International Journal of Mathematical Archive-7(6), 2016, 49-59
Available online through www.ijma.info ISSN 2229 – 5046

ON FUZZY IDEALS OF BCI-ALGEBRAS WITH DEGREES IN THE INTERVAL (0, 1]

M. A. HASHEMI*

Department of Mathematics,
Payame Noor University, P. O. Box: 19395-3697, Tehran, Iran.

(Received On: 29-04-16; Revised & Accepted On: 18-06-16)

ABSTRACT

In this paper, we introduced the notation of an enlarged quasi associative ideal and associative ideal and a fuzzy quasi associative ideal and associative ideal in BCI-algebras. Relations between enlarged quasi associative ideal, associative ideal and p-ideals are given and their properties are investigated.

***AMS**: 06F35, 03G25, 08A72.*

***Keywords:** Fuzzy ideal, fuzzy associative ideal, fuzzy quasi associative ideal, fuzzy p-ideal, enlarged p-ideal, enlarged quasi associative ideal, enlarged associative ideal.*

1. INTRODUCTION

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. Since 1965 Zadeh's [13] invention, the concept of fuzzy sets has been extensively applied to many mathematical field . On the other hand, the theory of BCI/BCK-algebras introduced by Iseki [5] and it has been raised by Imai and Iseki [4]. In 1999, Khalid and Ahmad [8] introduced fuzzy H-ideals in BCI-algebras. In 2002, Liu and Zhang [9] introduced fuzzy α - ideals in BCI-algebras and they found some relations between fuzzy p-ideals, fuzzy H-ideals and fuzzy α-ideals. In 2012, Hwang and Ahn introduced the notion of an enlarged p-ideal and a fuzzy p-ideal in BCI-algebras with degree. In this paper, we introduced the notation of an enlarged quasi associative ideal and associative ideal and a fuzzy quasi associative ideal and fuzzy associative ideal in BCI-algebras. Relations between enlarged quasi associative ideal, associative ideal and p-ideals are given and their properties are investigated.

2. PRELIMINARIES

By a BCI-algebra we mean an algebra $(X;*,0)$ of type $(2,0)$ satisfying the following axioms:

(1). $((x*y)*(x*z)*(z*y))=0,$

(2). $(x*(x*y)*y)=0,$

(3). $x*x=0,$

(4). $x*y=0$ and $y*x=0$ imply $x=y$

for all $x,y,z\in X$. We can define a partial ordering "$\le$" on X by $x\le y$ if and only if $x*y=0$.

The following statements are true in any BCI-algebra X.

(1.1) $(x*y)*z=(x*z)*y,$

(1.2) $x*0=x,$

(1.3) $(x*z)*(y*z)\le x*y,$

*Corresponding Author: M. A. Hashemi**
Department of Mathematics, Payame Noor University, P. O. Box: 19395-3697, Tehran, Iran.

(1.4) $x \le y$ implies $x * z \le y * z$ and $z * y \le z * x$,

(1.5) $0 * (x * y) = (0 * x) * (0 * y)$,

(1.6) $x * (x * (x * y)) = x * y$.

Definition 2.1: A non empty subset I of X is called an ideal of X if it satisfies:

(I_1) $0 \in I$,

(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

By ([15]), A nonempty subset I of X is called an p-ideal of X if it satisfies condition (I_1) and (I_3) $(x * z) * (y * z) \in I$ and $y \in I$ imply $x \in I$ and by easy calulation we can see that p-ideal is an ideal and by ([10]) we can say that a nonempty subset I of X is called an quasi associative ideal of X if it satisfies condition (I_1) and (I_4) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$. Putting $z = 0$ in (I_4), we can see that every quasi associative ideal is an ideal. Also, a nonempty subset I of X is called an associative ideal of X if it satisfies condition (I_1) and (I_5) $(x * z) * (0 * y) \in I$. and $z \in I$ imply $y * x \in I$ (see [10]).

In [10], Y. L. Liu and *et.al.* shown that the relation between p-ideals, quasi associative ideals and associative ideals and they proved some properties of them.

Definition 2.2: A fuzzy set μ of BCI-algebra X is called fuzzy ideal of X if it satisfies

(FI_1) $\mu(0) \ge \mu(x)$

(FI_2) $\mu(x) \ge \min\{\mu(x * y), \mu(y)\}$.

