SYMMETRIC BI- \( f \)-DERIVATIONS IN ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

In this paper, we introduce the concept of symmetric bi- \( f \)-derivation in an Almost Distributive Lattice (ADL) and derive some important properties of symmetric bi- \( f \)-derivations in ADLs.

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1. INTRODUCTION

The concept of derivation in an ADL was introduced in our earlier paper [11]. The notion of derivation in Lattices was first given in G.Szasz [15] in 1974. Earlier Posner[9] introduced derivations in ring theory and later several authors worked on it ([2], [5]). Several authors worked on derivations in Lattices ([1], [3], [4], [6], [7], [8], [16], [17] and [18]). We have introduced the concept of \( f \)-derivations in an ADL in our paper [12] and the concept of symmetric bi-derivations in an ADL in our paper [13]. The concept of symmetric bi- \( f \)-derivations in lattices was introduced by Kyung Ho Kim [6] in 2012.

In 1980, the concept of an Almost Distributive Lattice(ADL) was introduced by U.M.Swamy and G.C Rao[14]. In this paper, we introduce the concept of symmetric bi- \( f \)-derivation in an ADL and derive some important properties. We introduce the concept of an isotone symmetric bi- \( f \)-derivation in an ADL and establish a set of conditions which are sufficient for the trace of a symmetric bi- \( f \)-derivation on an ADL with a maximal element to become an isotone . Also, we prove \( D(x, y) = fx \land D(x \lor z, y) \) if \( D \) is isotone and \( D(x, y) = [fx \land D(x \lor z, y)] \lor D(x, y) \) if \( f \) is a join homomorphism or an increasing function on \( L \). We define a set \( F_a(L) \) for each \( a \in L \) and prove that it is a weak ideal if \( D \) is a join preserving symmetric bi- \( f \)-derivation on an ADL \( L \) with 0 where \( f \) is a join-homomorphism.

2. PRELIMINARIES

In this section, we recollect certain basic concepts and important results on Almost Distributive Lattices.

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Definition 2.1: [10] An algebra \((L, \lor, \land)\) of type \((2,2)\) is called an Almost Distributive Lattice if it satisfies the following axioms:
\[
\begin{align*}
L_1: \quad (a \lor b) \land c &= (a \land c) \lor (b \land c) \quad (RD \land) \\
L_2: \quad a \land (b \lor c) &= (a \land b) \lor (a \land c) \quad (LD \land) \\
L_3: \quad (a \lor b) \land b &= b \\
L_4: \quad (a \lor b) \land a &= a \\
L_5: \quad a \lor (a \land b) &= a
\end{align*}
\]

Definition 2.2: [10] Let \(X\) be any non-empty set. Define, for any \(x, y \in L, \ x \lor y = x\) and \(x \land y = y\). Then \((X, \lor, \land)\) is an ADL and such an ADL, we call discrete ADL.

Throughout this paper \(L\) stands for an ADL \((L, \lor, \land)\) unless otherwise specified.

Lemma 2.3: [10] For any \(a, b \in L,\) we have
\[
\begin{align*}
(i) \quad a \land a &= a \\
(ii) \quad a \lor a &= a \\
(iii) \quad (a \land b) \lor b &= b \\
(iv) \quad a \land (a \lor b) &= a \\
v \quad a \lor (b \land a) &= a. \\
v (vi) \quad a \lor b &= a \text{ if and only if } a \land b = b \\
v (vii) \quad a \lor b &= b \text{ if and only if } a \land b = a.
\end{align*}
\]

Definition 2.4: [10] For any \(a, b \in L,\) we say that \(a\) is less than or equal to \(b\) and write \(a \leq b\), if \(a \land b = a\) or, equivalently, \(a \lor b = b\).

