ABSTRACT

A set $D$ of a graph $G = (V, E)$ is a dominating set, if every vertex in $V(G) - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set. A dominating set $D$ is called a complementary tree nil dominating set, if the induced subgraph $< V(G) - D >$ is a tree and also the set $V(G) - D$ is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of $G$ and is denoted by $\gamma_{ctnd}(G)$. The connectivity $\kappa(G)$ of $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper, an upper bound for the sum of the complementary tree nil domination number and connectivity of a graph is found and the corresponding extremal graphs are characterized.

Key words: Domination number, Complementary tree nil domination number, Connectivity.

1. INTRODUCTION

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph $G$, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. A graph with $p$ vertices and $q$ edges is denoted by $G(p, q)$. The concept of domination in graphs was introduced by Ore [5]. A set $D \subseteq V(G)$ is said to be a dominating set of $G$, if every vertex in $V(G) - D$ is adjacent to some vertex in $D$. The cardinality of a minimum dominating set in $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. Muthammai, Bhanumathi and Vidhya [4] introduced the concept of complementary tree dominating set. A dominating set $D \subseteq V(G)$ is said to be a complementary tree dominating set (ctd-set) if the induced subgraph $< V(G) - D >$ is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of $G$ and is denoted by $\gamma_{ctd}(G)$. The connectivity $\kappa(G)$ of $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. Any undefined terms in this paper may be found in Harary[1].

The concept of complementary tree nil dominating set is introduced in [3]. A dominating set $D \subseteq V(G)$ is said to be a complementary tree nil dominating set (ctnd-set) if the induced subgraph $< V(G) - D >$ is a tree and the set $V(G) - D$ is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of $G$ and is denoted by $\gamma_{ctnd}(G)$.

In this paper, we find an upper bound for the sum of the complementary tree nil domination number and connectivity of a graph is found and the corresponding external graphs are characterized.

2. PRIOR RESULTS

Theorem 2.1: [1] For any connected graph $G$, $\kappa(G) \leq \delta(G)$.

Theorem 2.2: [3] For any connected graph $G$ with $p$ vertices, $2 \leq \gamma_{ctnd}(G) \leq p$, where $p \geq 2$.
Theorem 2.3: [3] Let $G$ be a connected graph with $p$ vertices. Then $\gamma_{ctnd}(G) = 2$ if and only if $G$ is a graph obtained by attaching a pendant edge at a vertex of degree $p - 2$ in $T + K_1$, where $T$ is a tree on $(p - 2)$ vertices.

Theorem 2.4: [3] For any connected graph $G$, $\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$, where $p \geq 2$.

Theorem 2.5: [3] Let $G$ be a connected graph with $p \geq 3$ and $\delta(G) = 1$. Then $\gamma_{ctnd}(G) = p - 1$ if and only if the subgraph of $G$ induced by vertices of degree at least $2$ is $K_2$ or $K_1$.

That is, $G$ is one of the graphs $K_{1,p-1}$ or $S_{m,n}$ ($m + n = p$, $m, n \geq 1$), where $S_{m,n}$ is a bistar which is obtained by attaching $m-1$ pendant edges at one vertex of $K_2$ and $n-1$ pendant edges at other vertex of $K_2$.

Theorem 2.6: [3] Let $G$ be a connected noncomplete graph with $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G) = p - 1$ if and only if each edge of $G$ is a dominating edge.

Theorem 2.7: [3] Let $T$ be a tree on $p$ vertices such that $\gamma_{ctnd}(T) \leq p - 2$. Then $\gamma_{ctnd}(T) = p - 2$ if and only if $T$ is one of the following graphs.

(i) $T$ is obtained from a path $P_n$ ($n \geq 4$ and $n < p$) by attaching pendant edges at least one of the end vertices of $P_n$.
(ii) $T$ is obtained from $P_3$ by attaching pendant edges either at both the end vertices or at all the vertices of $P_3$.

Notation 2.8: [3] Let $G$ be the class of connected graphs $G$ with $\delta(G) = 1$ having one of the following properties.

(a) There exist two adjacent vertices $u, v$ in $G$ such that $\text{deg}_{G}(u) = 1$ and $\langle V(G) - \{u, v\} \rangle$ contains $P_3$ as an induced subgraph such that end vertices of $P_3$ have degree at least $2$ and the central vertex of $P_3$ has degree at least $3$.

