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A NEW CLASS OF GENERALIZED CLOSED SETS USING GRILLS

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ABSTRACT

The aim of this paper is to apply the notion of ζ semi- open sets to obtain a new class of $\zeta\omega-$closed sets via grills. The properties of the above mentioned sets are investigated. Further the concept is extended to derive some applications of $\zeta\omega-$closed sets via Grills.

Key words and phrases: *Grill, topology τ_ζ, operator Φ, $\zeta\omega-$closed.*

1. INTRODUCTION AND PRELIMINARIES

The idea of grills on a topological space was first introduced by Choquet [2] in 1947. In [8], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Hatir and Jafari [4] have defined new classes of sets in grill topological spaces. Ahmad Al-Omari and Noiri [6] introduced and investigated the notions of $\zeta\alpha$ - open sets, ζ semi open sets and $\zeta\beta$ open sets in grill topological spaces.

Definition 1.1: [2] A collection ζ of non empty subsets of a topological spaces X is said to be a grill on X if
(i) $A\in\zeta$ and $A\subseteq B$ implies that $B\in\zeta$,
(ii) $A,B\subseteq X$ and $A\cup B\in\zeta$ implies that $A\in\zeta$ or $B\in\zeta$.

Definition 1.2: [8] Let (X,τ) be a topological space and ζ be a grill on X. We define a mapping $\Phi: P(X)\rightarrow P(X)$ denoted by $\Phi_\zeta(A,\tau)$ (for $A\in P(X)$) or $\Phi_\zeta(A)$ or simply $\Phi(A)$ called the operator associated with the grill ζ and the topology τ defined as follows: $\Phi(A)=\Phi_\zeta(X,\tau,\varsigma)$ $=\{x\in X \mid A\cap U\in\zeta$ for all $U\in\tau(x)$ for each $A\in P(X)\}$.

Definition 1.3: [8] Let ζ be a grill on X . We define a map $\psi: P(X)\rightarrow P(X)$ by $\psi(A)=A\cup\Phi(A)$ for all $A\in P(X)$.

Definition 1.4: [8] Corresponding to a grill ζ on topological space (X,τ) there exist a unique topology τ_ζ (say) on X given by $\tau_\zeta=\{U\subseteq X:\psi(X\setminus A)=X\setminus U\}$ where for any all $A\subseteq X, \psi(A)=A\cup\Phi(A)=\tau_\zeta-cl(A)$.

Definition 1.5: [10] Let X be a space and $(\phi\neq)A\subseteq X$. Then $[A]=\{B\subseteq X: A\cap B\neq\phi\}$ is a grill on X called principal grill generated by A.

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2. $\zeta\omega$ - CLOSED SETS

Definition 2.1: Let (X,τ) be a topological space and ζ be any grill on X. Then a subset A of X is called $\zeta\omega$ - closed if $\psi(A) \subseteq U$ whenever $A \subseteq U$ and U is ζ semi- open in X. A subset A of X is called $\zeta\omega$ - open if $X \setminus A$ is $\zeta\omega$ - closed.

Theorem 2.2: Every closed set in (X,τ) is $\zeta\omega$ - closed in (X,τ,ζ).

Proof: Let A be any closed set and U be any ζ semi - open set such that $cl(A) = A \subseteq U$ since A is closed. But $\psi(A) \subseteq cl(A)$, we have $\psi(A) \subseteq U$ whenever $A \subseteq U$. Hence A is $\zeta\omega$ - closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.3: Let $X = \{a,b,c\}$, $\tau = \{\phi,\{b\},\{b,c\},X\}$ with $\zeta = \{X,\{a\},\{c\},\{a,c\},\{a,b\}, \{b,c\}\}$. Let $A = \{b\}$ and $U = \{b\}$ where U is ζ semi open in X. Therefore, $\Phi(A) = \{\phi\}$ and $\psi(A) = A \cup \Phi(A) = \{b\} \subseteq U$. Then the set $\{b\}$ is $\zeta\omega$ – closed but not closed.

Theorem 2.4: Every τ_ζ - closed set in (X,τ,ζ) is $\zeta\omega$ - closed in (X,τ,ζ).