Definition 2.3: ([6]) A fuzzy set μ of BCI-algebra X is called fuzzy p-ideal of X if it satisfies (FI_1) and (FI_3) $\mu(x) \ge \min\{\mu(x * z) * (y * z), \mu(y)\}$.

Definition 2.4: ([8]) A fuzzy set μ of BCI-algebra X is called fuzzy quasi associative ideal of X if it satifies (FI_1) and (FI_4) $\mu(x * z) \ge \min\{\mu(x * (y * z)), \mu(y)\}$.

Definition 2.5: ([9]) A fuzzy set μ of BCI-algebra X is called fuzzy associative ideal of X if it satisfies (FI_1) and (FI_4) $\mu(y * x) \ge \min\{\mu(x * z) * (0 * y), \mu(z)\}$.

Proposition 2.6: ([10, 6]) Let μ be a fuzzy set in a BCI-algebra X. Then μ is a fuzzy associative (resp., quasi associative, p-ideal) ideal of X if and only if for all $t \in [0,1], \mu_\tau \ne \phi \Rightarrow \mu_\tau$ is an associative (resp., quasi associative, p-ideal) ideal of X, where $\mu_\tau = \{x \in X \,|\, \mu(x) \ge t\}$.

Definition 2.7: ([7]) Let I be a non-empty subset of a BCK/BCI-algebra X which is not necessary an ideal of X. We say that a subset I of X is an enlarged ideal of X related to I if it satisfies:

(EI_1) I is a subset of J

(EI_2) $0 \in J$,

(EI_3) for all $x \in X$ and $x * y \in I, y \in I$ imply $x \in J$.

Definition 2.8: ([3]) Let I be a non-empty subset of a BCI-algebra X which is not necessary a p-ideal of X. We say that a subset J of X is an enlarged p-ideal of X related to I if it satisfies:

(EI_1) I is a subset of J,

(EI_2) $0 \in J$,

(EI_4) for all $x, y, z \in X$, $(x * z) * (y * z) \in I$ and $y \in I$ imply $x \in J$.

Clearly, every p-ideal is an enlarged p-ideal of X related to itself. There exists an enlarged p-ideal of X related to any non-empty subset I of a BCI-algebra X (see [3]).

Remark 2.9: let λ and k be members of $(0,1]$, and let n and k denote a natural number and a real number, respectively, such that $k < n$ unless otherwise specified.

Definition 2.10: ([7]) A fuzzy subset μ of a BCK/BCI-algebra X is called a fuzzy ideal of X with degree (λ, k) if it satisfies:

(1) For all $x \in X, \mu(0) \geq \lambda\mu(x)$,

(2) For all $x, y \in X, \mu(x) \geq k \ \min\{\mu(x * y), \mu(y)\}$.

Definition 2.11: ([3]) A fuzzy subset μ of a BCK/BCI-algebra X is called a fuzzy p-ideal of X with degree (λ, k) if it satisfies:

(1) For all $x \in X, \mu(0) \geq \lambda\mu(x)$,

(2) For all $x, y \in X, \mu(x) \geq k \ \min\{\mu(x * z) * (y * z), \mu(y)\}$.

Now, if $\lambda \neq k$, then a fuzzy p-ideal with degree (λ, k) may not be a fuzzy p-ideal with degree (λ, k) and vice versa (see Example 3.8 of [3]).

3. FUZZY QUASI ASSOCIATIVE IDEALS OF BCI-ALGEBRAS WITH DEGREES IN THE INTERVAL $(0,1]$

Definition 3.1: Let I be a non-empty subset of a BCI-algebra X which is not necessary a quasi associative ideal of X. We say that a subset J of X is an enlarged quasi associative ideal of X related to I if it satisfies:

(EI_1) I is a subset of J,

(EI_2) $0 \in J$, (EI_3) for all $x, y, z \in X$, $x * (y * z) \in I$ and $y \in I$ imply $x * z \in J$.

Putting $z = 0$, then we can see that every enlarge quasi associative ideal is an ideal. Also, we can see that every quasi associative ideal is enlarge quasi associative ideal. But the following example show that an enlarged quasi associative ideal of X related to any non-empty subset I of a BCI-algebra X.

