ON A CLASS OF P-VALENT FUNCTIONS WITH ALTERNATING TYPE

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ABSTRACT

In this paper, we introduce a new subclass which are analytic and p-valent with alternating coefficients. Some results like coefficient estimation, radius of convexity, closure theorem, extreme points, convolution and inclusion property of p-valent functions are investigated.

Keywords: analytic function, p-valent function, radius of convexity, convolution property, inclusion property.

1. INTRODUCTION

Let \( \mathcal{A}(p) \) denote the class of normalized univalent functions of the form:

\[
f(z) = z^p + \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+p} z^{n+p}, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})
\]

which are analytic and p-valent in the unit disc \( \mathbb{E} = \{z : z \in \mathbb{C}; |z| < 1\} \).

A function \( f(z) \in \mathcal{A}(p) \) is said to in the class of \( \mathcal{S}^*_p(\alpha) \) p-valently starlike function of order \( \alpha \) (\( 0 \leq \alpha < p \)) if it satisfies, for \( z \in \mathbb{E} \), the condition

\[
Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha
\]

Furthermore, a function \( f(z) \in \mathcal{A}(p) \) is said to in the class \( \mathcal{K}_p(\alpha) \) of p-valently convex function of order \( \alpha \) (\( 0 \leq \alpha < p \)) if it satisfies, for \( z \in \mathbb{E} \), the condition

\[
Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha
\]

H. Özlem Güney and S. Sümmer Eker, [1], Yi-Hui Xu, Qing Yang and Jin-Lin Liu [2], K.S.Padmanabhan and Ganeshan[3] and S.L.Shukla and Dastrath[4] have studied the certain classes of analytic functions with negative coefficients. In this paper we introduce a new subclass \( S^*_p(\alpha, \beta, \xi, \gamma) \) of \( \mathcal{A}(p) \) defined by (1.1) and also satisfying condition:

\[
\frac{zf'(z)}{f(z)} + p < \beta
\]

\[
2\xi \left[ \frac{zf'(z)}{f(z)} + \alpha - \gamma \right] - \frac{zf'(z)}{f(z)} + p < \beta
\]

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where \( |z| < 1, p \in \mathbb{N}, 0 \leq \alpha < 1, 0 \leq \beta < 1, \frac{1}{2} < \xi \leq 1, \frac{1}{2} < \gamma \leq 1 \).

We obtain the results like coefficient estimation, radius of convexity, closure theorem, extreme points, convolution and inclusion property of analytic and p-valent functions alternating type.

2. COEFFICIENT ESTIMATION

Theorem 2.1: A function \( f(z) \in A(p) \) is in the class \( S^*(\alpha, \beta, \xi, \gamma) \) if and only if

\[
2p + \sum_{n=p}^{\infty} [(p + n + 1) + \beta(p + \alpha + 1) - \gamma(p + n + 1)] \leq 2\xi\beta[p(\xi - \gamma) + \alpha\xi]
\]

Proof: Assume that inequality \( (2.1) \) holds true and let \( |z| = 1 \). We show that \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \). From (1.4),

\[
2\xi \left| \frac{f'(z)}{f(z)} + p \right| - \gamma \left| \frac{f'(z)}{f(z)} + p \right| = \frac{2p + \sum_{n=p}^{\infty} (-1)^{n+1}(p + n + 1)a_{n+1}z^{n+1-p}}{2p[(\xi - \gamma) + 2\xi\alpha] - \sum_{n=p}^{\infty} (-1)^{n+1}[2\xi(\alpha + n + 1) - \gamma(p + n + 1)]a_{n+1}z^{n+1-p}} \leq \frac{2p + \sum_{n=p}^{\infty} (-1)^{n+1}(p + n + 1)a_{n+1}}{2p[(\xi - \gamma) + 2\xi\alpha] - \sum_{n=p}^{\infty} (-1)^{n+1}[2\xi(\alpha + n + 1) - \gamma(p + n + 1)]a_{n+1}}
\]

Above inequality is bounded above by \( \beta \) if,

\[
2p[(\xi - \gamma) + 2\xi\alpha] - \sum_{n=p}^{\infty} (-1)^{n+1}[2\xi(\alpha + n + 1) - \gamma(p + n + 1)]a_{n+1} \leq 2\beta[(\xi + \alpha) - p\gamma]
\]

Hence by maximum modulus theorem, we have \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \).

