A NEW CLASS OF GENERALIZED CLOSED SETS USING GRILLS

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(Received On: 02-07-16; Revised & Accepted On: 26-07-16)

ABSTRACT

The aim of this paper is to apply the notion of \( \zeta \) semi-open sets to obtain a new class of \( \zeta \omega \)–closed sets via grills. The properties of the above mentioned sets are investigated. Further the concept is extended to derive some applications of \( \zeta \omega \)–closed sets via Grills.

Key words and phrases: Grill, topology \( \tau_{\zeta} \), operator \( \Phi \), \( \zeta \omega \)–closed.

1. INTRODUCTION AND PRELIMINARIES

The idea of grills on a topological space was first introduced by Choquet [2] in 1947. In [8], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Hatir and Jafari [4] have defined new classes of sets in grill topological spaces. Ahmad Al-Omari and Noiri [6] introduced and investigated the notions of \( \zeta \alpha \) - open sets, \( \zeta \) semi open sets and \( \zeta \beta \) open sets in grill topological spaces.

Definition 1.1: [2] A collection \( \zeta \) of non empty subsets of a topological spaces \( X \) is said to be a grill on \( X \) if

(i) \( A \in \zeta \) and \( A \subseteq B \) implies that \( B \in \zeta \),

(ii) \( A, B \subseteq X \) and \( A \cup B \in \zeta \) implies that \( A \in \zeta \) or \( B \in \zeta \).

Definition 1.2: [8] Let \( (X, \tau) \) be a topological space and \( \zeta \) be a grill on \( X \). We define a mapping \( \Phi : P(X) \rightarrow P(X) \) denoted by \( \Phi_{\zeta}(A, \tau) \) (for \( A \in P(X) \)) or \( \Phi_{\zeta}(A) \) or simply \( \Phi(A) \) called the operator associated with the grill \( \zeta \) and the topology \( \tau \) defined as follows:

\[
\Phi(A) = \Phi_{\zeta}(X, \tau, \zeta) = \{ x \in X \mid A \cap U \in \zeta \; \text{for all} \; U \in \tau(x) \; \text{for each} \; A \in P(X) \}.
\]

Definition 1.3: [8] Let \( \zeta \) be a grill on \( X \). We define a map \( \psi : P(X) \rightarrow P(X) \) by \( \psi(A) = A \cup \Phi(A) \) for all \( A \in P(X) \).

Definition 1.4: [8] Corresponding to a grill \( \zeta \) on topological space \( (X, \tau) \) there exist a unique topology \( \tau_{\zeta} \) (say) on \( X \) given by \( \tau_{\zeta} = \{ U \subseteq X : \psi(X \setminus A) = X \setminus U \} \) where for any all \( A \subseteq X \), \( \psi(A) = A \cup \Phi(A) = \tau_{\zeta} - \text{cl}(A) \).

Definition 1.5: [10] Let \( X \) be a space and \( (\phi \neq) A \subseteq X \). Then \( [A] = \{ B \subseteq X : A \cap B \neq \phi \} \) is a grill on \( X \) called principal grill generated by \( A \).

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2. $\zeta_\omega$-CLOSED SETS

Definition 2.1: Let $(X, \tau)$ be a topological space and $\zeta$ be any grill on $X$. Then a subset $A$ of $X$ is called $\zeta_\omega$-closed if $\psi(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\zeta$ semi-open in $X$. A subset $A$ of $X$ is called $\zeta_\omega$-open if $X \setminus A$ is $\zeta_\omega$-closed.

Theorem 2.2: Every closed set in $(X, \tau)$ is $\zeta_\omega$-closed in $(X, \tau, \zeta)$.

Proof: Let $A$ be any closed set and $U$ be any $\zeta$ semi-open set such that $\text{cl}(A) = A \subseteq U$ since $A$ is closed. But $\psi(A) \subseteq \text{cl}(A)$, we have $\psi(A) \subseteq U$ whenever $A \subseteq U$. Hence $A$ is $\zeta_\omega$-closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.3: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ with $\zeta = \{X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\}$. Let $A = \{b\}$ and $U = \{b\}$ where $U$ is $\zeta$ semi-open in $X$. Therefore, $\Phi(A) = \emptyset$ and $\psi(A) = A \cup \Phi(A) = \{b\} \subseteq U$. Then the set $\{b\}$ is $\zeta_\omega$-closed but not closed.

Theorem 2.4: Every $\tau_\zeta$-closed set in $(X, \tau, \zeta)$ is $\zeta_\omega$-closed in $(X, \tau, \zeta)$.

