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COMPLEMENTARY TREE NIL DOMINATION NUMBER AND CONNECTIVITY OF GRAPHS

S. MUTHAMMAI, G. ANANTHAVALLI*

Government Arts College for Women (Autonomous), Pudukkottai-622001, India.

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ABSTRACT

A set D of a graph G = (V, E) is a dominating set, if every vertex in V(G) – D is adjacent to some vertex in D. The domination number γ(G) of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree nil dominating set, if the induced subgraph < V(G) – D > is a tree and also the set V(G) – D is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$. *The connectivity* κ *(G) of G is the minimum number of vertices whose removal results in a disconnected or trivial graph. In this paper, an upper bound for the sum of the complementary tree nil domination number and connectivity of a graph is found and the corresponding extremal graphs are characterized.*

Key words: *Domination number, Complementary tree nil domination number, Connectivity.*

1. INTRODUCTION

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A graph with p vertices and q edges is denoted by G(p, q). The concept of domination in graphs was introduced by Ore [5]. A set D $\subseteq$ V(G) is said to be a dominating set of G, if every vertex in V(G) $-$ D is adjacent to some vertex in D. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by γ(G). Muthammai, Bhanumathi and Vidhya [4] introduced the concept of complementary tree dominating set. A dominating set D $\subseteq$ V(G) is said to be a complementary tree dominating set (ctd-set) if the induced subgraph < V(G)- D > is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. The connectivity κ (G) of G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Any undefined terms in this paper may be found in Harary[1].

The concept of complementary tree nil dominating set is introduced in [3]. A dominating set D $\subseteq$ V(G) is said to be a complementary tree nil dominating set (ctnd-set) if the induced subgraph < V(G) $-$ D > is a tree and the set V(G) - D is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$.

In this paper, we find an upper bound for the sum of the complementary tree nil domination number and connectivity of a graph is found and the corresponding external graphs are characterized.

2. PRIOR RESULTS

Theorem 2.1: [1] For any connected graph G, $\kappa(G) \leq \delta$ (G).

Theorem 2.2: [3] For any connected graph G with p vertices, $2 \leq \gamma_{ctnd}(G) \leq$ p, where $p \geq 2$.

*Corresponding Author: G. Ananthavalli**
Government Arts College for Women (Autonomous), Pudukkottai-622001, India.

Theorem 2.3: [3] Let G be a connected graph with p vertices. Then $\gamma_{ctnd}(G)$ = 2 if and only if G is a graph obtained by attaching a pendant edge at a vertex of degree p - 2 in T + K_1, where T is a tree on (p – 2) vertices.

Theorem 2.4: [3] For any connected graph G, $\gamma_{ctnd}(G)$ = p if and only if G $\cong K_p$, where $p \geq 2$.

Theorem 2.5: [3] Let G be a connected graph with $p \geq 3$ and $\delta(G) = 1$. Then $\gamma_{ctnd}(G)$ = p -1 if and only if the subgraph of G induced by vertices of degree atleast 2 is K_2 or K_1.

That is, G is one of the graphs $K_{1, p-1}$ or $S_{m,n}$ (m + n = p, m, n $\geq$ 1), where $S_{m, n}$ is a bistar which is obtained by attaching m-1 pendant edges at one vertex of K_2 and n-1 pendant edges at other vertex of K_2.

Theorem 2.6: [3] Let G be a connected noncomplete graph with $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G)$ = p -1 if and only if each edge of G is a dominating edge.

Theorem 2.7: [3] Let T be a tree on p vertices such that $\gamma_{ctnd}(T) \leq p - 2$. Then $\gamma_{ctnd}(T)$ = p – 2 if and only if T is one of the following graphs.

(i) T is obtained from a path P_n (n $\geq$ 4 and n < p) by attaching pendant edges at atleast one of the end vertices of Pn.
(ii)T is obtained from P_3 by attaching pendant edges either at both the end vertices or at all the vertices of P_3.

Notation 2.8: [3] Let $\mathcal{G}$ be the class of connected graphs G with $\delta(G) = 1$ having one of the following properties.

