acl}(A) \subseteq X \cap \text{acl}(A_i) \) for each \( i \in \Lambda \) using theorem 4.17 for each \( \psi^* \alpha \)-closed set \( A_i \) we have \( X_1 \cap \text{acl}(A) \subseteq X \cap \text{acl}(A_i) \) \( \subseteq A_i \) for each \( i \in \Lambda \). Thus \( X_1 \cap \text{acl}(A) \subseteq \bigcap_{i \in \Lambda} A_i = A \). That is \( X_1 \cap \text{acl}(A) \) is \( \subseteq A \) and so by theorem 4.17 A is \( \psi^* \alpha \)-closed in \( (X, \tau) \).

Remark 4.19: Thus from theorem 4.1 and theorem 4.18 leads us into another class of closed sets namely \( \psi^* \alpha \)-closed sets which are closed under finite union and arbitrary intersection. Hence the class of \( \psi^* \alpha \)-closed sets form a topology.

Lemma 4.20: For a subset \( A \) of \( (X, \tau) \), \( \text{acl}(X-A) = \psi^* \text{cl}(A) \)

Theorem 4.21: A subset \( A \) of a topological space \( (X, \tau) \) is \( \psi^* \alpha \)-open if and only if \( U \subseteq \psi^* \text{cl}(A) \) whenever \( U \subseteq A \) and \( U \) is \( \psi g^* \)-closed.

Proof: (Necessity) Assume that \( A \) is \( \psi^* \alpha \)-open. Then \( A^* \) is \( \psi^* \alpha \)-closed. Let \( U \) be a \( \psi g^* \)-closed set in \( (X, \tau) \) contained in \( A \). Then \( U^* \) is a \( \psi g^* \)-open set in \( (X, \tau) \) containing \( A^* \). Since \( A^* \) is \( \psi^* \alpha \)-closed, \( \text{acl}(A^*) \subseteq U^* \) equivalently \( U \supseteq \psi^* \text{cl}(A) \).

Sufficiency: Assume that \( U \) is contained in \( \psi^* \text{cl}(A) \) whenever \( U \subseteq A \) is \( \psi g^* \)-closed in \( (X, \tau) \). Let \( A^* \) be contained in \( U \), where \( U \) is \( \psi g^* \)-open. Then \( U^* \subseteq \psi^* \text{cl}(A) \). By criteria, \( U^* \subseteq \psi^* \text{cl}(A) \). This implies \( \psi^* \text{cl}(A) = \psi g^* \text{cl}(U) \) that is \( \psi g^* \text{cl}(U) \) \( \subseteq U \). Therefore \( A^* \) is \( \psi^* \alpha \)-closed. Hence \( A \) is \( \psi^* \alpha \)-open in \( (X, \tau) \).

Proposition 4.22: If \( \psi^* \text{cl}(A) \subseteq B \subseteq A \) and \( A \) is \( \psi^* \alpha \)-open, then \( B \) is \( \psi^* \alpha \)-open.

Proof: Follows from lemma 4.20 and Theorem 4.8

Theorem 4.23: If \( A \) and \( B \) are \( \psi^* \alpha \)-open sets in \( (X, \tau) \), then \( A \cap B \) is \( \psi^* \alpha \)-open in \( (X, \tau) \).

Proof: Let \( A \) and \( B \) be \( \psi^* \alpha \)-open sets in \( (X, \tau) \). Then \( X \cap A \) and \( X \cap B \) are \( \psi^* \alpha \)-closed sets and \( (X-A) \cup (X-B) = X- (A \cap B) \) is \( \psi^* \alpha \)-closed in \( (X, \tau) \). Hence \( A \cap B \) is \( \psi^* \alpha \)-open.

Theorem 4.24: If \( A \) is \( \psi^* \alpha \)-open in \( (X, \tau) \) if and only if \( G=\psi^* \text{cl}(X) \) whenever \( G \) is \( \psi g^* \)-open and \( \psi^* \text{cl}(A) \cup A^* \subseteq G \).

Proof: (Necessity): Let \( A \) be \( \psi^* \alpha \)-open and \( G \) is \( \psi g^* \)-open and \( \psi^* \text{cl}(A) \cup A^* \subseteq G \). This gives \( G^- \subseteq \psi^* \text{cl}(A) \cup A^* = \psi^* \text{cl}(A^*) \). Hence \( A^* = \psi^* \alpha \)-closed and \( G^* \) is \( \psi g^* \)-closed by theorem 4.5, it follows that \( G^* = \phi \). Therefore \( G = \psi^* \text{cl}(X) \).

(Sufficiency): Suppose that \( F \) is \( \psi g^* \)-closed and \( F \subseteq A \). Then \( \psi^* \text{cl}(A) \cup A^* \subseteq \psi^* \text{cl}(A) \cup F^* \). As open implies \( \psi g^* \)-open, we get \( \psi^* \text{cl}(A) \) is \( \psi g^* \)-open and \( \psi^* \text{cl}(F) \) is \( \psi g^* \)-open. Hence \( \psi^* \text{cl}(A) \cup \psi^* \text{cl}(F) \) is \( \psi g^* \)-open. It follows by the hypothesis that \( \psi^* \text{cl}(A) \cup \psi^* \text{cl}(F) \) is \( \psi g^* \)-open and hence \( \psi^* \text{cl}(A) \cup \psi^* \text{cl}(F) \) is \( \psi^* \alpha \)-open. Therefore by theorem 4.21, \( A \) is \( \psi^* \alpha \)-open in \( (X, \tau) \).