(b) Let $P$ be the set of all pendant vertices in $G$ and let there exist a vertex $v \in V(G) - P$ having minimum degree in $V(G) - P$ and is not a support of $G$ such that $V(G) - (N_{v,p}[v] - P)$ contains $P_3$ as an induced subgraph such that the end vertices of $P_3$ have degree at least $2$ and the central vertex of $P_3$ has degree at least $3$.

Theorem 2.9: [3] Let $G$ be a connected graph with $\delta(G) = 1$ and $\gamma_{ctnd}(G) \neq p - 1$. Then $\gamma_{ctnd}(G) = p - 2$ if and only if $G$ does not belong to the class $G$ of graphs.

Theorem 2.10: [3] Let $G$ be a connected, noncomplete graph with $p$ vertices ($p \geq 4$) and $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G) = p - 2$ if and only if $G$ is one of the following graphs.

(a) A cycle on at least five vertices.
(b) A wheel with six vertices.
(c) $G$ is the one point union of complete graphs.
(d) $G$ is obtained by joining two complete graphs by edges.
(e) $G$ is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is a complete graph on $(p - 1)$ vertices.
(f) $G$ is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and $N(v)$ contains at least one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

3. MAIN RESULTS

Theorem 3.1: For any connected graph $G$, $\gamma_{ctnd}(G) + \kappa(G) \leq 2p - 1$, equality holds if and only if $G \cong K_p$.

Proof: $\gamma_{ctnd}(G) + \kappa(G) \leq p + \delta(G) \leq p + p - 1 = 2p - 1$.

Let $\gamma_{ctnd}(G) + \kappa(G) = 2p - 1$. Then $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 1$ and $G$ is a complete graph on $p$ vertices.

Hence $G \cong K_p$.

Conversely, if $G \cong K_p$, then $\gamma_{ctnd}(G) + \kappa(G) = 2p - 1$. 
Theorem 3.2: For any noncomplete graph $G$, \( \gamma_{ctnd}(G) + \kappa(G) \leq 2p - 3 \)

Proof: Since $G$ is not complete, by Theorem 3.1 \( \gamma_{ctnd}(G) + \kappa(G) \leq 2p - 2 \), by Theorem 3.1.
Assume \( \gamma_{ctnd}(G) + \kappa(G) = 2p - 2 \). Then either \( \gamma_{ctnd}(G) = p \) and \( \kappa(G) = p - 2 \) or \( \gamma_{ctnd}(G) = p - 1 \) and \( \kappa(G) = p - 1 \).

Case-1:
\( \gamma_{ctnd}(G) = p \) and \( \kappa(G) = p - 2 \).
\( \gamma_{ctnd}(G) = p \) if and only if $G \cong K_p$ on $p$ vertices. But \( \kappa(K_p) = p - 1 \). Therefore, no connected graph exists with \( \gamma_{ctnd}(G) = p \) and \( \kappa(G) = p - 2 \).

Case-2: \( \gamma_{ctnd}(G) = p - 1 \) and \( \kappa(G) = p - 1 \).
\( \kappa(G) = p - 1 \) if and only if $G \cong K_p$ on $p$ vertices. But \( \gamma_{ctnd}(K_p) = p \).
From Case 1 and Case 2, no connected graph $G$ exists with \( \gamma_{ctnd}(G) + \kappa(G) = 2p - 2 \).
Hence \( \gamma_{ctnd}(G) + \kappa(G) \leq 2p - 3 \).

Theorem 3.3: For any connected graph $G$, \( \gamma_{ctnd}(G) + \kappa(G) = 2p - 3 \) if and only if $G$ is isomorphic to the graph $K_p - Y$, where $Y$ is a matching in $K_p$ ($p \geq 3$).