(a) There exist two adjacent vertices u, v in G such that $\deg_G(u)$ = 1 and < V(G) - {u, v} > contains P_3 as an induced subgraph such that end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.
(b) Let P be the set of all pendant vertices in G and let there exist a vertex v $\in$ V(G) - P having minimum degree in V(G) - P and is not a support of G such that V(G) – ($N_{V-P}[v]$ - P) contains P_3 as an induced subgraph such that the end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

Theorem 2.9: [3] Let G be a connected graph with $\delta(G) = 1$ and $\gamma_{ctnd}(G) \neq$ p -1. Then $\gamma_{ctnd}(G)$ = p - 2 if and only if G does not belong to the class $\mathcal{G}$ of graphs.

Theorem 2.10: [3] Let G be a connected, noncomplete graph with p vertices (p $\geq$ 4) and $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G)$ = p – 2 if and only if G is one of the following graphs.

(a) A cycle on atleast five vertices.
(b) A wheel with six vertices.
(c) G is the one point union of complete graphs.
(d) G is obtained by joining two complete graphs by edges.
(e) G is a graph such that there exists a vertex v $\in$ V(G) such that G – v is a complete graph on (p – 1) vertices.
(f) G is a graph such that there exists a vertex v $\in$ V(G) such that G – v is K_{p-1} – e, (e $\in E(K_{p-1})$) and N(v) contains atleast one vertex of degree (p – 3) in K_{p-1} – e.

3. MAIN RESULTS

Theorem 3.1: For any connected graph G, $\gamma_{ctnd}(G) + \kappa(G) \leq 2p - 1$, equality holds if and only if G $\cong K_p$.

Proof: $\gamma_{ctnd}(G) + \kappa(G) \leq p + \delta(G) \leq p + p - 1 = 2p - 1$.

Let $\gamma_{ctnd}(G) + \kappa(G) = 2p - 1$. Then $\gamma_{ctnd}(G)$ = p and $\kappa(G)$ = p – 1 and G is a complete graph on p vertices.

Hence G $\cong K_p$.

Conversely, if G $\cong K_p$, then $\gamma_{ctnd}(G) + \kappa(G) = 2p - 1$.

Theorem 3.2: For any noncomplete graph G, $\gamma_{ctnd}(G) + \kappa(G) \leq 2p - 3$

Proof: Since G is not complete, by Theorem 3.1 $\gamma_{ctnd}(G) + \kappa(G) \leq 2p - 2$, by Theorem 3.1.
Assume $\gamma_{ctnd}(G) + \kappa(G) = 2p - 2$. Then either $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 2$ or $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 1$.

Case-1:
$\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 2$.
$\gamma_{ctnd}(G) = p$ if and only if G $\cong$ K_p on p vertices. But $\kappa(K_p) = p - 1$. Therefore, no connected graph exists with $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 2$.

Case-2: $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 1$.
$\kappa(G) = p - 1$ if and only if G $\cong K_p$ on p vertices. But $\gamma_{ctnd}(K_p) = p$.

From Case 1 and Case 2, no connected graph G exists with $\gamma_{ctnd}(G) + \kappa(G) = 2p - 2$.

Hence $\gamma_{ctnd}(G) + \kappa(G) \leq 2p - 3$.

Theorem 3.3: For any connected graph G, $\gamma_{ctnd}(G) + \kappa(G) = 2p - 3$ if and only if G is isomorphic to the graph $K_p - Y$, where Y is a matching in K_p ($p \geq 3$).

Proof: Let $\gamma_{ctnd}(G) + \kappa(G) = 2p - 3$. Then there are three cases to consider

(i) $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 3$
(ii) $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 2$
(iii) $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 1$

Case-1:
$\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 3$ or $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 1$.
$\gamma_{ctnd}(G) = p$ if and only if G $\cong$ K_p on p vertices. But $\kappa(K_p) = p - 1$. Therefore, no connected graph exists with $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 3$.
$\kappa(G) = p - 1$ if and only if G $\cong$ K_p on p vertices. But $\gamma_{ctnd}(K_p) = p$. Therefore, no connected graph exists with $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 1$.

Therefore there exists no connected graph in this case.

Case-2: $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 2$.

Since $\kappa(G) = p - 2$, $\delta(G) \geq p - 2$. If $\delta(G) = p - 1$, then G is a complete graph. Hence $\delta(G) = p - 2$. Then G is isomorphic to $K_p - Y$, where Y is a matching in K_p. But if $\delta(G) = 1$, then $\gamma_{ctnd}(G) = p - 1$ if and only if G is isomorphic to $K_{1, p-1}$ or $S_{m, n}$ and if $\delta(G) \geq 2$, then $\gamma_{ctnd}(G) = p - 1$ if and only if G is a graph in which each edge is a dominating edge.

