International Journal of Mathematical Archive-7(7), 2016, 1-8
Available online through www.ijma.info ISSN 2229 – 5046

Hopf Point Analysis of the Food Chain Model with Holling Type III Functional Response

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(Received On: 05-07-16; Revised & Accepted On: 22-07-16)

ABSTRACT

In this paper, an attempt has been made to understand the dynamics of a food chain model where the top predator has no dynamics of its own. In this regard, we consider a food chain model with Holling type III functional response. The dynamics behaviour of the model system around biolgically feasible equilibra are studied. Conditions for which the model enter into Hopf-bifuraction are worked out. To substantiate our analysis findings numerical simulations are carried out for hypothetical set of parameter values.

Keywords: *Food Chain Model, Functional Response, Stability, Hopf-bifurcation.*

1. INTRODUCTION

A food chain is a succession of organisms in a community that constitute a feeding sequence in which food energy is transferred from the source in plants through a series of organisms with different trophic levels. Food chain is started by photosynthesizing plant (called producers), then followed by herbivores, a succession of carnivore and finally decomposers. An interlocked pattern of food chain is called food web. Food web is very important in maintaining the stability of an ecosystem. Food chain hypothesis assumes only the simplest kind of Darwinian selection. A simple example of a food chain can be visualized as: producers – herbivores – carnivores. Some other three species food chain systems have also got the attention of the scientist. For example, in waste treatment process, the bacteria lives on the waste (or nutrient) while other organisms as ciliates feed on the bacteria. In many field situations, the plant – herbivore – parasitoid food chains become extremely important and it has been shown that parasitoid may determine fitness of the plant by destroying herbivore. Thus, three species food chain systems like nutrient – bacteria – ciliate, plant – herbivores – parasitoid, plant – pest – predator, et cetera are emerging in different branches of biology in their own right. Hsu, *et al.*, 2003 studied a ratio-dependent food chain model with Michalies – Menten type functional response. They presented that food chain model is rich in boundary dynamics and capable of generating extinction dynamics and the successful implementations of biological controls. Maiti, *et al.*, 2005; 2006; Kara and Can, 2006, Patra, *et al.* 2009; Pathak, *et al.*, 2009 proposed a delayed food chain model and showed that system is rich in boundary dynamics and successful implementation of biological control to reduce hazards of chemical pesticides.

After analysis of above research papers and their references, we have introduced a food chain model with Holliing type III functional response and studied the dynamical behavior of the system. This work is different from previous papers because in this paper we take density of top predator is constant and we also found the bifurcation point of the system (Agarwal, *et al.*, 2011, 2012, Jana, *et al.* 2016) which breaks the system into two parts.

2. THE MODEL

Consider a food chain model where the top predator has no dynamics of its own. The growth rate of the predator depends on the prey. Apart from the implicit competition between the predators due to sharing of food, competition is also assumed between them. The dynamical equations of this food chain are given as

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \frac{b_1 x^2 y}{a^2 + x^2}, \tag{2.1}$$
$$\frac{dy}{dt} = \frac{b_2 x^2 y}{a^2 + x^2} - dy - ey^2 - \frac{zy}{f+y}.$$

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With initial conditions

$$x(0) = x_0 \geq 0 \text{ and } y(0) = y_0 \geq 0.$$

Here x, y and z denotes the density of prey, predator and top predator. The model parameters r, K, b_1, b_2, a, d, e and f assume only positive values. The initial prey density is assumed to be strictly positive as the predator species will not survive in the absence of food. Further the initial condition for the prey species is assumed to be less than the carrying capacity K .The initial density of the predator is assumed to be positive.

3. EXISTENCE AND DISSIPATIVENESS

In analogy to the population dynamics, it is very important to observe the consequences that restrict the growth of the population. In this sense, study of dissipativeness of the model system (1) around different steady states is very much needed. For this, we find dissipativeness of the system in the following lemma:

Lemma 3.1: The set $\Omega = \{(x, y) : 0 < x < K, 0 < y < A\}$, where $A = \dfrac{b_2K^2 - a^2d}{a^2e}$, with condition $b_2K^2 > a^2d$.