To prove the converse, assume that

\[
\left| \frac{f'(z)}{f(z)} + p \right| = \beta < \frac{2p + \sum_{n=p}^{\infty} (-1)^{n+1}(p + n + 1)a_{n+1}z^{n+1-p}}{2p[(\xi - \gamma) + 2\xi\alpha] - \sum_{n=p}^{\infty} (-1)^{n+1}[2\xi(\alpha + n + 1) - \gamma(p + n + 1)]a_{n+1}z^{n+1-p}} < \beta
\]
Note that \( \text{Re}(z) \leq |z| \) for all \( z \), and so
\[
\text{Re} \left( 2p + \sum_{n=p}^{\infty} (-1)^{n+1} (p + n + 1) a_{n+1} z^{n+1-p} \right) < \beta
\]
(2.3)

Choosing value of \( z \) on real axis so that \( \frac{f'(z)}{f(z)} \) is real. Upon clearing the denominator in (2.3) and allowing \( Z \to 1 \) through the real values we obviously obtained required assertion (2.1).

**Corollary 2.1**

A function \( f(z) \in S^* (\alpha, \beta, \xi, \gamma) \) then
\[
a_{n+1} \leq \frac{2\xi \beta [p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(p + n + 1) + \beta (2\xi(\alpha + n + 1) - \gamma(p + n + 1))]} \quad \text{for } n \in \mathbb{N}_0 \quad \text{with equality for } f(z) \text{ given by,}
\]
\[
f(z) = z^p + \frac{2\xi \beta [p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(p + n + 1) + \beta (2\xi(\alpha + n + 1) - \gamma(p + n + 1))]} z^{p+n}, (n \in \mathbb{N}_0)
\]
(2.4)

**Corollary 2.2**

A function \( f(z) \in S^* (\alpha, \beta, \xi, \gamma) \) and \( p = 1 \) then
\[
a_{n+1} \leq \frac{2\xi \beta [p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(n + 2) + \beta (2\xi(\alpha + n + 1) - \gamma(n + 2))]} \quad \text{for } n \in \mathbb{N}_0 \quad \text{with equality for } f(z) \text{ given by,}
\]
\[
f(z) = z + \frac{2\xi \beta [p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(n + 2) + \beta (2\xi(\alpha + n + 1) - \gamma(n + 2))]} z^{1+n}, (n \in \mathbb{N}_0)
\]
(2.5)

**Corollary 2.3**

A function \( f(z) \in S^* (\alpha, \beta, 1, 1) \) then
\[
a_{n+1} \leq \frac{2\beta \alpha}{(-1)^{n+1}[(p + n + 1) + \beta (2(\alpha + n + 1) - \gamma(p + n + 1))]} \quad \text{for } n \in \mathbb{N}_0 \quad \text{with equality for } f(z) \text{ given by,}
\]
\[
f(z) = z^p + \frac{2\beta \alpha}{(-1)^{n+1}[(p + n + 1) + \beta (2(\alpha + n + 1) - \gamma(p + n + 1))]} z^{p+n}, (n \in \mathbb{N}_0)
\]
(2.6)

3. **RADIUS OF CONVEXITY AND STARLIKENESS**

**Theorem 3.1**

If \( f(z) \in A(p) \) is in the class \( S^* (\alpha, \beta, \xi, \gamma) \) then \( f(z) \) \( p \)-valently convex in
\[
0 < |z| < R_1 = \inf_n \left[ \frac{p^2 \left[ 2p + (-1)^{n+1} (p + n + 1) + 2\xi(p + n + 1) - \gamma(p + n + 1) \right]}{2\beta[\xi(p + \alpha) - \gamma(n + 1)]^2} \right]^{1/(n+1-p)}
\]
(3.1)