Example 3.2: Consider a BCI-algebra $X = \{0,1,2,3\}$ with the following Cayley table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $\{0,1,2\}$ is an enlarged quasi associative ideal of X related to $\{1\}$. But $\{0,1,2\}$ is not an quasi associative ideal, because $0 * (1 * 3) = 2, 1 \in \{0,1,2\}$ but $0 * 3 = 3 \notin \{0,1,2\}$.

Theorem 3.3: Let I be a non-empty subset of a BCI-algebra X. Every enlarged quasi associative ideal of X related to I is an enlarged ideal of X related to I.

Proof: Let J be an enlarged quasi associative ideal of X related to I. By putting $z = 0$ and for all $x, y \in X$ we have $x * (y * 0) = x * y \in I$ and $y \in I$ imply $x = x * 0 \in J$. Hence J is an enlarged ideal of X related to I.

The following example show that the converse of Theorem 3.3 is not true:

Example 3.4: ([3]) Consider a BCI-algebra $X = \{0,1,2,3,4\}$ with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	2	1	0	2
4	4	4	4	4	0

Note that $\{0,1,3\}$ is not both an ideal and an quasi associative ideal of X. Then $\{0,1,2,3\}$ is an enlarged ideal of X related to $\{0,1,3\}$ but not an enlarged quasi associative ideal of X related to $\{0,1,3\}$ since $4*(1*2)=0$, $3\in\{0,1,3\}$ and $4*2\notin\{0,1,3\}$.

Proposition 3.5: If μ is a fuzzy ideal of a BCI-algebra X with degree (λ,k), such that for all $x,y,z\in X$, $(x*y)*z=0$, then $\mu(x)\geq \lambda k^2 \min\{\mu(y),\mu(z)\}$.

Proof: Let $x,y,z\in X$ be such that $(x*y)*z=0$. Then

$$\begin{aligned}
\mu(x) &\geq k\min\{\mu(x*y),\mu(y)\}.\\
&\geq k\min\{k\min\{\mu((x*y)*z),\mu(z)\}\mu(y)\}\\
&\geq k\min\{k\min\{\mu(0),\mu(z)\}\mu(y)\}\\
&\geq k\min\{k\min\{\lambda\mu(z),\mu(z)\}\mu(y)\}\\
&\geq k\min\{k\lambda\mu(z),\mu(y)\}\\
&\geq k\min\{k\lambda\mu(z),k\lambda\mu(y)\}\\
&\geq \lambda k^2\min\{\mu(y),\mu(z)\}.
\end{aligned}$$

The proof is complete.

Corollary 3.6: Let μ be a fuzzy ideal of BCI-algebra X with degree (λ,k). If $(\ldots((x*a_1)*a_2)*\ldots)*a_n=0$, then $\mu(x)\geq \lambda k^n \min\{\mu(a_1),\ldots,\mu(a_n)\}$.

Proof: The proof is easy by induction.

Definition 3.7: A fuzzy subset μ of a BCK/BCI-algebra X is called a fuzzy quasi associative ideal of X with degree (λ,k) if it satisfies:

(1) For all $x\in X, \mu(0)\geq \lambda\mu(x)$,

(2) For all $x,y\in X, \mu(x*z)\geq k \min\{\mu(x*(y*z)),\mu(y)\}$.

Now, if $\lambda\neq k$, then a fuzzy quasi associative ideal with degree (λ,k) may not be a fuzzy quasi associative ideal with degree (λ,k) and vice versa.

Example 3.8: Consider a BCI-algebra $X=\{0,1,2,3\}$ with the following Cayley table:

*	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Define $\mu : X \to [0,1]$ as follows:

X	0	1	2	3
μ	1	1	0.5	0.5

Then μ is a fuzzy quasi associative ideal of X with degree $\left(\frac{3}{8},\frac{3}{8}\right)$, but it is neither a fuzzy quasi associative ideal of X nor a fuzzy quasi associative ideal of X with degree $\left(\frac{3}{4},\frac{3}{4}\right)$ since

$$\mu(2*3) = \mu(3) = 0.5 \geq \min\{\mu(2*(0*3)), \mu(0)\}$$

And

$$\mu(2*3) = \mu(3) = 0.5 \geq \frac{3}{4}\min\{\mu(2*(0*3)), \mu(0)\}.$$

Now, consider a BCI-algebra X and a fuzzy subset μ as in Example 3.8, By easy calculation, μ is a fuzzy ideal of X with degree $\left(\frac{1}{4},\frac{3}{4}\right)$. But it is not a fuzzy quasi associative ideal of X with degree $\left(\frac{1}{4},\frac{3}{4}\right)$. Because

$$\mu(2*3) = \mu(3) = 0.5 \geq \frac{3}{4}\min\{\mu(2*(0*3)), \mu(0)\}. .$$

Theorem 3.9: If μ is a fuzzy quasi associative ideal of a BCI-algebra X with degree $(\lambda.k)$, then μ is a fuzzy ideal of X with degree $(\lambda.k)$.

Proof: It's straightforward by getting $z = 0$ in Definition 3.7.

Theorem 3.10: Let μ be a fuzzy ideal of a BCI-algebra X with degree (λ, k), which it satisfies the condition $\mu(x*y) \geq \mu(x)$, for all $x, y \in X$, then μ is a fuzzy quasi associative ideal of X with degree (λ, k).

Proof: Assume that μ be a fuzzy ideal of X with degree (λ, k). For all $x, y, z \in X.$ we have

$$\begin{aligned} \mu(x*z) &\geq k\min\{\mu((x*z)*(y*z)), \mu(y*z)\} \\ &= k\min\{\mu((x*y)*z), \mu(y*z)\} \\ &\geq k\min\{\mu(x*(y*z)), \mu(y*z)\}. \end{aligned}$$

Hence μ is a fuzzy quasi associative ideal of X with degree (λ, k).

Theorem 3.11: Let μ be a fuzzy subset of associative BCI-algebra X such that μ is a fuzzy ideal with degree (λ, k). then μ is a fuzzy quasi associative ideal of X.

Proof: Let μ is a fuzzy ideal of X with degree (λ, k). then it satisfies the condition (1) of Definition 3.7. For all $x, y, z \in X$, we have

$$\begin{aligned} \mu(x*z) &\geq k\min\{\mu((x*z)*y), \mu(y)\} \\ &= k\min\{\mu((x*y)*z), \mu(y)\} \\ &= k\min\{\mu((x*(y*z))*y), \mu(y)\}. \end{aligned}$$

Hence μ is a fuzzy quasi associative ideal of X with degree (λ, k).

Theorem 3.12: Let μ be a fuzzy subset of X such that μ is a fuzzy ideal of X with degree (λ,k). Let the following properties be hold:

(1) μ is a fuzzy quasi associative ideal of X with degree (λ,k).

(2) for all $x,y\in X$, $\mu(x*y)\geq \lambda k\mu(x*(0*y))$.

(3) for all $x,y,z\in X$, $\mu((x*y)*z)\geq \lambda^2k^2\mu(\mu(x*(y*z)))$

Then (1) satisfies (2) and (2) satisfies (3)

Proof: (1)$\Rightarrow$ (2): Let μ is a fuzzy quasi associative ideal of X with degree (λ,k). Then, for all $x,y\in X$ we have

$$\begin{aligned}\mu(x*y)&\geq k\min\{\mu(x*(0*y)),\mu(0)\}\\&\geq k\min\{\mu(x*(0*y)),\lambda\mu(x*(0*y))\}\\&=\lambda k\mu(x*(0*y)).\end{aligned}$$

Therefore the inequality (2) is satisfied.

(2)$\Rightarrow$ (3): Suppose that (2) is satisfied. For all $x,y,z\in X$, we have

$$\begin{aligned}((x*y)*(0*z))*(x*(y*z))&=((x*y)*(x*(y*z)))*(0*z)\\&\leq((y*z)*y)*(0*z)\\&=((y*y)*z)*(0*z)\\&=(0*z)*(0*z)=0\end{aligned}$$

It follows from Proposition 3.10 that

$$\mu((x*y)*(0*z))\geq\lambda k\mu(x*(y*z)).$$

Now, by (2) we get $\mu((x*y)*z)\geq\lambda k\mu((x*y)*(0*z))\geq\lambda^2k^2\mu(x*(y*z))$.