Proof: Let $A$ be a $\tau_\zeta$-closed and then $\Phi(A) \subseteq A$ implies $A \cup \Phi(A) \subseteq A \cup A = A$. Let $A \subseteq U$ where $U$ is $\zeta$ semi open. Hence, $\psi(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\zeta$ semi-open. Therefore, $A$ is $\zeta_\omega$-closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.5: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\zeta = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then the set $\{a, b\}$ is $\zeta_\omega$-closed but not $\tau_\zeta$-closed.

Theorem 2.6: Every $\omega$-closed set in $(X, \tau)$ is $\zeta_\omega$-closed in $(X, \tau, \zeta)$.

Proof: Let $A$ be any $\omega$-closed set and $U$ be any $\zeta$ semi-open set containing $A$. Since every $\zeta$ semi-open set is semi open and $A$ is $\omega$-closed we have, $\text{cl}(A) \subseteq U$. But $\psi(A) \subseteq \text{cl}(A)$. Thus we have, $\psi(A) \subseteq U$ whenever $A \subseteq U$. Hence $A$ is $\zeta_\omega$-closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.7: In Example 2.3, the set $\{b\}$ is $\zeta_\omega$-closed but not $\omega$-closed.

Remark 2.8: In a grill space $(X, \tau, \zeta)$, $\zeta_\omega$-closed sets are generalization of $\omega$-closed sets which itself is a generalization of the closed set.

Theorem 2.9: Every $\zeta_\omega$-closed set in $(X, \tau, \zeta)$ is $\zeta g$-closed in $(X, \tau, \zeta)$.

Proof: Let $A \subseteq U$, $U$ is open and hence it is $\zeta$ semi open. Since $A$ is $\zeta_\omega$-closed, we have $\psi(A) \subseteq U$. But $\psi(A) \subseteq \text{cl}(A)$. Thus we have, $\psi(A) \subseteq U$ whenever $A \subseteq U$. Hence $A$ is $\zeta g$-closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.10: Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$ and $\zeta = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then the set $\{a, b\}$ is $\zeta g$-closed but not $\zeta_\omega$-closed.
Remark 2.11: In the case $[X]$ principal grill generated by $X$, it is known \([8]\) that \(\tau = \tau_{[X]}\) so that any \([X] - \zeta\omega\) closed set becomes simply an \(\omega\) closed set and vice versa.

**Theorem 2.12:** Let \((X, \tau, \zeta)\) be a topological space and \(\zeta\) be a grill on \(X\). Then for a subset \(A\) of \(X\), the following are equivalent:

- (i) \(A\) is \(\zeta\omega\) - closed.
- (ii) \(\psi(A) \subseteq U\) for \(\zeta\) semi open set \(U\) containing \(A\).
- (iii) For each \(x \in \psi(A)\), \(\zeta\text{scl}(\{x\} \cap A) \neq \emptyset\).
- (iv) \(\Phi(A) \setminus A\) contains no non empty \(\zeta\) semi closed set of \((X, \tau, \zeta)\).
- (v) \(\Phi(A) \setminus A\) contains no non empty \(\zeta\) semi closed set of \((X, \tau, \zeta)\).

**Proof:**

(i) \(\Rightarrow\) (ii): Let \(A\) be a \(\zeta\omega\) - closed. Then clearly, \(\psi(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\zeta\) semi open in \(X\).

(ii) \(\Rightarrow\) (iii): Suppose \(x \in \psi(A)\). If \(\zeta\text{scl}(\{x\} \cap A) = \emptyset\), then \(A \subseteq X \setminus \zeta\text{scl}(\{x\})\) where \(X \setminus \zeta\text{scl}(\{x\})\) is a \(\zeta\) semi open set. By assumption \(\psi(A) \setminus A \subseteq X \setminus \zeta\text{scl}(\{x\})\), which is a contradiction to \(x \in \psi(A)\). Hence \(\zeta\text{scl}(\{x\}) \cap A \neq \emptyset\). This proves (iv).

(iii) \(\Rightarrow\) (iv): Assume that \(F \subseteq \Phi(A) \setminus A\) where \(F\) is \(\zeta\) semi closed and \(F \neq \emptyset\). This gives \(F \subseteq \Phi(A)\). This contradicts (iv).

(iv) \(\Rightarrow\) (v): It follows from the fact that \(\psi(A) \setminus A = \Phi(A) \setminus A\).