Proof: Let A be a τ_ζ - closed and then $\Phi(A) \subseteq A$ implies $A \cup \Phi(A) \subseteq A \cup A = A.$. Let $A \subseteq U$ where U is ζ semi open. Hence, $\psi(A) \subseteq U$ whenever $A \subseteq U$ and U is ζ semi- open. Therefore, A is $\zeta\omega$ - closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.5: Let $X = \{a,b,c\}$, $\tau = \{\phi,\{a\},\{b,c\},X\}$ and $\zeta = \{\{a\},\{b\},\{a,b\}, \{a,c\}, \{b,c\}, X\}$. Then the set $\{a,b\}$ is $\zeta\omega$ – closed but not τ_ζ - closed.

Theorem 2.6: Every ω - closed set in (X,τ) is $\zeta\omega$ - closed in (X,τ,ζ).

Proof: Let A be any ω - closed set and U be any ζ semi open set containing A. Since every ζ semi open set is semi open and A is ω - closed we have, $cl(A) \subseteq U$. But $\psi(A) \subseteq cl(A)$. Thus we have, $\psi(A) \subseteq U$ whenever $A \subseteq U$. Hence A is $\zeta\omega$ – closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.7: In Example 2.3, the set $\{b\}$ is $\zeta\omega$ – closed but not ω - closed.

Remark 2.8: In a grill space (X,τ,ζ), $\zeta\omega$ – closed sets are generalization of ω - closed sets which itself is a generalization of the closed set.

Theorem 2.9: Every $\zeta\omega$ - closed set in (X,τ,ζ) is ζg - closed in (X,τ,ζ).

Proof: Let $A \subseteq U$, U is open and hence it is ζ semi open. Since A is $\zeta\omega$ - closed, we have $\psi(A) \subseteq U$. But $\Phi(A) \subseteq \psi(A) \subseteq U$. Hence A is ζg - closed.

The converse of the above Theorem is not true as seen from the following Example.

Example 2.10: Let $X = \{a,b,c\}$ with $\tau = \{\phi,\{a\},X\}$ and $\zeta = \{\{a\},\{b\},\{a,b\}, \{a,c\}, \{b,c\}, X\}$. Then the set $\{a,b\}$ is ζg – closed but not $\zeta\omega$ - closed.

Remark 2.11: In the case $[X]$ principal grill generated by X, it is known [8] that $\tau = \tau_{[X]}$ so that any $[X] - \zeta\omega -$ closed set becomes simply an ω closed set and vice –versa.

Theorem 2.12: Let (X,τ,ζ) be a topological space and ζ be a grill on X. Then for a subset A of X, the following are equivalent:
(i) A is $\zeta\omega$ - closed.
(ii) $\psi(A) \subseteq U$ for ζ semi open set U containing A.
(iii) For each $x \in \psi(A), \zeta scl(\{x\} \cap A) \neq \phi.$
(iv) $\psi(A) \setminus A$ contains no non empty ζ semi closed set of $(X,\tau,\zeta).$
(v) $\Phi(A) \setminus A$ contains no non empty ζ semi closed set of $(X,\tau,\zeta).$

Proof:
$(i) \Rightarrow (ii)$: Let A be a $\zeta\omega$ - closed. Then clearly, $\psi(A) \subseteq U$ whenever $A \subseteq U$ and U is ζ semi open in X.

$(ii) \Rightarrow (iii)$: Suppose $x \in \psi(A).$ If $\zeta scl(\{x\} \cap A) = \phi$, then $A \subseteq X \setminus \zeta scl(\{x\})$ where $X \setminus \zeta scl(\{x\}$ is a ζ semi open set. By assumption $\psi(A) \setminus A \subseteq X \setminus \zeta scl(\{x\})$, which is a contradiction to $x \in \psi(A)$. Hence $\zeta scl(\{x\}) \cap A \neq \phi.$ This proves (iv).

$(iii) \Rightarrow (iv)$: Assume that $F \subseteq \Phi(A) \setminus A$ where F is ζ semi closed and $F \neq \phi.$ This gives $F \subseteq \Phi(A)$. This contradicts (iv).