Definition 2.5: [10] For any \(a, b, c \in L,\) we have the following
\[
\begin{align*}
(i) \quad \text{The relation } \leq \text{ is a partial ordering on } L. \\
(ii) \quad a \lor (b \land c) &= (a \lor b) \land (a \lor c). \quad (LD \lor) \\
(iii) \quad (a \lor b) \lor a &= a \lor b = a \lor (b \lor a). \\
(iv) \quad (a \lor b) \land c &= (b \lor a) \land c. \\
v \quad \text{The operation } \land \text{ is associative in } L. \\
v (vi) \quad a \land b \land c &= b \land a \land c.
\end{align*}
\]

Theorem 2.6: [10] For any \(a, b \in L,\) the following are equivalent.
\[
\begin{align*}
(i) \quad (a \land b) \lor a &= a \\
(ii) \quad a \land (b \lor a) &= a \\
(iii) \quad (b \land a) \lor b &= b \\
(iv) \quad b \land (a \lor b) &= b \\
v \quad a \land b &= b \land a \\
v (vi) \quad a \lor b &= b \lor a \\
v (vii) \quad \text{the supremum of } a \text{ and } b \text{ exists in } L \text{ and equals to } a \lor b \\
v (viii) \quad \text{there exists } x \in L \text{ such that } a \leq x \text{ and } b \leq x \\
(ix) \quad \text{the infimum of } a \text{ and } b \text{ exists in } L \text{ and equals to } a \land b.
\end{align*}
\]

Definition 2.7: [10] \(L\) is said to be associative, if the operation \(\lor\) in \(L\) is associative.
Theorem 2.8: [10] The following are equivalent.

(i) \( L \) is a distributive lattice.

(ii) the poset \((L, \leq)\) is directed above.

(iii) \( a \wedge (b \vee a) = a \), for all \( a, b \in L \).

(iv) the operation \( \vee \) is commutative in \( L \).

(v) the operation \( \wedge \) is commutative in \( L \).

(vi) the relation \( \theta := \{(a, b) \in L \times L \mid a \wedge b = b\} \) is anti-symmetric.

(vii) the relation \( \theta \) defined in (vi) is a partial order on \( L \).

Lemma 2.9: [10] For any \( a, b, c, d \in L \), we have the following

(i) \( a \wedge b \leq b \) and \( a \leq a \vee b \)

(ii) \( a \wedge b = b \vee a \) whenever \( a \leq b \).

(iii) \( [a \vee (b \vee c)] \wedge d = [(a \vee b) \vee c] \wedge d \).

(iv) \( a \leq b \) implies \( a \wedge c \leq b \wedge c \), \( c \wedge a \leq c \wedge b \) and \( c \vee a \leq c \vee b \).

Definition 2.10: [10] An element \( 0 \in L \) is called zero element of \( L \), if \( 0 \wedge a = 0 \) for all \( a \in L \).

Lemma 2.11: [10] If \( L \) has \( 0 \), then for any \( a, b \in L \), we have the following

(i) \( a \vee 0 = a \), (ii) \( 0 \vee a = a \) and (iii) \( a \wedge 0 = 0 \).

(iv) \( a \wedge b = 0 \) if and only if \( b \wedge a = 0 \).

Definition 2.12: [14] Let \( L \) be a non-empty set and \( x_0 \in L \). Define, for \( x, y \in L \),

\[
\begin{align*}
    x \wedge y &= y & \text{if } x \neq x_0 \\
    &= x & \text{if } x = x_0 \\
    x \vee y &= x & \text{if } x \neq x_0 \\
    &= y & \text{if } x = x_0 ,
\end{align*}
\]

then \((L, \vee, \wedge, x_0)\) is an ADL with \( x_0 \) as zero element. This is called discrete ADL with zero.

An element \( x \in L \) is called maximal if, for any \( y \in L \), \( x \leq y \) implies \( x = y \).

We immediately have the following.

Lemma 2.13: [10] For any \( m \in L \), the following are equivalent:

(1) \( m \) is maximal

(2) \( m \vee x = m \) for all \( x \in L \)

(3) \( m \wedge x = x \) for all \( x \in L \).

Definition 2.14: [10] A nonempty subset \( I \) of \( L \) is said to be an ideal if and only if it satisfies the following:

(1) \( a, b \in I \Rightarrow a \vee b \in I \)

(2) \( a \in I, x \in L \Rightarrow a \wedge x \in I \).

Definition 2.15: [10] A nonempty subset \( I \) of \( L \) is said to be an initial segment of \( L \) if, \( a \in L \) and \( x \in L \) such that \( x \leq a \) imply that \( x \in L \).