(v) \(\Rightarrow\) (i): Let \(A \subseteq U\) where \(U\) is \(\zeta\) semi open such that \(\Phi(A) \not\subset U\). This gives \(\Phi(A) \cap (X - U) = \emptyset\) or \(\Phi(A) \setminus [X \setminus (X \setminus U)] = \emptyset\). This gives \(\Phi(A) \setminus A \neq \emptyset\). Moreover, \(\Phi(A) \setminus A = \Phi(A) \cap (X \setminus U)\) is \(\zeta\) semi closed set in \(X\) since \(\Phi(A) = \text{cl}(\Phi(A))\) is closed in \(X\) and \(X \setminus U \in \zeta \text{sC}(X)\). Also, \(\Phi(A) \setminus U \subseteq \Phi(A) \setminus A\). This gives that \(\Phi(A) \setminus A\) contains a non empty \(\zeta\) semi closed set. This contradicts (v).

This completes the proof.

**Corollary 2.13:** Let \((X, \tau)\) be a \(T_1\) space and \(\zeta\) be a grill on \(X\). Then every \(\zeta\omega\) - closed set is \(\tau_{\zeta}\) - closed.

**Corollary 2.14:** Let \((X, \tau, \zeta)\) be a grill topological space and \(A\) be a \(\zeta\omega\) - closed set. Then the following are equivalent:

- (i) \(A\) is \(\tau_{\zeta}\) - closed.
- (ii) \(\psi(A) \setminus A\) is \(\zeta\) semi - closed set in \((X, \tau, \zeta)\).
- (iii) \(\Phi(A) \setminus A\) is \(\zeta\) semi - closed set in \((X, \tau, \zeta)\).

**Proof:**

(i) \(\Rightarrow\) (ii): Let \(A\) be a \(\tau_{\zeta}\) - closed. Then \(\Phi(A) \setminus A = \psi(A) \setminus A\) gives \(\psi(A) \setminus A = \emptyset\). This proves that \(\psi(A) \setminus A\) is \(\zeta\) semi - closed.

(ii) \(\Rightarrow\) (iii): Since \(\Phi(A) \setminus A = \psi(A) \setminus A\) and so \(\Phi(A) \setminus A\) is \(\zeta\) semi closed in \(X\).

(iii) \(\Rightarrow\) (i): Let \(\Phi(A) \setminus A\) be a \(\zeta\) semi - closed set. Now, \(A\) is \(\zeta\omega\) - closed and by Theorem 2.12 (v), \(\Phi(A) \setminus A\) contains no non empty \(\zeta\) semi - closed set. Therefore, \(\Phi(A) \setminus A = \emptyset\). This proves \(\Phi(A) = A\) and hence \(A\) is \(\tau_{\zeta}\) - closed.
Theorem 2.15: In a grill topological space \((X, \tau, \zeta)\) an \(\zeta\omega\) - closed set and \(\tau\zeta\) - dense set in itself is \(\omega\) - closed.

Proof: Suppose \(A\) is \(\tau\zeta\) - dense in itself and \(\zeta\omega\) - closed in \(X\). Let \(U\) be any \(\zeta\) semi open set containing \(A\), then \(\psi(A) \subseteq U\). Since \(A\) is \(\tau\zeta\) - dense in itself by [3, Lemma 2.12], \(\Phi(A) = cl(\Phi(A)) = \psi(A) = cl(A)\), we get \(cl(A) \subseteq U\) whenever \(A \subseteq U\). This proves that \(A\) is \(\omega\) - closed.

Corollary 2.16: If \((X, \tau, \zeta)\) is any grill space where \(\{\}\) \(\Phi(X) = \emptyset\) then \(A\) is \(\zeta\omega\) - closed if and only if \(A\) is \(\omega\) - closed.

Proof: The proof follows from the fact that \(\{\}\) \(\Phi(X) = \emptyset\), \((A \subseteq X)\) \(\Phi(A) = \emptyset\) and so every subset of \(X\) is \(\tau\zeta\) - dense set in itself.

The following theorem gives another characterization of \(\zeta\omega\) - closed set.

Theorem 2.7: Let \((X, \tau, \zeta)\) be a grill topological space. Then \(AX \subseteq \zeta\omega\) - closed if and only if \(\forall F \text{ is } \tau\zeta\) - closed and \(N\) contains no non empty \(\zeta\) semi-closed set.

Proof: Necessity - If \(A\) is \(\zeta\omega\) - closed set then by Theorem 2.12, \(\forall F \subseteq X\) \(\text{ contains no non-empty } \zeta\) semi-closed set. Let \(F = \psi(A)\), then \(F\) is \(\tau\zeta\) - closed set and \(F - N = A \cup (\Phi(A)) \subseteq (\Phi(A) \setminus A) = A\).