The estimate is sharp for
\[
f(z) = z^p + \frac{2\xi \beta [p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(p + n + 1) + \beta (2\xi(\alpha + n + 1) - \gamma(p + n + 1))]} z^{p+n}, (n \in \mathbb{N}_0)
\]
(3.2)
Proof: It is sufficient to show that,
\[
\left| \frac{1+z}{z} \right| \leq 1 \text{ for } 0 < |z| < R
\]
\[
\left| \frac{1+z}{z} \right| \leq \frac{2p^2 z^{n-1} + \sum_{n=1}^{\infty} (-1)^n (1+n+p) a_{n+1} z^n}{z^{n-1} + \sum_{n=1}^{\infty} (-1)^n (1+n-p) a_{n+1} z^n}
\]
\[
\leq \frac{2p^2 + \sum_{n=1}^{\infty} (-1)^n (1+n+p) a_{n+1} |z|^{n-p+1}}{\sum_{n=1}^{\infty} (-1)^n (1+n-p) a_{n+1} |z|^{n-p+1}}
\]
The last expression is bounded by 1 provided,
\[
\sum_{n=1}^{\infty} \left( \frac{n+1}{p} \right)^2 a_{n+1} |z|^{n-p+1} \leq 1
\] (3.3)

Also from theorem 1, we have
\[
2p + \sum_{n=p}^{\infty} (-1)^{n+1} [(p+n+1) + \beta(2\xi(\alpha+n+1) - \gamma(p+n+1))] a_{n+1}
\]
\[
\leq 2\beta[\xi(p+\alpha) - p\gamma]
\]
Thus (3.3) is satisfied if,
\[
\sum_{n=1}^{\infty} \left( \frac{n+1}{p} \right)^2 a_{n+1} |z|^{n-p+1} \leq \frac{2p + \sum_{n=p}^{\infty} (-1)^{n+1} [(p+n+1) + \beta(2\xi(\alpha+n+1) - \gamma(p+n+1))] a_{n+1}}{2\beta[\xi(p+\alpha) - p\gamma]} - \gamma
\] (3.4)

Solving for |z| we get,
\[
|z| = \inf_n \left[ \frac{p^2 \left[ 2p + (-1)^{n+1} (p+n+1) + 2\xi(p+n+1) - \gamma(p+n+1) \right]^{\frac{1}{n+1-p}}}{2\beta[\xi(p+\alpha) - p\gamma](n+1)^2} \right]^{-\frac{1}{n+1-p}}
\] (3.5)

Substituting |z| < R₁ in (3.5) we obtained required assertion (3.1).

Corollary 3.1: A function \( f(z) \in S_+(\alpha, \beta, \xi, 1) \) then \( f(z) \) is convex in the disc
\[
0 < |z| < R_2 = \inf_n \left[ \frac{p^2 \left[ 2p + (-1)^{n+1} (p+n+1) + 2\xi(p+n+1) - (p+n+1) \right]^{\frac{1}{n+1-p}}}{2\beta[\xi(p+\alpha) - p](n+1)^2} \right]^{-\frac{1}{n+1-p}}
\] (3.6)

The estimate is sharp for the function
\[
f(z) = z^p + \frac{2\xi[\beta(p-1)+\alpha\xi]}{(-1)^{n+1}[(p+n+1)+\beta(2\xi(\alpha+n+1)-(p+n+1))]} z^{p+n} \quad (n \in \mathbb{N}_0)
\] (3.7)
Corollary 3.2: A function \( f(z) \in S^*(\alpha, \beta, 1, 1) \) then \( f(z) \) is convex in the disc
\[
0 < |z| < R_3 = \inf_n \left[ \frac{p^2 \left[ 2p + (-1)^{n+1} (p + n + 1) + 2(p + n + 1) - (p + n + 1) \right]}{2\beta[(p + \alpha) - p](n + 1)^2} \right]^{1/(n+1-p)}
\] (3.8)

The estimate is sharp for the function
\[
f(z) = z^p + \frac{2\beta \alpha}{(-1)^{n+1}[(p + n + 1) + \beta(2(\alpha + n + 1) - (p + n + 1))] z^{p+n}}
\] (3.9)

Corollary 3.3: A function \( f(z) \in S^*(\alpha, \beta, 1, 1) \) then \( f(z) \) is convex in the disc
\[
0 < |z| < R_4 = \inf_n \left[ \frac{p^2 \left[ 2p + (-1)^{n+1} (p + n + 1) + 2(p + n + 1) - (p + n + 1) \right]}{2\beta[(p + \alpha) - p](n + 1)^2} \right]^{1/(n+1-p)}
\] (3.10)