Proof: From the first equation of the system (2.1), we get

$$\frac{dx}{dt} \leq rx - \frac{rx^2}{K}. \tag{3.1}$$

According to comparison principle, it follows that

$$x_{\max} = K. \tag{3.2}$$

From the second equation of the system (2.1), we get

$$\frac{dy}{dt} \leq \left(\frac{b_2x^2}{a^2 + x^2} - d\right)y - ey^2. \tag{3.3}$$

According to comparison principle, we get

$$y_{\max} = \frac{b_2K^2 - a^2d}{a^2e} = A. \tag{3.4}$$

With positive condition

$$b_2K^2 > a^2d. \tag{3.5}$$

Therefore all solutions of system (2.1) enter into the region

$$\Omega = \{(x, y) : 0 < x < K, 0 < y < A\},$$

Where $$A = \frac{b_2K^2 - a^2d}{a^2e}.$$

This completes the proof of lemma. Thus, the model system (2.1) is dissipative. In the next Section we present the equilibrium analysis of the model.

4. EQUILIBRIUM ANALYSIS OF THE SYSTEM

There are three non negative equilibrium points are exist. (i) the trivial equilibrium point $E_0(0,0)$ always exists. (ii) the predator free equilibrium point $E_1(K,0)$.(iii)the interior equilibrium point $E_2(x^*, y^*)$. The existence of equilibrium points $E_0(0,0)$ and $E_1(K,0)$ are obvious. We show the existence of interior equilibrium point $E_2(x^*, y^*)$as follows.

Existence of $E_2(x^*, y^*)$

Here x^* and y^* are the positive solutions of the following algebraic equations

$$r\left(1-\frac{x^*}{K}\right)-\frac{b_1 x^* y^*}{a^2+x^{*2}}=0, \tag{4.1}$$

$$\frac{b_2 x^{*2}}{a^2+x^{*2}}-d-ey^*-\frac{z}{f+y^*}=0. \tag{4.2}$$

From equation (4.2), we get

$$y^*=\frac{r\left(a^2+x^{*2}\right)}{b_1 x^*}\left(1-\frac{x^*}{K}\right), \tag{4.3}$$

The solution of equation is always positive if $x^* < K$.

Putting the value of y^* from equation (4.3) in equation (4.1), we get

$$F\left(x^*\right)=P_1x^{*8}+P_2x^{*7}+P_3x^{*6}+P_4x^{*5}+P_5x^{*4}+P_6x^{*3}+P_7x^{*2}+P_8x^*+P_9. \tag{4.4}$$

Where

$$P_1=-\frac{er^2}{K}, P_2=\frac{2er^2}{K}, P_3=-\frac{r}{K}\left(b_1b_2-db_1+\frac{2era^2}{K}\right)-er^2+\frac{er}{K}\left(f_1b_1-\frac{ra^2}{K}\right),$$

$$P_4=\frac{5a^2er^2}{K}+\left(b_1b_2-db_1\right)r+er\left(\frac{ra^2}{K}-f_1b_1\right), P_5=-3a^2er^2-zb_1^2+\frac{r}{K}\left(da^2b_1-\frac{era^4}{K}\right)+$$

$$\left(f_1b_1-\frac{ra^2}{K}\right)\left(b_1b_2-db_1+\frac{2era^2}{K}\right), P_6=\frac{er^2a^4}{K}-rda^2b_1-2a^2er\left(f_1b_1-\frac{ra^2}{K}\right)+ra^2\left(b_1b_2-\right.$$

$$\left.db_1+\frac{2era^2}{K}\right), P_7=-3er^2a^4-za^2b_1^2-da^2b_1\left(f_1b_1-\frac{ra^2}{K}\right), P_8=-era^4\left(f_1b_1-\frac{ra^2}{K}\right)+ra^2$$