Theorem 3.13: Let μ be a fuzzy subset of X such that μ is a fuzzy ideal of X with degree (λ,k) and for all $x,y,z\in X$, $\mu((x*y)*z)\geq\lambda^ik^j\mu(x*(y*z))$. Then μ is a fuzzy quasi associative ideal of X with degree (λ,t), where $t\leq\lambda^ik^{j+1}$.

Proof: For all $x,y,z\in X$ we have

$$\begin{aligned}\mu(x*z)&\geq k\min\{\mu((x*z)*y),\mu(y)\}\\&=k\min\{\mu((x*y)*z),\mu(y)\}\\&\geq k\min\{\lambda^ik^i\mu(x*(y*z)),\mu(y)\}\\&\geq k\min\{\lambda^ik^j\mu(x*(y*z)),\lambda^ik^j\mu(y)\}\\&=\lambda^ik^{j+1}\min\{\mu(x*(y*z)),\mu(y)\}\\&\geq t\min\{\mu(x*(y*z)),\mu(y)\}.\end{aligned}$$

Therefore, μ is a fuzzy quasi associative ideal of X with degree (λ,t).

Corollary 3.14: Let μ be a fuzzy subset of X such that μ is a fuzzy ideal of X with degree (λ,k) and for all $x,y,z\in X$, $\mu((x*y)*z)\geq\mu(x*(y*z))$. Then μ is a fuzzy quasi associative ideal of X with degree (λ,k).

Theorem 3.15: Let μ be a fuzzy subset of a BCI-algebra X. For all $t\in[0,1]$ with $t\leq\max\{\lambda,k\}$, if $U(\mu;t)$ is an enlarged quasi associative ideal of X related to $U\left(\mu;\dfrac{t}{\max\{\lambda,k\}}\right)$. then μ is a fuzzy quasi associative ideal of X with degree (λ,k).

Proof: Suppose that $\mu(0)<t\leq\lambda\mu(x)$ for some $x\in X$ and $t\in(0,\lambda]$. Then $\dfrac{t}{\max\{\lambda,k\}}\leq\dfrac{t}{\lambda}\leq\mu(x)$. Thus, we have $x\in\left(\mu;\dfrac{t}{\max\{\lambda,k\}}\right)$, i.e., $U\left(\mu;\dfrac{t}{\max\{\lambda,k\}}\right)\neq\phi$. Since $U(\mu;t)$ is an enlarged quasi associative ideal of X related to $U\left(\mu;\dfrac{t}{\max\{\lambda,k\}}\right)$. we have $0\in U(\mu;t)$, i.e., $\mu(0)\geq t$. This is a contradiction, and thus $\mu(0)\geq\lambda\mu(x)$ for all $x\in X$. Now, assume that there exist $a,b,c\in X$ such that $\mu(a*c)<k\min\{\mu(a*(b*c)),\mu(b)\}$. If we take $t:=k\min\{\mu(a*(b*c)),\mu(b)\}$, then $t\in(0,k]\subseteq(0,\max\{\lambda,k\}]$. Hence $a*(b*c)\in U\left(\mu;\dfrac{t}{k}\right)\subseteq U\left(\mu;\dfrac{t}{\max\{\lambda,k\}}\right)$ and

$b\in U\left(\mu;\dfrac{t}{k}\right)\subseteq U\left(\mu;\dfrac{t}{\max\{\lambda,k\}}\right)$. It follows from Definition 4.1 that $a*c\in U(\mu;t)$ so that $\mu(a*c)\geq t$. which is impossible. Therefore, for all $x,y,z\in X$. we have $\mu(a*c)\geq k\min\{\mu(a*(b*c)),\mu(b)\}$. Thus μ is a fuzzy quasi associative ideal of X with degree (λ,k).

Corollary 3.16: Let μ be a fuzzy subset of a BCI-algebra X. For any $t\in[0,1]$ with $t\leq\dfrac{k}{n}$, if $U(\mu;t)$ is an enlarged quasi associative ideal of X related to $U\left(\mu;\dfrac{n}{k}t\right)$, then μ is a fuzzy quasi associative ideal of X with degree $\left(\dfrac{k}{n},\dfrac{k}{n}\right)$.