Subcase-2.1: $\delta(G) = 1$
Since $\delta(G) = p - 2$, p = 3. G is isomorphic to $K_3 - Y$, where Y is a matching in K_3. That is, G $\cong P_3$. But $\gamma_{ctnd}(G) = p - 1$ if and only if G is isomorphic to $K_{1, p-1}$ or $S_{m, n}$. Since p = 3, G $\cong K_{1,2} \cong P_3$ and G $\ncong S_{m, n}$ ($m, n \geq 2$). Hence G $\cong P_3$.

Subcase-2.2: $\delta(G) \geq 2$
Then G is isomorphic to $K_p - Y$, where Y is a matching in K_p ($p \geq 4$). But $\gamma_{ctnd}(G) = p - 1$ if and only if G is a graph in which each edge is a dominating edge.

Hence G is isomorphic to $K_p - Y$ where Y is a matching in K_p ($p \geq 3$).

Conversely, if G is isomorphic to $K_p - Y$ where Y is a matching in K_p ($p \geq 3$), then $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 2$. Hence $\gamma_{ctnd}(G) + \kappa(G) = 2p - 3$.

Notation 3.4: Following notations are used in this paper.

(i) $K_3(1,0,0)$ is a graph obtained by attaching a pendant edge at one of the vertices of K_3.
(ii) G_1 is a graph obtained from $K_{2,3}$ by joining the vertices of degree 3 by an edge.
(iii) G_2 is a graph obtained from $K_{3,3}$ by joining any three independent vertices by atleast two edges.
(iv) G_3 is a graph with atleast 7 vertices such that V(G) can be partitioned into two sets X and V – X such that $|X| = p - 3$, each edge of $< X >$ is a dominating edge and $< V - X >$ is independent.
(v) G_4 is a graph with atleast 6 vertices such that V(G) can be partitioned into two sets X and V – X such that $|X| = p - 4$ and each edge of $< X >$ is a dominating edge and $< V - X >$ is independent.
(vi) G_5 is a graph obtained from $K_{4,4}$ by joining any four independent vertices by atleast five edges.

Theorem 3.5: For any connected graph G, $\gamma_{ctnd}(G) + \kappa(G) = 2p - 4$ if and only if G is one of the following graphs. $K_{1,3}$, P_4, $K_{2,3}$, $K_{3,3}$, G_1, G_2 and G_3.

Proof: Let $\gamma_{ctnd}(G) + \kappa(G) = 2p - 4$. Then there are four cases to consider

(i) $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 4$
(ii) $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 3$
(iii) $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 2$
(iv) $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 1$

Case-1:
$\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 4$ or $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 1$.
$\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$ on p vertices. But $\kappa(K_p) = p - 1$. Therefore no connected graph exists with $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 4$.
$\kappa(G) = p - 1$ if and only if $G \cong K_p$ on p vertices. But $\gamma_{ctnd}(K_p) = p$. Therefore, no graph exists with $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 1$.

Therefore there exist no connected graphs in this case.

Case-2: $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 3$.

Since $\kappa(G) = p - 3$, $\delta(G) \geq p - 3$. If $\delta(G) = p - 1$, then G is a complete graph. If $\delta(G) = p - 2$, then G is isomorphic to $K_p - Y$, where Y is a matching in K_p and $\gamma_{ctnd}(G) = p - 1$. But $\kappa(G) = p - 2$. Therefore no connected graph exists.

Hence $\delta(G) = p - 3$.

Subcase-2.1: $\delta(G) = 1$.
Since $\gamma_{ctnd}(G) = p - 1$, $G \cong K_{1,p-1}$ or $S_{m,n}$.

If $G \cong K_{1,p-1}$, then $\kappa(K_{1,p-1}) = 1$ and $\kappa(G) = p - 3$ implies p = 4 and hence $G \cong K_{1,3}$ and $G \cong S_{m,n}$, $\kappa(S_{m,n}) = 1$ and $\kappa(G) = p - 3$ implies p = 4 and hence $G \cong S_{2,2}$. But $S_{2,2} \cong P_4$.

Subcase-2.2: $\delta(G) \geq 2$.
Then G is a connected graph in which each edge is a dominating edge.
Let $X = \{v_1, v_2, \ldots, v_{p-3}\}$ be a vertex cut of G and let $V - X = \{x_1, x_2, x_3\}$. Then $< V - X > \cong \overline{K_3}$, $K_1 \cup K_2$ or K_3. If $< V - X > \cong K_1 \cup K_2$, then the edge in K_2 is not a dominating edge.