Definition 2.16: [13] A nonempty subset \( I \) of \( L \) is said to be a weak ideal if and only if it satisfies the following:

(1) \( a, b \in I \Rightarrow a \vee b \in I \)

(2) \( I \) is an initial segment of \( L \).

Observe that every ideal of \( L \) is a weak ideal, but not converse.
Definition 2.17: [10] A function \( f : L \rightarrow L \) is said to be an ADL homomorphism if it satisfies the following:

1. \( f(x \land y) = fx \land fy \),
2. \( f(x \lor y) = fx \lor fy \) for all \( x, y \in L \).

Definition 2.18: A function \( d : L \rightarrow L \) is called an isotone, if \( dx \leq dy \) for any \( x, y \in L \) with \( x \leq y \).

3. SYMMETRIC bi-\( f \)-Derivations IN ADLs

We begin this section with the following definition of a symmetric map and a symmetric bi-derivation in an ADL.

Definition 3.1: [13] A mapping \( D : L \times L \rightarrow L \) is called symmetric if \( D(x, y) = [y \land D(x, y)] \lor [x \land D(y, z)] \) for all \( x, y, z \in L \).

Observe that a symmetric bi-derivation \( D \) on \( L \) also satisfies
\[
D(x, y \land z) = [z \land D(x, y)] \lor [y \land D(x, z)] \text{ for all } x, y, z \in L.
\]

The following definition introduces the notion of a symmetric bi-\( f \)-derivation on ADLs.

Definition 3.3: A symmetric function \( D : L \times L \rightarrow L \) is called a symmetric bi-\( f \)-derivation on \( L \), if there exists a function \( f : L \rightarrow L \) such that
\[
D(x, y \land z) = [fx \land D(x, y)] \lor [fy \land D(y, z)] \text{ for all } x, y, z \in L.
\]

Example 3.4: Let \( f : L \rightarrow L \) be a function such that \( f(x \land y) = fx \land fy \) for all \( x, y \in L \). Let \( a \in L \) and define a function \( D : L \times L \rightarrow L \) by \( D(x, y) = fx \land fy \land a \) for all \( x, y \in L \). Then \( D \) is a symmetric bi-\( f \)-derivation on \( L \).

Example 3.5: Every symmetric bi-derivation on \( L \) is a symmetric bi-\( f \)-derivation, where \( f : L \rightarrow L \) is the identity map.

But, a symmetric bi-\( f \)-derivation need not be a symmetric bi-derivation. For, consider the following example.

Example 3.6: Let \( L \) be discrete ADL with 0 and \( 0 \neq a \in L \). Define a function \( f : L \rightarrow L \) by \( fx = a \) for all \( x \in L \) and \( D : L \times L \rightarrow L \) by \( D(x, y) = a \) for all \( x, y \in L \), then \( D \) is a symmetric bi-\( f \)-derivation on \( L \) but not a symmetric bi-derivation.

Example 3.7: Let \( L \) be a discrete ADL with at least two elements. Define a function \( D : L \times L \rightarrow L \) by \( D(x, y) = x \land y \) for all \( x, y \in L \), then \( D \) is not a symmetric bi-\( f \)-derivation on \( L \). Since, it is not a symmetric map.

Lemma 3.8: Let \( D \) be a symmetric bi-\( f \)-derivation on \( L \). Then the following hold:

1. \( D(x, y) = fx \land D(x, y) \) for all \( x, y \in L \)
2. \( D(x \land z, y) = [fx \lor fy] \land D(x \land z, y) \) for all \( x, y, z \in L \)
3. If \( L \) has 0, then \( f0 = 0 \) implies \( D(0, y) = 0 \) for all \( y \in L \)

Proof: (1) Let \( x, y \in L \).

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Then \( D(x, y) = D(x \land x, y) = [fx \land D(x, y)] \lor [fx \land D(x, y)] = fx \land D(x, y). \)

(2) Let \( x, y, z \in L \). Then
\[
[fx \lor fz] \land D(x \land z, y) = [fx \lor fz] \land [[fx \land D(x, y)] \lor [fx \land D(z, y)]]
= [fx \land D(x, y)] \lor [fx \land D(z, y)] = D(x \land z, y).
\]

(3) Suppose \( L \) has 0 and \( f \neq 0 \). Then, by (1) above, \( D(0, y) = f \neq 0 \land D(0, y) = 0 \land D(0, y) = 0 \).