Theorem 3.17: Let $t\in[0,1]$ be such that $U(\mu;t)(\neq\phi)$ is not necessary an quasi associative ideal of a BCI-algebra X. If μ is a fuzzy quasi associative ideal of X with degree (λ,k), then $U(\mu;t\min\{\lambda,k\})$ is an enlarged quasi associative ideal of X related to $U(\mu;t)$.

Proof: Since $t\min\{\lambda,k\}\leq t$, we have $U(\mu;t)\subseteq U(\mu;t\min\{\lambda,k\})$. Since $U(\mu;t)\neq\phi$. there exists $x\in U(\mu;t)$ and so $\mu(x)\geq t$. By Definition 3.7(1), we have

$$t\min\{\lambda,k\}\leq\lambda t\leq\lambda\mu(x)\leq\mu(0).$$

Therefore $0 \in U(\mu; t\min\{\lambda, k\})$. Now, suppose that $x, y, z \in X$ be such that $x*(y*z) \in U(\mu;t)$ and $y \in U(\mu;t)$. Then $\mu(x*(y*z)) \geq t$ and $\mu(y) \geq t$. It follows from Definition 3.7(2) that

$$\mu(x*z) \geq k\min\{\mu(x*(y*z)), \mu(y)\} \geq kt \geq t\min\{\lambda, k\}.$$

so that $x \in U(\mu; t\min\{\lambda, k\})$. Hence $U(\mu; t\min\{\lambda, k\})$ is an enlarged quasi associative ideal of X related to $U(\mu;t)$..

4. FUZZY ASSOCIATIVE IDEALS OF BCI-ALGEBRAS WITH DEGREES IN THE INTERVAL $(0,1]$

Definition 4.1: Let I be a non-empty subset of a BCI-algebra X which is not necessary an associative ideal of X. We say that a subset J of X is an enlarged associative ideal of X related to I if it satisfies:

(EAI_1) I is a subset of J,

(EAI_2) $0 \in J$,

(EAI_3) for all $x \in X, 0*x \in J$ imply $x \in J$,

(EAI_4) for all $x, y, z \in X, (x*z)*(0*y) \in I$ and $z \in I$ imply $y*x \in J$.

Putting $z = 0$, then we can see that every enlarge associative ideal is an ideal. Also, we can see that every associative ideal is enlarge associative ideal. But the following example show that an enlarged associative ideal of X related to any non-empty subset I of a BCI-algebra X.

Example 4.2: Consider a BCI-algebra $X = \{0,1,2,3\}$ with the following Cayley table:

*	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Then $\{0,1\}$ is an enlarged associative ideal of X related to $\{1\}$. But $\{0,1\}$ is not an associative ideal, because $(0*0)*(0*3) = 0, 0 \in \{0,1\}$ but $3*0 = 3 \notin \{0,1\}$.

Theorem 4.3: Let I be a non-empty subset of a BCI-algebra X. Every enlarged associative ideal of X related to I is an enlarged ideal of X related to I.

Proof: Let J be an enlarged associative ideal of X related to I. For all $x, y \in X, x*y \in I$ and $y \in I$. we have

$$(x*y)*(0*0) = x*y \in I \text{ and } y \in I \text{ imply } 0*x \in J.$$

By, (EAI_3) we have, $x \in J$. Hence J is an enlarged ideal of X related to I..

Example 3.4, show that the converse of Theorem 4.3 is not true, because $\{0,1,2,3\}$ is an enlarged ideal of X related to $\{0,1,2\}$ but not an enlarged associative ideal of X related to $\{0,1,3\}$ since $(0*3)*(0*2) = 0, 3 \in \{0,1,3\}$ and $2*0 \notin \{0,1,3\}$.

Definition 4.4: A fuzzy subset μ of a BCK/BCI-algebra X is called a fuzzy associative ideal of X with degree (λ, k) if it satisfies:

(1) For all $x \in X, \mu(0) \geq \lambda\mu(x)$,

(2) For all $x \in X, \mu(x) \geq \mu(0*x)$,

(3) For all $x, y \in X, \mu(y*x) \geq k\min\{\mu((x*z)*(0*y))\mu(z)\}$.

Now, if $\lambda \neq k,$ then a fuzzy associative ideal with degree (λ, k) may not be a fuzzy associative ideal with degree (λ, k) and vice versa.