Proof: Let \( \gamma_{ctnd}(G) + \kappa(G) = 2p - 3 \). Then there are three cases to consider
(i) \( \gamma_{ctnd}(G) = p \) and \( \kappa(G) = p - 3 \)
(ii) \( \gamma_{ctnd}(G) = p - 1 \) and \( \kappa(G) = p - 2 \)
(iii) \( \gamma_{ctnd}(G) = p - 2 \) and \( \kappa(G) = p - 1 \)

Case-1:
\( \gamma_{ctnd}(G) = p \) and \( \kappa(G) = p - 3 \) or \( \gamma_{ctnd}(G) = p - 2 \) and \( \kappa(G) = p - 1 \).
\( \gamma_{ctnd}(G) = p \) if and only if $G \cong K_p$ on $p$ vertices. But \( \kappa(K_p) = p - 1 \). Therefore, no connected graph exists with \( \gamma_{ctnd}(G) = p \) and \( \kappa(G) = p - 3 \).
\( \kappa(G) = p - 1 \) if and only if $G \cong K_p$ on $p$ vertices. But \( \gamma_{ctnd}(K_p) = p \). Therefore, no connected graph exists with \( \gamma_{ctnd}(G) = p - 2 \) and \( \kappa(G) = p - 1 \).
Therefore there exists no connected graph in this case.

Case-2: \( \gamma_{ctnd}(G) = p - 1 \) and \( \kappa(G) = p - 2 \).
Since \( \kappa(G) = p - 2 \), \( \delta(G) \geq p - 2 \). If \( \delta(G) = p - 1 \), then $G$ is a complete graph. Hence \( \delta(G) = p - 2 \). Then $G$ is isomorphic to $K_p - Y$, where $Y$ is a matching in $K_p$. But if \( \delta(G) = 1 \), then \( \gamma_{ctnd}(G) = p - 1 \) if and only if $G$ is isomorphic to $K_{1,p-1}$ or $S_{m,n}$ and if \( \delta(G) \geq 2 \), then \( \gamma_{ctnd}(G) = p - 1 \) if and only if $G$ is a graph in which each edge is a dominating edge.

Subcase-2.1: \( \delta(G) = 1 \)
Since \( \delta(G) = p - 2 \), \( p = 3 \). $G$ is isomorphic to $K_3 - Y$, where $Y$ is a matching in $K_3$. That is, $G \cong P_3$. But \( \gamma_{ctnd}(G) = p - 1 \) if and only if $G$ is isomorphic to $K_{1,p-1}$ or $S_{m,n}$. Since \( p = 3 \), $G \cong K_{1,2} \cong P_3$ and $G \not\cong S_{m,n}$ (m, n \( \geq 2 \)). Hence $G \cong P_3$.

Subcase-2.2: \( \delta(G) \geq 2 \)
Then $G$ is isomorphic to $K_p - Y$, where $Y$ is a matching in $K_p$ ($p \geq 4$). But \( \gamma_{ctnd}(G) = p - 1 \) if and only if $G$ is a graph in which each edge is a dominating edge.
Hence $G$ is isomorphic to $K_p - Y$ where $Y$ is a matching in $K_p$ ($p \geq 3$).

Conversely, if $G$ is isomorphic to $K_p - Y$ where $Y$ is a matching in $K_p$ ($p \geq 3$), then $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 2$. Hence $\gamma_{ctnd}(G) + \kappa(G) = 2p - 3$.

**Notation 3.4:** Following notations are used in this paper.

(i) $K_3(1,0,0)$ is a graph obtained by attaching a pendant edge at one of the vertices of $K_3$.

(ii) $G_1$ is a graph obtained from $K_2, 3$ by joining the vertices of degree 3 by an edge.

(iii) $G_2$ is a graph obtained from $K_3, 3$ by joining any three independent vertices by at least two edges.

(iv) $G_3$ is a graph with at least 7 vertices such that $V(G)$ can be partitioned into two sets $X$ and $V - X$ such that $|X| = p - 3$, each edge of $<X>$ is a dominating edge and $<V - X>$ is independent.

(v) $G_4$ is a graph with at least 6 vertices such that $V(G)$ can be partitioned into two sets $X$ and $V - X$ such that $|X| = p - 4$ and each edge of $<X>$ is a dominating edge and $<V - X>$ is independent.

(vi) $G_5$ is a graph obtained from $K_4, 4$ by joining any four independent vertices by at least five edges.