$$\left(-da^2b_1+\frac{era^4}{K}\right), P_9=-er^2a^6.$$

From equation (4.4), we have

$$F(0)=P_9<0. \tag{4.5}$$

$$F(K)=P_1K^8+P_2K^7+P_3K^6+P_4K^5+P_5K^4+P_6K^3+P_7K^2+P_8K+P_9>0. \tag{4.6}$$

Thus there exists a x^*, $0<x^*<K$, such that $F(x^*)=0$. Now, the sufficient condition for the uniqueness of E_2 is $F'\left(x^*\right)>0$ .From (4.4) we can find and $F'\left(x^*\right)$ as follows,

$$F'\left(x^*\right)=8P_1x^{*7}+7P_2x^{*6}+6P_3x^{*5}+5P_4x^{*4}+4P_5x^{*3}+3P_6x^{*2}+2P_7x^*+P_8>0. \tag{4.7}$$

By using (4.4) in (4.7) we can check that $F'\left(x^*\right)$ is positive.

This completes the existence of E_2 .

5. STABILITY ANALYSIS OF THE SYSTEM

To discuss the local stability of system (2.1), we compute the variational matrix $V(E)$of system (2.1). The entries of general variational matrix are given by differentiating the right side of system (2.1) with respect to x and y , i.e.

$$V(E)=\begin{bmatrix} r-\frac{2rx}{K}-\frac{2a^2b_1xy}{\left(a^2+x^2\right)^2} & -\frac{b_1x^2}{a^2+x^2} \\ \frac{2a^2b_2xy}{\left(a^2+x^2\right)^2} & \frac{b_2x^2}{a^2+x^2}-d-2ey+\frac{z}{(f+y)^2} \end{bmatrix}. \tag{5.1}$$

Accordingly, the linear stability analysis about the equilibrium points $E_i, i = 0,1,2$ gives the following results:

1. The equilibrium point E_0 is saddle point if $\frac{z}{f^2} < d$ in $x - y$ plane otherwise it is unstable manifold in $x - y$ plane.
2. The equilibrium point E_1 is stable manifold in $x - y$ plane if $\frac{b_2 K^2}{a^2 + K^2} + \frac{z}{f^2} < d$ otherwise it is unstable manifold in $x - y$ plane.
3. The characteristic polynomial corresponding equilibrium point E_2 is given as
$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{5.2}$$
Where
$$a_{11} = r - \frac{2rx^*}{K} - \frac{2a^2 b_1 x^* y^*}{(a^2 + x^{*2})^2}, a_{12} = -\frac{b_1 x^{*2}}{a^2 + x^{*2}}, a_{21} = \frac{2a^2 b_2 x^* y^*}{(a^2 + x^{*2})^2},$$
$$a_{22} = \frac{b_2 x^{*2}}{a^2 + x^{*2}} - d - 2ey^* + \frac{z}{(f + y^*)^2}.$$

The equilibrium point E_2, if it exists, is locally asymptotically stable if $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0.$

Obviously, there are only two equilibrium points while the E_1 is stable. As it begins to lose its stability, the equilibrium point E_2 starts to appear in the system. For further analysis, the possibility of occurrence of a local Hopf bifurcation is explored. In fact, it requires two conditions. One is the presence of an equilibrium point with a pair of purely imaginary eigenvalues for a particular value of the bifurcation parameter and another one is the change in its stability for a variation of the bifurcation parameter. Here, we will focus on the equilibrium point E_2 only.

6. BIFURCATION ANALYSIS OF THE SYSTEM

For finding the conditions of bifurcation point of the system (2.1), we follow following steps:

Step-1: Find a_{22} as the function of the interesting bifurcation parameter f.

Step-2: Putting $\lambda = i\omega$ in equation (5.2), we get
$$\omega^2 + (a_{11} + a_{22})i\omega - a_{11}a_{22} + a_{12}a_{21} = 0. \tag{5.3}$$

Comparing the real and imaginary part of the equation (5.3), we get the positive value of ω as $\omega = \sqrt{a_{11}a_{22} - a_{12}a_{21}}$ and critical value of $f = f^* = -y^* + \sqrt{\frac{z}{a_{11} + s}}i$. Where $s = -d - 2ey^* + \frac{b_2 x^{*2}}{a^2 + x^2}$.
This indicates the occurrence of a pair of purely imaginary eigenvalues.