Example 4.5: In Example 3.8, we can see that μ is a fuzzy associative ideal of X with degree $\left(\frac{3}{8}, \frac{3}{8}\right)$, but it is neither a fuzzy associative ideal of X nor a fuzzy associative ideal of X with degree $\left(\frac{3}{4}, \frac{3}{4}\right)$, since

$$\mu(2*3) = \mu(3) = 0.5 \geq \min\{\mu((3*0)*(0*2)), \mu(0)\}. \quad .$$

and

$$\mu(2*3) = \mu(3) = 0.5 \geq \frac{3}{4}\min\{\mu((3*0)*(0*2)), \mu(0)\}.$$

Now, consider a BCI-algebra X and a fuzzy subset μ as in Example 3.8. By easy calculation, μ is a fuzzy ideal of X with degree $\left(\frac{1}{4}, \frac{3}{4}\right)$. But it is not a fuzzy associative ideal of X with degree $\left(\frac{1}{4}, \frac{3}{4}\right)$. Because

$$\mu(2*3) = \mu(3) = 0,\ 5 \geq \frac{3}{4}\min\{\mu((3*0)*(0*2)), \mu(0)\}. \quad .$$

Theorem 4.6: If μ is a fuzzy associative ideal of a BCI-algebra X with degree (λ, k), then μ is a fuzzy ideal of X with degree (λ, k).

Proof: By getting $z = y,\ \ y = 0$ in Definition 4.4, the proof is complete.

Theorem 4.7: Let μ be a fuzzy subset of X such that μ is a fuzzy ideal of X with degree (λ, k). Let the following properties be hold:

(1) μ is a fuzzy associative ideal of X with degree (λ, k),

(2) For all $x, y \in X,\ \mu(y*(x*z)) \geq \lambda k \mu((x*z)*(0*y))$.

(3) For all $x, y, z \in X,\ \mu(y*x) \geq \lambda k \mu(x*(0*y))$.

Then (1) satisfies (2) and (2) satisfies (3).

Proof: (1)$\Rightarrow$ (2): Assume that μ is a fuzzy associative ideal of X with degree (λ, k). We have

$$\begin{aligned}\mu(y*(x*z)) &\geq k\min\{\mu(((x*z)*0)*(0*y)), \mu(0)\} \\ &\geq k\min\{\mu(((x*z)*0)*(0*y)), \lambda\mu(((x*z)*0)*(0*y))\} \\ &= \lambda k \mu(((x*z)*0)*(0*y)).\end{aligned}$$

(2)$\Rightarrow$ (3): By getting $z = 0$ in (2), we have (3).

Theorem 4.8: Let μ be a fuzzy subset of a BCI-algebra X. For all $t \in [0,1]$ with $t \leq \max\{\lambda, k\}$, if $U(\mu; t)$ is an enlarged associative ideal of X related to $U\left(\mu; \frac{t}{\max(\lambda, k)}\right)$, then μ is a fuzzy associative ideal of X with degree (λ, k).

Proof: The proof is the similar of Theorem 3.15.

Corollary 4.9: Let μ be a fuzzy subset of a BCI-algebra X. For any $t \in [0,1]$ with $t \leq \frac{k}{n}$, if $U(\mu;t)$ is an enlarged associative ideal of X related to $U\left(\mu;\frac{n}{k}t\right)$, then μ is a fuzzy associative ideal of X with degree $\left(\frac{k}{n},\frac{k}{n}\right)$.

Theorem 4.10: Let $t \in [0,1]$ be such that $U(\mu;t)(\neq \phi)$ is not necessary an associative ideal of a BCI-algebra X. If μ is a fuzzy associative ideal of X with degree (λ,k), then $U(\mu;t\min\{\lambda,k\})$ is an enlarged associative ideal of X related to $U(\mu;t)$.

Proof: The proof is the similar of Theorem 3.17.

Theorem 4.11: If μ is a fuzzy associative ideal of a BCI-algebra X with degree (λ,k), then μ is a fuzzy p-ideal of X with degree (λ,k).

Proof: Assume that μ be a fuzzy associative ideal of X with degree (λ,k). Setting $x=0$ in Theorem 4.7, we have $\mu(x) \geq \lambda k \mu(0*(0*x))$. Hence, by Theorem 3.15 of [3], μ is fuzzy p-ideal with degree (λ,k).