**Theorem 3.5:** For any connected graph $G$, $\gamma_{ctnd}(G) + \kappa(G) = 2p - 4$ if and only if $G$ is one of the following graphs. $K_{1,3}, P_4, K_2, 3, K_3, 3, G_1, G_2$ and $G_3$.

**Proof:** Let $\gamma_{ctnd}(G) + \kappa(G) = 2p - 4$. Then there are four cases to consider

(i) $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 4$

(ii) $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 3$

(iii) $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 2$

(iv) $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 1$

**Case-1:**

$\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 4$ or $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 1$.

$\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$ on $p$ vertices. But $\kappa(K_p) = p - 1$. Therefore no connected graph exists with $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 4$.

$\kappa(G) = p - 1$ if and only if $G \cong K_p$ on $p$ vertices. But $\gamma_{ctnd}(K_p) = p$. Therefore, no graph exists with $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 1$.

Therefore there exist no connected graphs in this case.

**Case-2:**

$\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 3$.

Since $\kappa(G) = p - 3$, $\delta(G) \geq p - 3$. If $\delta(G) = p - 1$, then $G$ is a complete graph. If $\delta(G) = p - 2$, then $G$ is isomorphic to $K_p - Y$, where $Y$ is a matching in $K_p$ and $\gamma_{ctnd}(G) = p - 1$. But $\kappa(G) = p - 2$. Therefore no connected graph exists.

Hence $\delta(G) = p - 3$.

**Subcase-2.1:** $\delta(G) = 1$.

Since $\gamma_{ctnd}(G) = p - 1$, $G \cong K_{1,p-1}$ or $S_{m,n}$.

If $G \cong K_{1,p-1}$, then $\kappa(K_{1,p-1}) = 1$ and $\kappa(G) = p - 3$ implies $p = 4$ and hence $G \cong K_{1,3}$ and $G \cong S_{m,n}$. $\kappa(S_{m,n}) = 1$ and $\kappa(G) = p - 3$ implies $p = 4$ and hence $G \cong S_{2,2}$. But $S_{2,2} \cong P_4$.

**Subcase-2.2:** $\delta(G) \geq 2$.

Then $G$ is a connected graph in which each edge is a dominating edge.

Let $X = \{v_1, v_2, \ldots, v_{p_3}\}$ be a vertex cut of $G$ and let $V - X = \{x_1, x_2, x_3\}$. Then $<V - X> \cong K_{3,3}$, $K_1 \cup K_2$ or $K_3$. If $<V - X> \cong K_4 \cup K_2$, then the edge in $K_2$ is not a dominating edge.
Subcase-2.2.1: \(<V − X> \cong \overline{K}_3\)
Since each edge of \(G\) is a dominating edge, every vertex of \(V − X\) is adjacent to all the vertices in \(X\). If \(|X| = 2\), then \(G \cong K_{2,3}\) or \(G\) is a graph obtained from \(K_{2,3}\) by joining the vertices of degree 3 by an edge. Therefore \(G \cong K_{2,3}\) or \(G_1\).

If \(|X| = 3\), then \(G \cong K_{3,3}\) or \(G\) is a graph obtained from \(K_{3,3}\) by joining the vertices of degree 3 by edges.

If \(G\) is isomorphic to a graph obtained from \(K_{3,3}\) by joining any three independent vertices by exactly one edge, then \(\gamma_{ctnd}(G) = p - 2\).

Therefore \(G\) is a graph \(K_{3,3}\) or to the graph obtained from \(K_{3,3}\) by joining any three independent vertices by at least two edges and hence \(G \cong K_{3,3}\) or \(G_2\).

If \(|X| \geq 4\), then \(p - 3 \geq 4\) implies \(p \geq 7\). If \(X\) is independent, then \(\kappa(G) = 3 \neq p - 3\).

If \(X\) is not independent and if there exists at least one edge in \(<X>\) which is not a dominating edge, then either \(\gamma_{ctnd}(G) = p - 2\) or \(\kappa(G) = 3 \neq p - 3\).

Therefore each edge of \(<X>\) is a dominating edge and \(G \cong G_3\).

Subcase-2.2.2: \(<V − X> \cong K_3\)
Since each edge of \(G\) is a dominating edge, every vertex of \(V − X\) is adjacent to all the vertices in \(X\).