Sufficiency - Let \(U\) be any \(\zeta\) semi - open set in \(X\) containing \(A\), then \(\Phi(F) \subseteq F\) as \(\Phi(F) \subseteq F\) is \(\tau\zeta\) - closed gives \(\Phi(A) \cap (X \setminus U) \subseteq \Phi(F) \cap (X \setminus U) \subseteq F \cap (X \setminus U) \subseteq N\) where \(\Phi(A) \cap (X \setminus U)\) is \(\zeta\) semi-closed set. By hypothesis \(\Phi(A) \cap (X \setminus U) = \emptyset\) or \(\psi(A) \subseteq U\) implies that \(A\) is \(\zeta\omega\) - closed set.

Theorem 2.8: Let \((X, \tau, \zeta)\) be a grill topological space. Then every subset of \(X\) is \(\zeta\omega\) - closed if and only if every \(\zeta\) semi-open set is \(\tau\zeta\) - closed.

Proof: Necessity - Suppose every subset of \(X\) is \(\zeta\omega\) - closed. Let \(U\) be a \(\zeta\) semi-open set then \(\psi(U) \subseteq U\). Hence \(U\) is \(\tau\zeta\) - closed.

Sufficiency - Suppose that every \(\zeta\) semi-open set is \(\tau\zeta\) - closed. Let \(A\) be non empty subset of \(X\) contained in a \(\zeta\) semi-open set \(U\). Then \(\psi(A) \subseteq \psi(U)\) implies \(\psi(A) \subseteq U\). This proves that \(A\) is \(\zeta\omega\) - closed.

Theorem 2.9: A set \(A\) is \(\zeta\omega\) - open if and only if \(\forall F \subseteq \tau\zeta - \text{int}(A)\) whenever \(F\) is \(\zeta\) semi closed and \(F \subseteq A\).

Proof: Necessity - Suppose \(F \subseteq \tau\zeta - \text{int}(A)\), where \(F\) is \(\zeta\) semi closed and \(F \subseteq A\). Let \(A^C \subseteq U\), where \(U\) is \(\zeta\) semi open. Then \(U^C \subseteq A\) and \(U^C\) is \(\zeta\) semi closed. Therefore, \(U^C \subseteq \tau\zeta - \text{int}(A)\). Since \(U^C \subseteq \tau\zeta - \text{int}(A)\) we have \((\tau\zeta - \text{int}(A))^C \subseteq U\). That is, \(\psi(A^C) \subseteq U\), since \(\psi(A^C) = (\tau\zeta - \text{int}(A))^C\). Thus \(A^C\) is \(\zeta\omega\) - closed, that is \(A\) is \(\zeta\omega\) - open.

Sufficiency - Suppose that \(A\) is \(\zeta\omega\) - open. \(F \subseteq A\) and \(F\) is \(\zeta\) semi closed. Then \(F^C\) is \(\zeta\) semi - open and \(A^C \subseteq F^C\). Therefore, \(\psi(A^C) \subseteq F^C\) and so \(F \subseteq \tau\zeta - \text{int}(A)\), since \(\psi(A^C) = (\tau\zeta - \text{int}(A))^C\).
3. SOME CHARACTERIZATIONS OF $\zeta$-NORMAL AND $\zeta$-REGULAR SPACES

In this section we introduce $\zeta$-regular and $\zeta$-normal spaces via grills.

**Definition 3.1:** A grill space $(X, \tau, \zeta)$ is said to be an $\zeta$-normal if for every pair of disjoint closed sets $A$ and $B$, there exist $\zeta$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Theorem 3.2:** Let $X$ be a normal space and $\zeta$ be a grill on $X$ then for each pair of disjoint closed sets $A$ and $B$, there exist disjoint $\zeta$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Proof:** It is obvious since every open set is $\zeta$-open.

**Theorem 3.3:** Let $X$ be a normal space and $\zeta$ be a grill on $X$, then for each closed set $A$ and an open set $V$ containing $A$, there exists a $\zeta$-open set $U$ such that $A \subseteq U \subseteq \psi(U) \subseteq V$.