$(iv) \Rightarrow (v)$: It follows from the fact that $\psi(A) \setminus A = \Phi(A) \setminus A.$

$(v) \Rightarrow (i)$: Let $A \subseteq U$ where U is ζ semi - open such that $\Phi(A) \not\subset U.$ This gives $\Phi(A) \cap (X - U) = \phi$ or $\Phi(A) \setminus [X \setminus (X \setminus U)] = \phi.$ This gives $\Phi(A) \setminus A \neq \phi.$ Moreover, $\Phi(A) \setminus A = \Phi(A) \cap (X \setminus U)$ is ζ semi closed set in X since $\Phi(A) = cl(\Phi(A))$ is closed in X and $X \setminus U \in \zeta sC(X).$ Also, $\Phi(A) \setminus U \subseteq \Phi(A) \setminus A.$ This gives that $\Phi(A) \setminus A$ contains a non empty ζ semi closed set . This contradicts $(v).$
This completes the proof.

Corollary 2.13: Let (X,τ) be a T_1 space and ζ be a grill on X. Then every $\zeta\omega$ - closed set is τ_ζ - closed.

Corollary 2.14: Let (X,τ,ζ) be a grill topological space and A be a $\zeta\omega$ - closed set. Then the following are equivalent:
(i) A is τ_ζ - closed.
(ii) $\psi(A) \setminus A$ is ζ semi - closed set in (X,τ,ζ).
(iii) $\Phi(A) \setminus A$ is ζ semi - closed set in (X,τ,ζ).

Proof:
$(i) \Rightarrow (ii)$: Let A be a τ_ζ -closed. Then $\Phi(A) \setminus A = \psi(A) \setminus A$ gives $\psi(A) \setminus A = \phi.$ This proves that $\psi(A) \setminus A$ is ζ semi- closed.

$(ii) \Rightarrow (iii)$: Since $\Phi(A) \setminus A = \psi(A) \setminus A$ and so $\Phi(A) \setminus A$ is ζ semi closed in X.

$(iii) \Rightarrow (i)$: Let $\Phi(A) \setminus A$ be a ζ semi- closed set. Now, A is $\zeta\omega$ - closed and by Theorem 2.12 (v), $\Phi(A) \setminus A$ contains no non empty ζ semi - closed set. Therefore, $\Phi(A) \setminus A = \phi.$ This proves $\Phi(A) = A$ and hence A is τ_ζ - closed.

Theorem 2.15: In a grill topological space (X,τ,ζ) an $\zeta\omega$ - closed set and τ_ζ - dense set in itself is ω - closed.

Proof: Suppose A is τ_ζ - dense in itself and $\zeta\omega$ - closed in X . Let U be any ζ semi open set containing A, then $\psi(A) \subseteq U$. Since A is τ_ζ - dense in itself by [3, Lemma 2.12], $\Phi(A) = cl(\Phi(A) = \psi(A) = cl(A),$ we get $cl(A) \subseteq U$ whenever $A \subseteq U.$ This proves that A is ω - closed.

Corollary 2.16: If (X,τ,ζ) is any grill space where $\zeta = P(X)\setminus\{\phi\}$ then A is $\zeta\omega$ - closed if and only if A is ω - closed.

Proof: The proof follows from the fact that $\zeta = P(X)\setminus\{\phi\}$, $\Phi(A) = cl(A) \supset A$ and so every subset of X is τ_ζ - dense set in itself.

The following theorem gives another characterization of $\zeta\omega$ - closed set.