If \(|X| = 2\), then \(G \cong K_5, G_1\) if \(<X> \cong K_2\) or \(G \cong K_5 - e\) if \(<X> \cong 2K\). If \(G \cong K_5\), then \(\gamma_{ctnd}(G) = p\).

If \(G \cong K_5 - e\), \(\gamma_{ctnd}(G) = 3 \neq p - 3\). Therefore, no connected graph exists in this case.

If \(|X| \geq 3\), then \(p - 3 \geq 3\) implies \(p \geq 6\). Let \(X\) can be partitioned into two sets \(S\) and \(V - S\). Assume \(S = \{v_1, v_2, v_3\}\) be a set of three independent vertices in \(X\) and \(V - S = (X - S) \cup (V - X)\). As in the Subcase 2.2.1, each edge in \(V - S\) is a dominating edge and \(G \cong K_{2,3}, K_{3,3}, G_1, G_2\) or \(G_3\).

Case-3: \(\gamma_{ctnd}(G) = p - 2\) and \(\kappa(G) = p - 2\).
Since \(\kappa(G) = p - 2\), \(\delta(G) \geq p - 2\). If \(\delta(G) = p - 1\), then \(G\) is a complete graph. Hence \(\delta(G) = p - 2\). Then \(G\) is isomorphic to \(K_p - Y\), where \(Y\) is a matching in \(K_p (p \geq 3)\). But in this case \(\gamma_{ctnd}(G) = p - 1\). Therefore, no connected graph exists in this case.

Hence \(G \cong K_{1,3}, P_4, K_{2,3}, K_{3,3}, G_1, G_2\) or \(G_3\).

Conversely, if \(G \cong K_{1,3}, P_4, K_{2,3}, K_{3,3}, G_1, G_2\) or \(G_3\), then \(\gamma_{ctnd}(G) = p - 1\) and \(\kappa(G) = p - 3\) and hence \(\gamma_{ctnd}(G) + \kappa(G) = 2p - 4\).

Theorem 3.6: For any connected graph \(G\), \(\gamma_{ctnd}(G) + \kappa(G) = 2p - 5\) if and only if \(G\) is isomorphic to one of the following graphs
(a) \(K_{1,4}, S_{2,3}, K_{2,4}, K_{3,4}, K_{4,4}, G_4, G_5, G_6, K_{4,(1,0,0)}, C_5, W_6\).
(b) \(G\) is a graph obtained from \(K_{p-1}\) by joining a vertex \(v \notin V(K_{p-1})\) to exactly \(p - 3\) vertices of \(K_{p-1}\).
(c) \(G\) is a graph obtained from \(K_{p-1} - e\) by joining a vertex \(v \notin V(K_{p-1} - e)\) to exactly \(p - 3\) vertices of \(K_{p-1} - e\).

Proof: Let \(\gamma_{ctnd}(G) + \kappa(G) = 2p - 5\). Then there are four cases to consider
(i) \(\gamma_{ctnd}(G) = p\) and \(\kappa(G) = p - 5\)
(ii) \(\gamma_{ctnd}(G) = p - 1\) and \(\kappa(G) = p - 4\)
(iii) \(\gamma_{ctnd}(G) = p - 2\) and \(\kappa(G) = p - 3\)
(iv) \(\gamma_{ctnd}(G) = p - 3\) and \(\kappa(G) = p - 2\)
(v) \(\gamma_{ctnd}(G) = p - 4\) and \(\kappa(G) = p - 1\)
Case-1:
\[ \gamma_{ctnd}(G) = p \text{ and } \kappa(G) = p - 5 \text{ or } \gamma_{ctnd}(G) = p - 4 \text{ and } \kappa(G) = p - 1. \]

\[ \gamma_{ctnd}(G) = p \text{ if and only if } G \cong K_p \text{ on } p \text{ vertices. But } \kappa(K_p) = p - 1. \text{ Therefore, no connected graph exists with } \gamma_{ctnd}(G) = p \text{ and } \kappa(G) = p - 5. \]

\[ \kappa(G) = p - 1 \text{ if and only if } G \cong K_p \text{ on } p \text{ vertices. But } \gamma_{ctnd}(K_p) = p. \text{ Therefore, no graph exists with } \gamma_{ctnd}(G) = p - 4 \text{ and } \kappa(G) = p - 1. \]

Therefore there exist no connected graphs in this case.