**Corollary 3.9:** If \( d \) is the trace of a symmetric bi-\( f \)-derivation \( D \), then \( dx = fx \land dx \) for all \( x \in L \).

**Theorem 3.10:** If \( d \) is the trace of a symmetric bi-\( f \)-derivation on an associative ADL \( L \), then \( d(x \land y) = (fy \land dx) \lor D(x, y) \lor (fx \land dy) \).

**Proof:** Let \( x, y \in L \). Then
\[
d(x \land y) = D(x \land y, x \land y)
= [(fx \land D(x, y)] \lor [fx \land D(x, y)]
= [((fx \land D(x, y)] \lor [fx \land D(x, y)]) \lor [fx \land D(x, y)]
= (fy \land dx) \lor D(x, y) \lor (fx \land dy).
\]

**Corollary 3.11:** If \( d \) is the trace of a symmetric bi-\( f \)-derivation on an ADL \( L \), then \( fy \land dx \leq d(x \land y) \).

**Proof:** Let \( x, y \in L \). Then
\[
d(x \land y) = D(x \land y, x \land y)
= [(fx \land D(x, y)] \lor [fx \land D(x, y)]
= [((fx \land D(x, y)] \lor [fx \land D(x, y)]) \lor [fx \land D(x, y)]
= (fy \land dx) \lor D(x, y) \lor (fx \land dy).
\]

Thus \( fy \land dx \leq (fy \land dx) \lor D(x, y) \leq d(x \land y) \).

**Theorem 3.12:** Let \( m \) be a maximal element of \( L \) and \( d \) be the trace of a symmetric bi-\( f \)-derivation \( D \) on \( L \) such that \( fm \) is also a maximal element. Then the following are equivalent.

1. \( d \) is an isotope map on \( L \)
2. \( dx = fx \land dm \) for all \( x \in L \)
3. \( d(x \land y) = dx \land dy \) for all \( x, y \in L \)
4. \( d(x \lor y) = dx \lor dy \) for all \( x, y \in L \).

**Proof:** (1) \(\Rightarrow\) (2): Let \( x \in L \). By Corollary 3.11, \( fx \land dm \leq d(m \land x) = dx \).

On the other hand, since \( d \) is an isotope, \( d(m \land x) \leq dm \). Thus \( fm \land dx \leq d(x \land m) \leq dm \).

Therefore, \( dx = fx \land dx = fm \land fx \land dx = fx \land fm \land dx \leq fx \land dm \). Hence \( dx = fx \land dm \).

(2) \(\Rightarrow\) (3): Let \( x, y \in L \). Then \( d(x \land y) = x \land y \land dm = x \land dm \land y \land dm = dx \land dy \).

Then \( d(x \land y) = f(x \land y) \land d \land m \land fy \land dm = fx \land dm \land fy \land dm = dx \land dy \).

(2) \(\Rightarrow\) (4): Let \( x, y \in L \). Then \( d(x \lor y) = (x \lor y) \land dm = (x \land dm) \lor (y \land dm) = dx \lor dy \).

Then \( d(x \lor y) = f(x \lor y) \land dm = (fx \lor fy) \land dm = (fx \land dm) \lor (fy \land dm) = dx \lor dy \).

(3) \(\Rightarrow\) (1) and (4) \(\Rightarrow\) (1) are trivial.
Lemma 3.13: Let \( D \) be a symmetric bi-\( f \)-derivation on \( L \). Then the following hold:

1. If \( D \) is isotone, then \( D(x, y) = fx \land D(x \lor z, y) \)
2. If \( f \) is a join homomorphism, then \( D(x, y) = [fx \land D(x \lor z, y)] \lor D(x, y) \)
3. If \( f \) is increasing, then \( D(x, y) = [fx \land D(x \lor z, y)] \lor D(x, y) \)

Proof: Let \( x, y, z \in L \).