Theorem 2.7: Let (X,τ,ζ) be a grill topological space. Then $A(\subseteq X)$ is $\zeta\omega$ - closed if and only if $A = F \setminus N$, where F is τ_ζ - closed and N contains no non empty ζ semi- closed set

Proof: Necessity - If A is $\zeta\omega$ - closed set then by Theorem 2.12, $N = \psi(A)\setminus(A)$ contains no non - empty ζ semi-closed set. Let $F = \psi(A),$ then F is τ_ζ - closed set and $F - N = A \cup (\Phi(A)) \setminus (\Phi(A) \setminus A) = A.$

Sufficiency -Let U be any ζ semi - open set in X containing A , then $F \setminus N \subseteq U$ implies $F \cap (X \setminus U) \subseteq F \cap (X \setminus (F \setminus N)) = F \cap (X \setminus F) \cup N = F \cap N \subseteq N.$ By hypothesis $A \subseteq F$ and $\Phi(F) \subseteq F$ as F is τ_ζ - closed gives $\Phi(A) \cap (X \setminus U) \subseteq$ $\Phi(F) \cap (X \setminus U) \subseteq$ $F \cap (X \setminus U) \subseteq N$ where $\Phi(A) \cap (X \setminus U)$ is ζ semi- closed set. By hypothesis $\Phi(A) \cap$ $(X \setminus U) = \phi$ or $\psi(A) \subseteq U$ implies that A is $\zeta\omega$ - closed set.

Theorem 2.8: Let (X,τ,ζ) be a grill topological space. Then every subset of X is $\zeta\omega$ - closed if and only if every ζ semi-open set is τ_ζ - closed.

Proof: Necessity - Suppose every subset of X is $\zeta\omega$ - closed. Let U be a ζ semi- open set then U is $\zeta\omega$ - closed and $\psi(U) \subseteq U.$ Hence U is τ_ζ - closed.

Sufficiency - Suppose that every ζ semi- open set is τ_ζ - closed. Let A be non empty subset of X contained in a ζ semi- open set U . Then $\psi(A) \subseteq \psi(U)$ implies $\psi(A) \subseteq U.$ This proves that A is $\zeta\omega$ - closed.

Theorem 2.9: A set A is $\zeta\omega$ – open if and only if $F \subseteq \tau_\zeta - \text{int}(A)$ whenever F is ζ semi closed and $F \subseteq A.$

Proof: Necessity - Suppose that $F \subseteq \tau_\zeta - \text{int}(A)$, where F is ζ semi closed and $F \subseteq$ A . Let $A^c \subseteq U,$ where U is ζ semi open. Then $U^c \subseteq A$ and U^c is ζ semi closed. Therefore, $U^c \subseteq \tau_\zeta - \text{int}(A)$ .Since $U^c \subseteq \tau_\zeta - \text{int}(A)$, we have $(\tau_\zeta - \text{int}(A))^c \subseteq U$ .That is, $\psi(A^c) \subseteq U,$ since $\psi(A^c)$ $= (\tau_\zeta - \text{int}(A))^c$. Thus A^c is $\zeta\omega$ - closed, that is A is $\zeta\omega$ - open.

Sufficiency - Suppose that A is $\zeta\omega$ - open. $F \subseteq A$ and F is ζ semi closed. Then F^c is ζ semi - open and $A^c \subseteq F^c.$ Therefore, $\psi(A^c) \subseteq F^c$ and so $F \subseteq \tau_\zeta - \text{int}(A)$, since $\psi(A^c) = (\tau_\zeta - \text{int}(A))^c$.

3. SOME CHARACTERIZATIONS OF $\zeta\omega$-NORMAL AND $\zeta\omega$–REGULAR SPACES

In this section we introduce $\zeta\omega$ - regular and $\zeta\omega$ -normal spaces via grills.

Definition 3.1: A grill space (X,τ,ζ) is said to be an $\zeta\omega$ – normal if for every pair of disjoint closed sets A and B, there exist $\zeta\omega$ – open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 3.2: Let X be a normal space and ζ be a grill on X then for each pair of disjoint closed sets A and B, there exist disjoint $\zeta\omega$ - open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Proof: It is obvious since every open set is $\zeta\omega$ - open.

Theorem 3.3: Let X be a normal space and ζ be a grill on X, then for each closed set A and an open set V containing A, there exists a $\zeta\omega$ - open set U such that $A \subseteq U \subseteq \psi(U) \subseteq V$.