The estimate is sharp for the function
\[
f(z) = z^p + \frac{2\beta \alpha}{(-1)^{n+1}[(p + n + 1) + \beta(2(\alpha + n + 1) - (p + n + 1))] z^{p+n}}
\] (3.11)

Theorem 3.2: If \( f(z) \in A(p) \) is in the class \( S^*(\alpha, \beta, \xi, \gamma) \) then \( f(z) \) \( p \)-valently convex in
\[
0 < |z| < R_5 = \inf_n \left[ \frac{p^2 \left[ 2p + (-1)^{n+1} (p + n + 1) + 2\xi(p + n + 1) - \gamma(p + n + 1) \right]}{2\beta[\xi(p + \alpha) - p\gamma](n + 1)^2} \right]^{1/(n+1-p)}
\] (3.12)

Proof: It is sufficient to show that,
\[
\left| \frac{f''(z)}{f'(z)} + p \right| \leq 1 \quad \text{for} \quad 0 < |z| < R_5
\]
The rest of the details fairly straightforward and are thus omitted.

4. EXTREME POINTS

Theorem 4.1: If \( f_{p-n}(z) = z^p \) and
\[
f_{p+n}(z) = z^p + \frac{2\xi\beta[p(\xi - \gamma) + \alpha\xi]}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] z^{p+n}} \quad (n \in \mathbb{N}_0)
\] (4.1)

Then \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \) if and only if it can be expressed in the form
\[
f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{n+1}(z) \quad \text{and} \quad \sum_{n=-1}^{\infty} \lambda_{p+n} = 1
\]

Proof: Assume
\[
f(z) = \sum_{n=-1}^{\infty} \lambda_{p+n} f_{n+1}(z)
\]
Using equation (1.1),
\[ f(z) = z^p + \sum_{n=p}^{\infty} \lambda_{n+1} \frac{2\xi\beta[p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] Z^{1+n}} \quad (n \in N_0) \]  
(4.2)

Notice that,
\[ \sum_{n=-1}^{\infty} \lambda_{n+1} = 1 - \lambda_{p-1} \leq 1 \]

Which implies that then \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \).

Conversely, let \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \). Then by corollary (2.1)
\[ a_{n+1} = \frac{2\xi\beta[p(\xi - \gamma) + \alpha \xi]}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}, \quad (n \in N_0) \]

Setting,
\[ \lambda_{n+1} = \frac{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] a_{n+1}}{2\xi\beta[p(\xi - \gamma) + \alpha \xi]} \]
and
\[ \lambda_{p-1} = 1 - \sum_{n=p}^{\infty} \lambda_{n+1} f_{n+1}(z) \]

We obtained \( f(z) = \sum_{n=-1}^{\infty} \lambda_{n+1} f_{n+1}(z) \)

We complete the proof of theorem.

5. CLOSURE THEOREM

Theorem 5.1: If
\[ f_j(z) = z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1,j} z^{n+1}, (a_{n+1,j} \geq 0, j = 1, 2, 3, \ldots) \]  
(5.1)

be in the class \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \). Then the function \( g(z) = \sum_{n=p}^{\infty} c_j f_j(z) \) also belongs to the class \( f(z) \in S^*(\alpha, \beta, \xi, \gamma) \)

if \( \sum_{n=p}^{\infty} c_j = 1 \).

Proof: Let
\[ g(z) = \sum_{n=p}^{\infty} c_j z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1,j} z^{n+1} \]
\[ = Z^p + \sum_{n=p}^{\infty} \sum_{j=p}^{1} c_j (-1)^{n} a_{n+1,j} Z^{n+1} \]
\[ = Z^p + \sum_{n=p}^{\infty} (-1)^{n+1} C_{n+1,j} z^{n+1} \]

Where \( C_{n+1,j} = \sum_{j=1}^{1} C_j a_{n+1,j} \)
Notice that \( f \in S^* (\alpha, \beta, \xi, \gamma) \) since
\[
\sum_{n=p}^{\infty} (-1)^{n+1} [(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \\
\frac{2\xi \beta[p(\xi - \gamma) + \alpha \xi]}{a_{n+1,j}}
\]
\[
= \sum_{j=1}^{\infty} c_j \sum_{n=p}^{\infty} (-1)^{n+1} [(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \\
\frac{2\xi \beta[p(\xi - \gamma) + \alpha \xi]}{a_{n+1,j}}
\]
\[
\leq \sum_{j=1}^{\infty} c_j = 1 \text{ since } f_j(s) \in S^* (\alpha, \beta, \xi, \gamma)
\]