(1) Suppose \( D \) is an isotone function on \( L \).

Then \( D(x, y) \leq D(x \lor z, y) \). Thus \( D(x, y) \land fx \land D(x \lor z, y) = D(x, y) \).

Therefore \( D(x, y) \leq fx \land D(x \lor z, y) \).

Now, \( D(x, y) = D((x \lor z) \land x, y) = [fx \land D(x \lor z, y)] \lor [f(x \lor z) \land D(x, y)] \).

Thus \( fx \land D(x \lor z, y) \leq D(x, y) \). Hence \( D(x, y) = fx \land D(x \lor z, y) \).

(2) Let \( f \) be a join-homomorphism on \( L \). Then

\[
D(x, y) = D((x \lor z) \land x, y) \\
= [fx \land D(x \lor z, y)] \lor [f(x \lor z) \land D(x, y)] \\
= [fx \land D(x \lor z, y)] \lor (fx \land f z) \land D(x, y) \\
= [fx \land D(x \lor z, y)] \lor ([fx \land D(x, y)] \lor [fz \land D(x, y)]) \\
= [fx \land D(x \lor z, y)] \lor [D(x, y) \lor [fz \land D(x, y)]] \\
= [fx \land D(x \lor z, y)] \lor D(x, y).
\]

(3) Let \( f \) be an increasing function on \( L \). Then \( fx \leq f(x \lor z) \).

Now,

\[
D(x, y) = D((x \lor z) \land x, y) \\
= [fx \land D(x \lor z, y)] \lor [f(x \lor z) \land D(x, y)] \\
= [fx \land D(x \lor z, y)] \lor [fx \land D(x, y)] \\
= [fx \land D(x \lor z, y)] \lor D(x, y).
\]

Definition 3.14: Let \( D \) be a symmetric bi-\( f \)-derivation on \( L \) and \( a \in L \). We define \( F_a(L) = \{ x \in L \mid D(a, x) \land fx = fx \} \).

Lemma 3.15: Let \( D \) be a symmetric bi-\( f \)-derivation on \( L \) where \( f \) is an increasing function and \( a \in L \). Then \( F_a(L) \) is an initial segment in \( L \).

Proof: Let \( x, y \in L \) with \( x \leq y \) and \( y \in Fix_a(L) \). Since \( f \) is an increasing function, \( fx \leq fy \).

Now,

\[
D(x, a) \land fx = D(x \land y, a) \land fx \\
= [[fy \land D(x, a)] \lor [fx \land D(y, a)]] \land fx \\
= [[fy \land fx \land D(x, a)] \lor [fx \land fy \land D(y, a)]] \land fx \\
= [[fx \land D(x, a)] \lor [fx \land D(y, a) \land fy]] \land fx \\
= [D(x, a) \lor [fx \land fy]] \land fx \\
= [D(x, a) \lor fx] \land fx \\
= fx.
\]
Lemma 3.16: Let $D$ be a join preserving symmetric bi-$f$-derivation on $L$ where $f$ is a join-homomorphism and $a \in L$. Then $x \lor y \in F_a(L)$ for all $x, y \in F_a(L)$.

Proof: Let $x, y \in F_a(L)$. Then
\[ D(x \lor y, a) \land f(x \lor y) = [D(x, a) \lor D(y, a)] \land f(x \lor y) \]
\[ = [(D(x, a) \lor f(x)] \land [(D(y, a) \lor f(x)] \land f(x \lor y) \]
\[ = [(D(x, a) \lor f(x)] \land [(D(y, a) \lor f(x)] \land f(x \lor y) \]
\[ = [f(x \lor y) \land f(x \lor y) \]
\[ = f(x \lor y) \]
\[ = f(x \lor y). \]

Hence $x \lor y \in F_a(L)$.

Finally we conclude this paper with the following theorem, which is a direct consequence of Lemma 3.15 and Lemma 3.16.

Theorem 3.17: Let $L$ be an ADL with 0 and $D$ be a join preserving symmetric bi-$f$-derivation on $L$ where $f$ is a join-homomorphism and $a \in L$. Then $F_a(L)$ is a weak ideal of $L$.

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