Step-3: Verify the transversality condition by differentiating equation (5.2) with respect to bifurcation parameter f, we get
$$\frac{d\lambda}{df} = -\frac{2z(\lambda - a_{11})}{(f + y^*)^3 (2\lambda - a_{11} - a_{22})}. \tag{5.4}$$

Using the value of ω in the equation (5.4) and collect the real part of λ, we have following equation
$$\left.\frac{d(\mathrm{Re}\,\lambda)}{df}\right|_{f=f^*} = -\frac{2z\{2\omega^2 + a_{11}(a_{11} + a_{22})\}}{(f^* + y^*)^3 \{4\omega^2 + (a_{11} + a_{22})^2\}}. \tag{5.5}$$

Step-4: Determine the occurrence of a local Hopf bifurcation by considering the sign of (5.5). If (5.5) is nonzero then a local Hopf bifurcation occurs. In addition, if (5.5) is positive, the system behaviour changes from equilibrium state to oscillatory. In contrast, if (5.5) is negative then the system behaviour changes from oscillatory to equilibrium state.

7. NUMERICAL RESULTS

The global stability of the non-linear model system (2.1), in the positive octant, is investigated numerically. The numerical integration for model system (2.1) is carried out for various choices of biologically feasible parameter values and for different sets of initial conditions. In all the cases being considered here the data is chosen such that the persistence conditions are satisfied. Model system (2.1) is solved numerical using the Runge–Kutta method. The dynamic behavior and its corresponding time series of the model system (2.1) are decided on the following data set:

$$r = 10, K = 200, b_1 = 0.04, a = 0.1, b_2 = 0.03, d = 0.4, e = 0.02, f = 2, z = 90. \tag{7.1}$$

The interior equilibrium point of the model system (2.1) corresponding to the feasible parameters values are same as (7.1) is:

$$E_2(2.7337,\ 1.1520).$$

The characteristic polynomial and characteristic roots of the model system corresponding to the interior equilibrium point are given as:

$$\lambda^2 + 18.3693\lambda + 84.0636 = 0. \tag{7.2}$$

$$\lambda_1 = -9.72658, \lambda_2 = -8.64267. \tag{7.3}$$

From equations (5.3) and (5.4), we find the critical value of bifurcation point f and its behaviour .The critical value of bifurcation point f is $f^* = 1.82682$ and $\left.\frac{d(\operatorname{Re}\lambda)}{df}\right|_{f=f^*} = -3.50557.$ The sign of equation (5.5) is negative then the system behaviour changes from oscillatory to equilibrium state. The figure 1 shows this dynamical behaviour of the system (2.1). From equation (7.3), it is clear that all characteristic roots of the characteristic polynomial (7.2) are negative. So, the model system (2.1) is shows stable behavior at the interior equilibrium point. Figure (2) shows the stability of the system (2.1) and figure (3) shows the time series graph of the system at $f = 2 > 1.82682$. When the value of $f = 0.6 < 1.82682$, the system become unstable (refer Figure (4) and Figure 5).So in this paper f is a bifurcation point of the system which breaks the system in two parts as:

$f < 1.82682$, the system (2.1) becomes unstable.

$f > 1.82682$, the system (2.1) becomes stable.