**Theorem 5.2:** Let \( f_j = Z^p + \sum_{n=p}^{\infty} \frac{1}{m} a_{n+1,j} Z^{n+1}, a_{n+1,j} \geq 0, j = 1, 2, 3, \ldots \) be in the class \( S^* (\alpha, \beta, \xi, \gamma) \). Then the function \( h(z) = \frac{1}{m} \sum_{n=p}^{\infty} f_j(z) \) also belongs to the class \( S^* (\alpha, \beta, \xi, \gamma) \).

**Proof:** We have,
\[
h(z) = \frac{1}{m} \sum_{n=p}^{\infty} f_j(z)
\]
\[
h(z) = Z^p + \sum_{n=p}^{\infty} \frac{1}{m} \sum_{j=1}^{\infty} a_{n+1,j} Z^{n+1}
\]
\[
= Z^p + \sum_{n=p}^{\infty} d_k Z^k \text{ where } d_k = \frac{1}{m} \sum_{j=1}^{\infty} a_{n+1,j}
\]
Since \( f_j \in S^* (\alpha, \beta, \xi, \gamma) \) from theorem 1, we have
\[
\sum_{n=p}^{\infty} (-1)^{n+1} [(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \\
\frac{2\xi \beta[p(\xi - \gamma) + \alpha \xi]}{a_{n+1,j}} \leq 1 \quad (5.2)
\]
Now \( h(z) \in S^* (\alpha, \beta, \xi, \gamma) \) since
\[
\sum_{n=p}^{\infty} (-1)^{n+1} [(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \\
\frac{2\xi \beta[p(\xi - \gamma) + \alpha \xi]}{d_k}
\]
\[
= \sum_{n=p}^{\infty} (-1)^{n+1} [(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \\
\frac{1}{m} \sum_{j=1}^{\infty} a_{n+1,j}
\]
\[
= \frac{1}{m} \sum_{j=1}^{\infty} \sum_{n=p}^{\infty} (-1)^{n+1} [(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \\
\frac{1}{m} \sum_{j=1}^{\infty} a_{n+1,j}
\]
\[
= 1 \text{ by theorem (5.1)}
\]
Therefore \( f(z) \in S^* (\alpha, \beta, \xi, \gamma) \)

**6. CONVOLUTION AND INCLUSION PROPERTY**

For
\[
f(z) = Z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} Z^{n+1}, a_{n+1} \geq 0
\]
\[
g(z) = Z^p + \sum_{n=p}^{\infty} (-1)^{n+1} b_{n+1} Z^{n+1}, b_{n+1} \geq 0
\]
in \( f(z) \in S^* (\alpha, \beta, \xi, \gamma) \) the convolution of \( f(z) \ast g(z) \) is defined by,

\[
f(z) \ast g(z) = z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} b_{n+1} z^{n+1}, \quad a_{n+1} b_{n+1} \geq 0
\]

**Theorem 6.1:** Let \( f(z) \) and \( g(z) \) belongs to \( S^* (\alpha, \beta, \xi, \gamma) \) the convolution of \( f(z) \ast g(z) \in S^* (\alpha, \beta, \xi, \gamma) \) for

\[
\eta \geq \frac{2\xi\beta[p(\xi - \gamma) + \alpha\xi](p + n + 1)}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] - 2\xi\beta[p(\xi - \gamma) + \alpha](2\xi(\alpha + n + 1) - \gamma(p + n + 1))}
\]

**Proof:** Since \( f(z) \) and \( g(z) \) belongs to \( S^* (\alpha, \beta, \xi, \gamma) \) and so

\[
\sum_{n=p}^{\infty} (-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] a_{n+1} \leq 1
\]

and

\[
\sum_{n=p}^{\infty} (-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] b_{n+1} \leq 1
\]