Case-2: \[ \gamma_{ctnd}(G) = p - 1 \text{ and } \kappa(G) = p - 4. \]

Since \[ \kappa(G) = p - 4, \delta(G) \geq p - 4. \] If \[ \delta(G) = p - 1, \text{ then } G \text{ is a complete graph. If } \delta(G) = p - 2, \text{ then } G \text{ is isomorphic to } K_{p - 1, y}, \text{ where } Y \text{ is a matching in } K_p. \text{ But } \kappa(G) = p - 2. \text{ Therefore no connected graph exists.} \]

Suppose \[ A(G) = p - 3. \text{ Let } X = \{v_1, v_2, \ldots, v_{p - 4}\} \text{ be a vertex cut of } G \text{ and let } V - X = \{x_1, x_2, x_3, x_4\}. \]

If \[ <V - X> \text{ contains an isolated vertex, then } \delta(G) \leq p - 4. \]

If \[ <V - X> \cong 2K_2, \text{ then } \gamma_{ctnd}(G) \leq p - 2. \text{ Therefore no connected graph exists.} \]

Hence \[ A(G) = p - 4. \]

Subcase-2.1: \[ A(G) = 1. \]

Since \[ \gamma_{ctnd}(G) = p - 1, \text{ G } \cong K_{1,p-1} \text{ or } S_{m,n}. \]

If \[ G \cong K_{1,p-1} \text{ or } S_{m,n}, \text{ then } \kappa(K_{1,p-1}) = \kappa(S_{m,n}) = 1 \text{ and } \kappa(G) = p - 4 \text{ implies } p = 5 \text{ and hence } G \cong K_{1,4} \text{ or } S_{2,3}. \]

Subcase-2.2: \[ \delta(G) \geq 2. \]

Then \[ G \text{ is a graph in which each edge is a dominating edge. Let } X = \{v_1, v_2, \ldots, v_{p - 4}\} \text{ be a vertex cut of } G \text{ and let } V - X = \{x_1, x_2, x_3, x_4\}. \text{ Then } <V - X> \cong K_4, K_2 \cup K_2, P_3 \cup K_1, 2K_2, K_3 \cup K_3, P_4, K_{1,3}, K_{3,1,0,0}, C_4, K_4 - e, K_4. \]

If \[ <V - X> \cong K_2 \cup K_2, P_3 \cup K_1, 2K_2, K_3 \cup K_3, P_4, K_{1,3}, K_{3,1,0,0}, \text{ then } \gamma_{ctnd}(G) \leq p - 2, \text{ since in these graphs each edge is not a dominating edge. Therefore no connected graph exists in this case.} \]

Subcase-2.2.1: \[ <V - X> \cong K_4. \]

Since each edge of \[ G \text{ is a dominating edge, } \text{ every vertex of } V - X \text{ is adjacent to all the vertices in } X. \text{ If } \mid X \mid = 2, \text{ then } G \cong K_{2,4} \text{ or } G \text{ is a graph obtained from } K_{2,4} \text{ by joining the vertices of degree } 4 \text{ by an edge. Therefore } G \cong K_{2,4} \text{ or } G_4. \]

If \[ \mid X \mid = 3, \text{ then } G \cong K_{3,4} \text{ or } G \text{ is a graph obtained from } K_{3,4} \text{ by joining the vertices of degree } 4 \text{ by edges.} \]

If \[ G \text{ is isomorphic to the graph obtained from } K_{3,4} \text{ by joining any three independent vertices by exactly one edge, then } \gamma_{ctnd}(G) = p - 2. \]

Therefore \[ G \text{ is isomorphic to the } K_{3,4} \text{ or a graph obtained from } K_{3,4} \text{ by joining any three independent vertices by at least two edges. Therefore } G \cong K_{3,4} \text{ or } G_4. \]

If \[ \mid X \mid = 4, \text{ then } G \cong K_{4,4} \text{ or } G \text{ is a graph obtained from } K_{4,4} \text{ by joining the independent vertices by edges.} \]

If \[ G \text{ is isomorphic to a graph obtained from } K_{4,4} \text{ by joining any four independent vertices by at most four edges then } \gamma_{ctnd}(G) \leq p - 2. \]

Therefore \[ G \text{ is isomorphic to } K_{4,4} \text{ or the graph obtained from } K_{4,4} \text{ by joining any four independent vertices by at least five edges. Therefore } G \cong K_{4,4} \text{ or } G_4. \]
If \(|X| \geq 5\), then \(p - 4 \geq 5 \implies p \geq 9\).