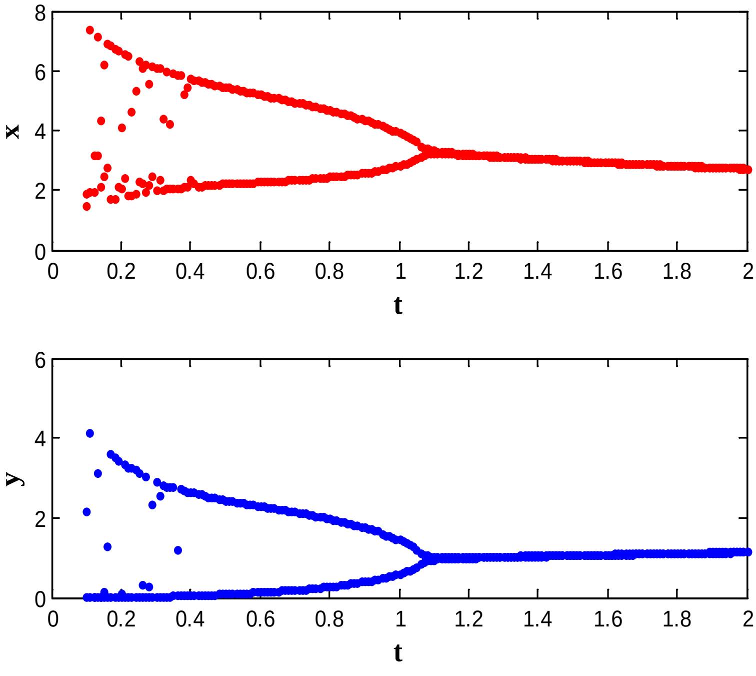

Figure-1: Bifurcation Graph for values of f from 0.1 to 2 and other values of parameters are same as (7.1)

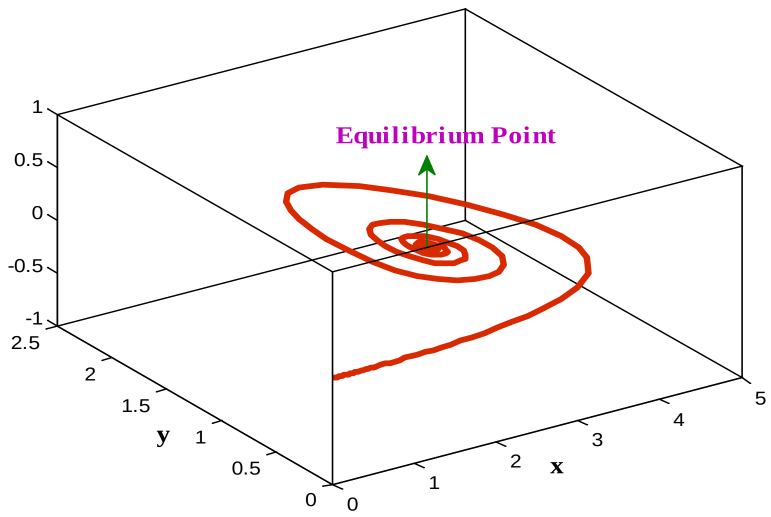

Figure-2: Stable limit cycle and other parameters are same as (7.1).

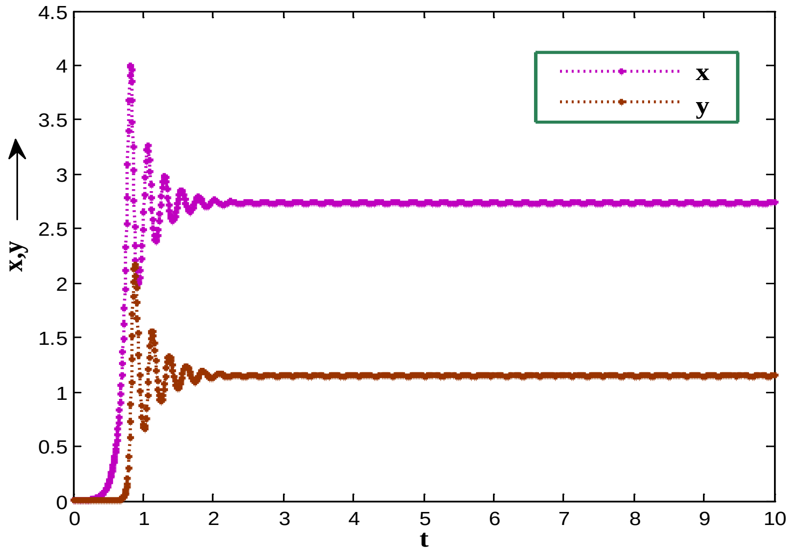

Figure-3: Graph of x and y versus t and other parameters are same as (7.1).