**Proof:** Let $A$ be a closed set and $V$ be an open set containing $A$. Since $A$ and $X \setminus V$ are disjoint closed sets, there exist disjoint $\zeta$-open sets $U$ and $V$ such that $A \subseteq U$ and $X \setminus V \subseteq W$. Again $U \cap V = \phi$ implies that $U \cap \tau_\zeta = \text{int}(W) = \phi$ and so $\psi(U) \subseteq X \setminus \tau_\zeta \subseteq \text{int}(W)$. Since $X \setminus V$ is closed and $W$ is $\zeta$-open, $X \setminus V \subseteq W$ implies that $X \setminus V \subseteq \tau_\zeta \subseteq \text{int}(W)$ and so $X \setminus \tau_\zeta \subseteq \text{int}(W) \subseteq V$. Thus, we have, $A \subseteq U \subseteq \psi(U) \subseteq X \setminus \tau_\zeta \subseteq \text{int}(W) \subseteq V$ where $U$ is a $\zeta$-open set.

**Remark 3.4:** The following Theorem gives characterizations of a normal space in terms of $\omega$-open sets, which is a consequence of Theorems 3.2, 3.3 and Remark 2.11 if one takes $\zeta = [X]$.

**Theorem 3.5:** Let $X$ be a normal space and $\zeta$ be a grill on $X$ then for each pair of disjoint closed sets $A$ and $B$, there exist disjoint $\omega$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Theorem 3.6:** Let $X$ be a normal space and $\zeta$ be a grill on $X$ then for each closed set $A$ and an open set $V$ containing $A$, there exists an $\omega$-open set $U$ such that $A \subseteq U \subseteq \text{cl}(U) \subseteq V$.

**Definition 3.7:** A grill space $(X, \tau, \zeta)$ is said to be $\zeta$-$\omega$-regular if for each pair consisting of a point $x$ and a closed set $B$ not containing $x$, there exist disjoint $\zeta$-$\omega$-open sets $U$ and $V$ such that $x \in U$ and $B \subseteq V$.

**Remarks 3.8:** It is obvious that every regular space is $\zeta$-$\omega$-regular.

**Theorem 3.9:** Let $(X, \tau, \zeta)$ a grill space. Then the following are equivalent:

(i) $(X, \tau, \zeta)$ is $\zeta$-$\omega$-regular.

(ii) For every closed set $B$ not containing $x \in U$, there exists disjoint $\zeta$-$\omega$-open set $U$ and $V$ of $X$ such that $x \in U$ and $B \subseteq V$.

(iii) For every open set $V$ containing $x \in X$, there exists an $\zeta$-$\omega$-open set $U$ such that $x \in U \subseteq \psi(U) \subseteq V$.

**Proof:**

(i) $\Rightarrow$ (ii): It is clear, since every open set is $\zeta$-$\omega$-open.

(ii) $\Rightarrow$ (iii): Let $V$ be an open subset such that such that $x \in V$. Then $X \setminus V$ is a closed set not containing $x$. Therefore, there exist disjoint $\zeta$-$\omega$-open sets $U$ and $W$ such that $x \in U$ and $X \setminus V \subseteq W$. Now, $X \setminus V \subseteq W$ implies that $X \setminus V \subseteq \tau_\zeta \subseteq \text{int}(W)$ and so $X \setminus \tau_\zeta \subseteq \text{int}(W) \subseteq V$. Again $U \cap W = \phi$ implies that $U \cap \tau_\zeta \subseteq \text{int}(W) = \phi$ and so, $\psi(U) \subseteq X \setminus \tau_\zeta \subseteq \text{int}(W)$. Therefore $x \in U \subseteq \psi(U) \subseteq V$. This proves (iii)
(iii) $\Rightarrow$ (i): Let $B$ be a closed set not containing $x$. By hypothesis, there exists a $\zeta\omega$-open set $U$ of $X$ such that $x \in U \subseteq \psi(U) \subseteq X \setminus B$. If $W = X \setminus \psi(U)$ then $U$ and $W$ are disjoint $\zeta\omega$-open sets such that $x \in U$ and $B \subseteq W$. This proves $(i)$.

**Theorem 3.10:** If every $\zeta$ semi open subset of a grill space $(X, \tau, \zeta)$ is $\tau_\zeta$-closed, then $(X, \tau, \zeta)$ is $\zeta\omega$-regular.

**Proof:** Suppose every $\zeta$ semi open subset of a grill space $(X, \tau, \zeta)$ is $\tau_\zeta$-closed. Then by Theorem 2.8, every subset of $X$ is $\zeta\omega$-closed and hence every subset of $X$ is $\zeta\omega$-open. If $B$ is a closed set not containing $x$, then $\{x\}$ and $B$ are the required disjoint $\zeta\omega$-open sets containing $x$ and $B$ respectively. Therefore, $(X, \tau, \zeta)$ is $\zeta\omega$-regular.

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**Source of support:** Nil, **Conflict of interest:** None Declared

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