We need to find small number \( \eta \) such that

\[
\sum_{n=p}^{\infty} (-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] a_{n+1} b_{n+1} \leq 1
\]

Using Cauchy Schwartz inequality; we have

\[
\sum_{n=p}^{\infty} (-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \leq \frac{2\xi\beta[p(\xi - \gamma) + \alpha\xi]}{\sqrt{a_{n+1}b_{n+1}}} \leq 1
\]

Thus it is enough to show that,

\[
\sum_{n=p}^{\infty} (-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] a_{n+1} b_{n+1} \leq \sqrt{a_{n+1}b_{n+1}}
\]

That is

\[
\sqrt{a_{n+1}b_{n+1}} \leq \eta(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] \beta(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]
\]

In view of (6.3) and (6.4) it is enough to show that,

\[
\frac{2\xi\beta[p(\xi - \gamma) + \alpha\xi]}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}
\]

On simplifying, we get

\[
\eta \geq \frac{2\xi\beta[p(\xi - \gamma) + \alpha\xi](p + n + 1)}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))] - 2\xi\beta[p(\xi - \gamma) + \alpha](2\xi(\alpha + n + 1) - \gamma(p + n + 1))}
\]

We complete the proof of theorem.
Next we state that another inclusion theorem for the class $S^{*}(\alpha, \beta, \xi, \gamma)$.

Theorem 6.2: Let $f(z), g(z) \in S^{*}(\alpha, \beta, \xi, \gamma)$ then

$$h(z) = z^p + \sum_{n=p}^{\infty} (a_{n+1}^2 + b_{n+1}^2) z^{n+1} \text{ in } S^{*}(\alpha, \beta, \xi, \gamma)$$

where

$$\delta \geq \frac{4\xi\beta[p(\xi - \gamma) + \alpha \xi](p + n + 1)}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]^2 - 4\xi\beta[p(\xi - \gamma) + \alpha](2\xi(\alpha + n + 1) - \gamma(p + n + 1))}$$

(6.5)

Proof: $f(z), g(z) \in S^{*}(\alpha, \beta, \xi, \gamma)$ and hence

$$\sum_{n=p}^{\infty} \left[ \frac{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}{2\xi\beta[p(\xi - \gamma) + \alpha \xi]} \right] a_{n+1}^2 \leq 1 \quad (6.6)$$

Similarly,

$$\sum_{n=p}^{\infty} \left[ \frac{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}{2\xi\beta[p(\xi - \gamma) + \alpha \xi]} \right] b_{n+1}^2 \leq 1 \quad (6.7)$$

We have to show that,

$$\sum_{n=p}^{\infty} \left[ \frac{(-1)^{n+1}[(p + n + 1) + \delta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}{2\xi\delta[p(\xi - \gamma) + \alpha \xi]} \right] (a_{n+1}^2 + b_{n+1}^2) \leq 1 \quad (6.8)$$

Adding (6.5) and (6.6), we get

$$\sum_{n=p}^{\infty} \left[ \frac{(-1)^{n+1}[(p + n + 1) + \delta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}{2\xi\delta[p(\xi - \gamma) + \alpha \xi]} \right] (a_{n+1}^2 + b_{n+1}^2) \leq 1 \quad (6.9)$$

it is enough to show that,

$$\left[ \frac{(-1)^{n+1}[(p + n + 1) + \delta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]}{2\xi\delta[p(\xi - \gamma) + \alpha \xi]} \right] \leq 1 \quad (6.8)$$

This implies that

$$\delta \geq \frac{4\xi\beta[p(\xi - \gamma) + \alpha \xi](p + n + 1)}{(-1)^{n+1}[(p + n + 1) + \beta(2\xi(\alpha + n + 1) - \gamma(p + n + 1))]^2 - 4\xi\beta[p(\xi - \gamma) + \alpha](2\xi(\alpha + n + 1) - \gamma(p + n + 1))}$$

(6.5)

We complete the proof of theorem.

In this paper, we derived interesting properties of subclass $S^{*}(\alpha, \beta, \xi, \gamma)$ which are analytic and p-valent. The results like coefficient estimation, radius of convexity, closure theorem, extreme points, convolution and inclusion property of subclass $S^{*}(\alpha, \beta, \xi, \gamma)$ are obtained.
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