If \(X\) is independent then \(\kappa(G) = 4 \neq p - 4\). If \(X\) is not independent and if there exists at least one edge in \(<X>\) which is not a dominating edge, then \(\gamma_{ctnd}(G) = p - 2\).

Therefore each edge of \(<X>\) is a dominating edge. Therefore \(G \cong G_4\).

**Subcase-2.2.2:** \(<V - X> \cong C_4, K_{1,3}, K_4 - e>\).

Since each edge of \(G\) is a dominating edge, every vertex of \(V - X\) is adjacent to all the vertices in \(X\).

If each edge of \(<X>\) is a dominating edge, then \(\kappa(G) \neq p - 4\).

If \(X\) is independent and \(|X| \neq 4\), then \(\kappa(G) = 4 \neq p - 4\).

If \(X\) is independent and \(|X| = 4\), then \(G \cong G_5\).

If \(X\) is not independent and if there exists at least one edge in \(<X>\) which is not a dominating edge, then \(\gamma_{ctnd}(G) = p - 2\).

If \(|X| \geq 4\), then \(p - 4 \geq 4 \implies p \geq 8\). Let \(G\) be a graph such that \(X\) can be partitioned into two sets \(S\) and \(V - S\). Let \(S = \{v_1, v_2, v_3, v_4\}\) be a set of four independent vertices in \(X\) and \(V - S = (X - S) \cup (V - X)\). As in the subcase 2.2.1, if each edge in \(V - S\) is a dominating edge, then \(G \cong G_4\).

**Subcase-2.2.3:** \(<V - X> \cong K_4\).

Since each edge of \(G\) is a dominating edge, every vertex of \(V - X\) is adjacent to all the vertices in \(X\). If \(<X>\) is complete, then \(G \cong K_4\). But \(\gamma_{ctnd}(K_4) = p\).

As in the subcase 2.2.2, if \(X\) is independent and \(|X| = 4\), then \(G \cong G_5\).

As in the subcase 2.2.1, if each edge in \(V - S\) is a dominating edge, then \(G \cong G_4\).

Hence \(G\) is isomorphic to the graph \(K_{1,4}, S_{2,3}, K_{2,4}, K_{3,4}, K_{4,4}, G_4, G_5\).

**Case-3:** \(\gamma_{ctnd}(G) = p - 2\) and \(\kappa(G) = p - 3\).

Since \(\kappa(G) = p - 3\), \(\delta(G) \geq p - 3\). If \(\delta(G) = p - 1\), then \(G\) is a complete graph. If \(\delta(G) = p - 2\), then \(G\) is isomorphic to \(K_p - Y\), where \(Y\) is a matching in \(K_p\) and \(\kappa(G) = p - 2\). But \(\gamma_{ctnd}(G) = p - 1\). Therefore no connected graph exists.

By Theorem 2.7, Notation 2.8, Theorem 2.9, and Theorem 2.10, \(\gamma_{ctnd}(G) = p - 2\) if and only if

1. \(G \cong T\), where \(T\) is a tree either obtained from a path \(P_n\) (\(n \geq 4\) and \(n < p\)) by attaching pendant edges at least one of the end vertices of \(P_n\).

   Or obtained from \(P_3\) by attaching pendant edges at either both the end vertices or all the vertices of \(P_3\).