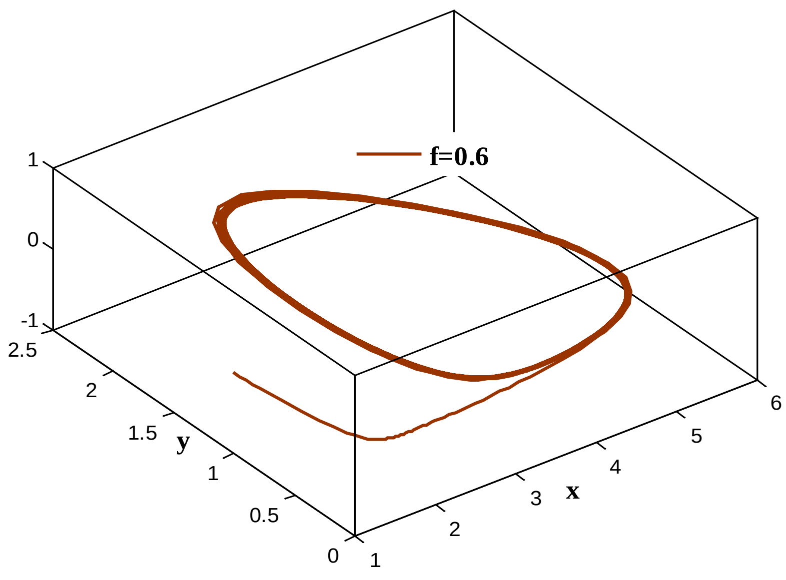

Figure-4: Dynamic behavior of the system (2.1) when $f = 0.6$ and other values of parameters are same as (7.1).

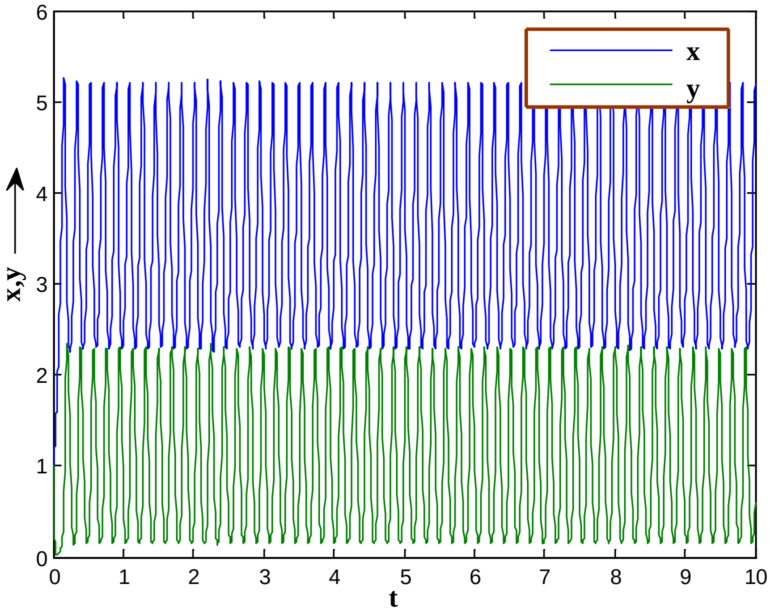

Figure-6: Time Series Graph for the system (2.1) when $f = 0.6$ and other values of parameters are same as (7.1)

8. DISCUSSIONS AND CONCLUSIONS

A nonlinear mathematical model is proposed and analyzed to see the dynamic behaviour of the food chain model with Holling type III functional response. Using stability theory of differential equations, we have obtained conditions for the existence of different equilibria and discussed their stabilities in local manner. From our analysis, we have found the bifurcation point of the system which split the system in two parts. In this paper f is the bifurcation point. When $f < 1.82682$, the system (2.1) becomes unstable and when $f > 1.82682$, the system (2.1) becomes stable. The dynamical behaviour of the system changes from oscillatory to equilibrium state. It has been theoretically and numerically shown that a local Hopf bifurcation is possible. Therefore, the system (2.1) permits limit cycle behaviour.

ACKNOWLEDGEMENT

Author Dr. Rachana Pathak thankfully acknowledges the NBHM (2/40(29)/2014/R\&D-11/14138) for the financial assistance in the form of PDF.

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Source of support: Nil, Conflict of interest: None Declared