Proof: Let A be a closed set and V be an open set containing A. Since A and $X \setminus V$ are disjoint closed sets, there exist disjoint $\zeta\omega$ - open sets U and W such that $A \subseteq U$ and $X \setminus V \subseteq W$. Again $U \cap W = \phi$ implies that $U \cap \tau_\zeta$ - $\mathrm{int}(W) = \phi$ and so $\psi(U) \subseteq X - \tau_\zeta$ - $\mathrm{int}(W)$. Since $X \setminus V$ is closed and W is $\zeta\omega$ - open, $X \setminus V \subseteq W$ implies that $X \setminus V \subseteq \tau_\zeta$ -$\mathrm{int}(W)$ and so $X \setminus \tau_\zeta$ -$\mathrm{int}(W) \subseteq V$. Thus, we have,
$A \subseteq U \subseteq \psi(U) \subseteq X \setminus \tau_\zeta$ -$\mathrm{int}(W) \subseteq V$ where U is a $\zeta\omega$ - open set.

Remark 3.4: The following Theorem gives characterizations of a normal space in terms of ω – open sets, which is a consequence of Theorems 3.2, 3.3 and Remark 2.11 if one takes $\zeta = [X]$.

Theorem 3.5: Let X be a normal space and ζ be a grill on X then for each pair of disjoint closed sets A and B, there exist disjoint ω - open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 3.6: Let X be a normal space and ζ be a grill on X then for each closed set A and an open set V containing A, there exists an ω - open set U such that $A \subseteq U \subseteq cl(U) \subseteq V$.

Definition 3.7: A grill space (X,τ,ζ) is said to be $\zeta\omega$ - regular if for each pair consisting of a point x and a closed set B not containing x, there exist disjoint $\zeta\omega$ - open sets U and V such that $x \in U$ and $B \subseteq V$.

Remarks 3.8: It is obvious that every regular space is $\zeta\omega$ - regular.

Theorem 3.9: Let (X,τ,ζ) a grill space. Then the following are equivalent:

(i) (X,τ,ζ) is $\zeta\omega$ - regular .

(ii) For every closed set B not containing $x \in U$, there exists disjoint $\zeta\omega$ - open set U and V of X such that $x \in U$ and $B \subseteq V$.

(iii) For every open set V containing $x \in X$, there exists an $\zeta\omega$ - open set U such that $x \in U \subseteq \psi(U) \subseteq V$.

Proof:
$(i) \Rightarrow (ii)$: It is clear, since every open set is $\zeta\omega$ - open.

$(ii) \Rightarrow (iii)$: Let V be an open subset such that such that $x \in V$. Then $X \setminus V$ is a closed set not containing x. Therefore, there exist disjoint $\zeta\omega$– open sets U and W such that $x \in U$ and $X \setminus V \subseteq W$. Now, $X \setminus V \subseteq W$ implies that $X \setminus V \subseteq \tau_\zeta - \mathrm{int}(W)$ and so $X \setminus \tau_\zeta - \mathrm{int}(W) \subseteq V$. Again $U \cap W = \phi$ implies that $U \cap \tau_\zeta - \mathrm{int}(W) = \phi$ and so, $\psi(U) \subseteq X \setminus \tau_\zeta - \mathrm{int}(W)$. Therefore $x \in U \subseteq \psi(U) \subseteq V$. This proves (iii)

$(iii) \Rightarrow (i)$: Let B be a closed set not containing x. By hypothesis, there exists a $\zeta\omega$ - open set U of X such that $x \in U \subseteq \psi(U) \subseteq X \setminus B$. If $W = X \setminus \psi(U)$ then U and W are disjoint $\zeta\omega$ - open sets such that $x \in U$ and $B \subseteq W$. This proves (i).

Theorem 3.10: If every ζ semi open subset of a grill space (X, τ, ζ) is τ_ζ - closed, then (X, τ, ζ) is $\zeta\omega$ – regular**.**

Proof: Suppose every ζ semi open subset of a grill space (X, τ, ζ) is τ_ζ - closed.
Then by Theorem 2.8, every subset of X is $\zeta\omega$ – closed and hence every subset of X is $\zeta\omega$ - open. If B is a closed set not containing x, then $\{x\}$ and B are the required disjoint $\zeta\omega$ – open sets containing x and B respectively. Therefore, (X, τ, ζ) is $\zeta\omega$ – regular.

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