2. \(G \not\in \mathcal{G}\), if \(\delta(G) = 1\)

3. If \(\delta(G) \geq 2\), then \(G\) is one of the following graphs.

   (i) A cycle on at least five vertices.
   (ii) A wheel with six vertices.
   (iii) \(G\) is the one point union of complete graphs.
   (iv) \(G\) is obtained by joining two complete graphs by an edge.
   (v) \(G\) is a graph such that there exists a vertex \(v \in V(G)\) such that \(G - v\) is a complete graph on \((p - 1)\) vertices.
   (vi) \(G\) is a graph such that there exists a vertex \(v \in V(G)\) such that \(G - v\) is \(K_{p - 1} - e\), \((e \in E(K_{p - 1}))\) and \(N(v)\) contains at least one vertex of degree \((p - 3)\) in \(K_{p - 1} - e\).
Case-3.1: $G \cong T$, 
$\kappa(T) = 1$ and $\kappa(G) = p - 3$ implies $p = 4$. But this case is not possible, since $p \geq 5$.

Therefore no connected graph exists in this case.

Case-3.2: $G \not\cong T$ and $\delta(G) = 1$ 
$\kappa(G) = p - 3$ implies $p = 4$. Therefore $G \cong K_1(1,0,0)$.

Case-3.3: $\delta(G) \geq 2$.

Subcase-3.3.1: A cycle on at least five vertices.
$\kappa(C_p) = 2$ and $\kappa(G) = p - 3$ implies $p = 5$. Hence $G \cong C_5$.

Subcase-3.3.2: A wheel with six vertices.
In this case, $\kappa(G) = p - 3$. Hence $G \cong W_6$.

Subcase-3.3.3: $G$ is the one point union of complete graphs.
In this case, $\kappa(G) = 1 = p - 3$ implies $p = 4$. Therefore no connected graph exists in this case, since $p \geq 5$.

Subcase-3.3.4: $G$ is obtained by joining two complete graphs by an edge.
In this case, $\kappa(G) = 2 = p - 3$ implies $p = 5$. Therefore no connected graph exists in this case, since $p \geq 6$.

Subcase-3.3.5: $G$ is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is a complete graph on $(p - 1)$ vertices.
In this case, $\kappa(G) = \deg(v) = p - 3$. Therefore $G$ is a graph obtained from $K_{p-1}$ by joining a vertex $v \not\in V(K_{p-1})$ to exactly $(p - 3)$ vertices of $K_{p-1}$.

Subcase-3.3.6: $G$ is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and $N(v)$ contains at least one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

Since $\kappa(G) = p - 3$, $G$ is a graph obtained from $K_{p-1} - e$ by joining a vertex $v \not\in V(K_{p-1} - e)$ to exactly $(p - 3)$ vertices of $K_{p-1} - e$.

Case-4: $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 2$.

Since $\kappa(G) = p - 2$, $\delta(G) \geq p - 2$. If $\delta(G) = p - 1$, then $G$ is a complete graph. If $\delta(G) = p - 2$, then $G$ is isomorphic to $K_p - Y$, where $Y$ is a matching in $K_p$ and $\kappa(G) = p - 2$. But $\gamma_{ctnd}(G) = p - 1$.

Therefore no connected graph exists in this case.

Hence $G$ is isomorphic to one of the following graphs
(a) $K_{1,4}, S_{2,1}, K_{2,4}, K_{3,4}, K_{4,4}, G_4, G_5, K_5(1,0,0), C_5, W_6$.
(b) $G$ is a graph obtained from $K_{p-1}$ by joining a vertex $v \not\in V(K_{p-1})$ to exactly $(p - 3)$ vertices of $K_{p-1}$.
(c) $G$ is a graph obtained from $K_{p-1} - e$ by joining a vertex $v \not\in V(K_{p-1} - e)$ to exactly $(p - 3)$ vertices of $K_{p-1} - e$.

Conversely, if $G \cong K_{1,4}, S_{2,1}, K_{2,4}, K_{3,4}, K_{4,4}, G_4, G_5, C_5, W_6$, then $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 4$.

If $G \cong K_5(1,0,0), C_5, W_6$, $G$ is a graph obtained from $K_{p-1}$ by joining a vertex $v \not\in V(K_{p-1})$ to exactly $(p - 3)$ vertices of $K_{p-1}$, or $G$ is a graph obtained from $K_{p-1} - e$ by joining a vertex $v \not\in V(K_{p-1} - e)$ to exactly $(p - 3)$ vertices of $K_{p-1} - e$, then $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 3$.

Hence $\gamma_{ctnd}(G) + \kappa(G) = 2p - 5$. 

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