5. \( \psi^* \alpha \)-CLOSURE

Definition 5.1: The \( \psi^* \alpha \)-closure of \( A \) (briefly \( \psi^* \text{acl}(A) \)) of a topological space \( (X, \tau) \) is defined as follows.
\[ \psi^* \text{acl}(A) = \bigcap \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \psi^* \alpha \text{-closed in } (X, \tau) \} \]

Proposition 5.2: For a subset \( A \) of a topological space \( (X, \tau) \), \( A \subseteq \psi^* \text{acl}(A) \subseteq \text{cl}(A) \)

Proof: Follows from proposition 3.2

Remark 5.3: If \( A \) is \( \psi^* \alpha \)-closed in \( (X, \tau) \), then \( \psi^* \text{acl}(A) = A \).

Theorem 5.4: Let \( A \) be a subset of \( X \) and \( x \in X \), then \( x \in \psi^* \text{acl}(A) \) if and only if for every \( \psi^* \alpha \)-open set \( U \) containing \( x \), \( U \cap A = \phi \).

Proof: (Necessity): Let \( x \in \psi^* \text{acl}(A) \) and there exists a \( \psi^{\alpha} \)-open set \( U \) containing \( x \) such that \( U \cap A = \phi \). Since \( A \subseteq U^* \), \( \psi^* \text{acl}(A) \subseteq U^* \) and hence \( x \notin \psi^* \text{acl}(A) \), which is a contradiction. Hence \( U \cap A = \phi \).

(Sufficiency): Assume the given condition. Suppose that \( x \in \psi^* \text{acl}(A) \). Then there exists a \( \psi^* \alpha \)-closed set \( F \) containing \( A \) such that \( x \notin F \). Then \( x \in F^* \) and \( F^* \) is \( \psi^* \alpha \)-open. By assumption, \( F^* \cap A = \phi \). Since \( A \subseteq F^* \), \( F^* \cap A = \phi \) is which is a contradiction. Therefore \( x \in \psi^* \text{acl}(A) \).
Proposition 5.5: Let A and B be any two subsets of (X, τ). Then the following statements are true
(a) \( \psi^* \text{cl}(\emptyset) = \emptyset \) and \( \psi^* \text{cl}(X) = X \).
(b) If \( A \subseteq B \), then \( \psi^* \text{cl}(A) \subseteq \psi^* \text{cl}(B) \).
(c) \( \psi^* \text{cl}(A) \cup \psi^* \text{cl}(B) = \psi^* \text{cl}(A \cup B) \)
(d) \( \psi^* \text{cl}((A \cap B) \subseteq \psi^* \text{cl}(A) \cap \psi^* \text{cl}(B) \)
(e) \( \psi^* \text{cl}(\psi^* \text{cl}(A)) = \psi^* \text{cl}(A) \).

Proof: (a) and (b) follow from the definition of \( \psi^* \text{-closure} \).
(c) Since \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \), by (b) \( \psi^* \text{cl}(A) \subseteq \psi^* \text{cl}(A \cup B) \) and \( \psi^* \text{cl}(B) \subseteq \psi^* \text{cl}(A \cup B) \). Hence \( \psi^* \text{cl}(A) \cup \psi^* \text{cl}(B) \subseteq \psi^* \text{cl}(A \cup B) \). To prove the reverse inclusion, let \( x \in \psi^* \text{cl}(A \cup B) \) and suppose that \( x \notin \psi^* \text{cl}(A) \cup \psi^* \text{cl}(B) \). Then \( x \notin \psi^* \text{cl}(A) \) and \( x \notin \psi^* \text{cl}(B) \). Therefore there exist a \( \psi^* \text{-closed set} \) U and V in X such that \( A \subseteq U \), \( B \subseteq V \), \( x \notin U \) and \( x \notin V \). Hence we have \( A \cup B \subseteq U \cup V \) and \( x \notin U \cup V \). By Theorem 4.1, \( U \cup V \) is a \( \psi^* \text{-closed set} \) and hence \( x \notin \psi^* \text{cl}(A \cup B) \), which is a contradiction. Hence \( \psi^* \text{cl}(A \cup B) \subseteq \psi^* \text{cl}(A) \cup \psi^* \text{cl}(B) \).
(d) Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), by (b) \( \psi^* \text{cl}(A \cap B) \subseteq \psi^* \text{cl}(A) \) and \( \psi^* \text{cl}(A \cap B) \subseteq \psi^* \text{cl}(B) \). Hence \( \psi^* \text{cl}(A \cap B) \subseteq \psi^* \text{cl}(A) \cap \psi^* \text{cl}(B) \).
(e) Follows from the definition of \( \psi^* \text{-closure} \).

Remark 5.6: The reverse inclusion of (d) is not true in general as seen from the following example.

Example 5.7: Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, \{a\}, \{a, b, c\}, X\} \). If \( A = \{a\} \) and \( B = \{d\} \), then \( \psi^* \text{cl}(A) = X \) and \( \psi^* \text{cl}(B) = \{d\} \). \( A \cap B = \emptyset \), \( \psi^* \text{cl}(A \cap B) = \emptyset \). But \( \psi^* \text{cl}(A) \cap \psi^* \text{cl}(B) = \{d\} \).

Theorem 5.8: The \( \psi^* \text{-closure} \) is a Kuratowski closure operator on \((X, \tau)\).

Proof: From \( \psi^* \text{cl}(\emptyset) = \emptyset \), \( \psi^* \text{cl}(A \cup \psi^* \text{cl}(A)) = \psi^* \text{cl}(A) \), and \( \psi^* \text{cl}(\psi^* \text{cl}(A)) = \psi^* \text{cl}(A) \) we can say that \( \psi^* \text{-closure} \) is a Kuratowski closure operator on \((X, \tau)\).

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