Theorem 4.12: If μ is a fuzzy associative ideal of a BCI-algebra X with degree (λ,t), then μ is a fuzzy quasi associative ideal with degree (λ,t), where $t \leq \lambda^2 k^2$..

Proof: Assume that μ be a fuzzy associative ideal of X with degree (λ,k). For all $x, y \in X$. we have

$$\begin{aligned}\Big(0*\big(0*\big(y*(0*x)\big)\big)\Big)*\big(x*(0*y)\big) &= \big(0*(0*y)\big)*\Big(0*\big(0*(0*x)\big)\Big)*\big(x*(0*y)\big)\\ &= \Big(\big(0*(0*y)\big)*(0*x)\Big)*\big(x*(0*y)\big)\\ &\leq \big(x*(0*y)\big)*\big(x*(0*y)\big)=0,\end{aligned}$$

We have

$$\mu\Big(0*\big(0*\big(y*(0*x)\big)\big)\Big) \geq \lambda k \mu\big(x*(0*y)\big),$$

as Proposition 3.5. Also, by Definition 4.4, we have

$$\mu\big(y*(0*x)\big) \geq \lambda k \mu\big(x*(0*y)\big),$$

and by Theorem 4.7 (3), we have

$$\mu(x*y) \geq \lambda^2 k^2 \mu\big(x*(0*y)\big).$$

Hence, by Theorem 3.13, μ is a fuzzy quasi associative ideal with degree (λ,t), where $t \leq \lambda^2 k^2$. .

5. CONCLUSIONS

We introduced the notion of a enlarge quasi associative ideal and enlarge quasi associative ideals of a BCI-algebra and also, fuzzy quasi associative and associative ideal with degree (λ,k). Some properties of them are provided. Relations between fuzzy p-ideals, quasi associative ideals and associative ideals with degree (λ,k) are given. For future works, we can work on some other fuzzy ideals in BCI-algebra, for example, fuzzy implicative ideals, fuzzy positive implicative ideals, ... Also, we canfind some relations between fuzzy p-ideal, fuzzy quasi associative ideal and fuzzy associative ideal with degree (λ,k) and other fuzzy ideals such as fuzzy implicative, positive implicative ideals... with degree (λ,k).

REFERENCES

1. C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.
2. Y. Huang, BCI-Algebras, Science Press, Beijing, 2006.
3. Y. S. Hwang and S. S. Ahn, Fuzzy p-ideals of BCI-algebras with degrees in the interval $(0,1]$, Commun. Korean Math. Soc. 27 (2012), no. 4, 701-708.
4. Y. Imai and K. Iseki, On axiom system of propositional calculi, Proc. Japan Academy 42 (1966), 19-22.
5. K. Iseki, On BCI-algebras, Math. Math. Sem. Notes Kobe Univ. 8 (1980), 125- 130.
6. Y. B. Jun and J. Meng, Fuzzy p-ideals in BCI-algebras, Math. Japon. 40 (1994), no. 2, 271-282.
7. Y. B. Jun, E. H. Roh, and K. J. Lee, Fuzzy sub algebras and ideals of BCK/BCI-algebras with degree in the interval $(0,1]$, Fuzzy Sets and Systems, submitted.
8. Khalid, H.M. and Ahmad, B., Fuzzy H-ideals in BCI-algebras, Fuzzy Sets and Systems 101 (1999), 153-158.
9. Y. L. Liu and X. H. Zhang, Fuzzy a-ideals in BCI-algebras, Adv. in Math. (China), 31 (2002), 65-73.
10. Y. L. Liu, J. Meng, X. H. Zhang and Z. C. Yue, q-ideals and a-ideals in BCI- algebras, Southeast Asian Bull. Math., 24 (2000), 243-253.
11. A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512-517.
12. O. G. Xi, Fuzzy BCK-algebras, Math. Japon. 36 (1991), no. 5, 935-942.
13. L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965), 338-353.
14. Zadeh, L.A., Is there a need for fuzzy logic?, Inform. Sci. 178 (2008), 2751-2779.
15. X. Zhang, J. Hao, and S. A. Bhatti, On p-ideals of a BCI-algebra, Punjab Univ. J. Math. 27 (1994), 121-128.

Source of support: Nil, Conflict of interest: None Declared