Subcase-2.2.1: $< V - X > \cong \overline{K}_3$
Since each edge of G is a dominating edge, every vertex of V – X is adjacent to all the vertices in X. If $|X| = 2$, then $G \cong K_{2,3}$ or G is a graph obtained from $K_{2,3}$ by joining the vertices of degree 3 by an edge. Therefore $G \cong K_{2,3}$ or G_1

If $|X| = 3$, then $G \cong K_{3,3}$ or G is a graph obtained from $K_{3,3}$ by joining the vertices of degree 3 by edges.

If G is isomorphic to a graph obtained from $K_{3,3}$ by joining any three independent vertices by exactly one edge, then $\gamma_{ctnd}(G) = p - 2$.

Therefore G is a graph $K_{3,3}$ or to the graph obtained from $K_{3,3}$ by joining any three independent vertices by atleast two edges and hence $G \cong K_{3,3}$ or G_2.

If $|X| \geq 4$, then $p - 3 \geq 4$ implies $p \geq 7$. If X is independent, then $\kappa(G) = 3 \neq p - 3$.

If X is not independent and if there exists at least one edge in < X > which is not a dominating edge, then either $\gamma_{ctnd}(G) = p - 2$ or $\kappa(G) = 3 \neq p - 3$.

Therefore each edge of < X > is a dominating edge and $G \cong G_3$.

Subcase-2.2.2: $< V - X > \cong K_3$
Since each edge of G is a dominating edge, every vertex of V – X is adjacent to all the vertices in X.

If $|X| = 2$, then $G \cong K_5$, if $< X > \cong K_2$, or $G \cong K_5 - e$, if $< X > \cong \overline{K}_2$. If $G \cong K_5$, then $\gamma_{ctnd}(G) = p$.

If $G \cong K_5 - e$, $\kappa(G) = 3 \neq p - 3$. Therefore, no connected graph exists in this case.

If $|X| \geq 3$, then $p - 3 \geq 3$ implies $p \geq 6$. Let X can be partitioned into two sets S and V – S. Assume $S = \{v_1, v_2, v_3\}$ be a set of three independent vertices in X and $V - S = (X - S) \cup (V - X)$. As in the Subcase 2.2.1, each edge in V – S is a dominating edge and $G \cong K_{2,3}$, $K_{3,3}$, G_1, G_2 or G_3.

Case-3: $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 2$.
Since $\kappa(G) = p - 2$, $\delta(G) \geq p - 2$. If $\delta(G) = p - 1$, then G is a complete graph. Hence $\delta(G) = p - 2$. Then G is isomorphic to $K_p - Y$, where Y is a matching in K_p $(p \geq 3)$. But in this case $\gamma_{ctnd}(G) = p - 1$. Therefore, no connected graph exists in this case.

Hence $G \cong K_{1,3}$, P_4, $K_{2,3}$, $K_{3,3}$, G_1, G_2 or G_3.

Conversely, if $G \cong K_{1,3}$, P_4, $K_{2,3}$, $K_{3,3}$, G_1, G_2 or G_3, then $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 3$ and hence $\gamma_{ctnd}(G) + \kappa(G) = 2p - 4$.

Theorem 3.6: For any connected graph G, $\gamma_{ctnd}(G) + \kappa(G) = 2p - 5$ if and only if G is isomorphic to one of the following graphs

(a) $K_{1,4}$, $S_{2,3}$, $K_{2,4}$, $K_{3,4}$, $K_{4,4}$, G_4, G_5, G_6, $K_3(1,0,0)$, C_5, W_6.
(b) G is a graph obtained from K_{p-1} by joining a vertex $v \notin V(K_{p-1})$ to exactly $(p - 3)$ vertices of K_{p-1}.
(c) G is a graph obtained from $K_{p-1} - e$ by joining a vertex $v \notin V(K_{p-1} - e)$ to exactly $(p - 3)$ vertices of $K_{p-1} - e$.

Proof: Let $\gamma_{ctnd}(G) + \kappa(G) = 2p - 5$. Then there are four cases to consider

(i) $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 5$
(ii) $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 4$
(iii) $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 3$
(iv) $\gamma_{ctnd}(G) = p - 3$ and $\kappa(G) = p - 2$
(v) $\gamma_{ctnd}(G) = p - 4$ and $\kappa(G) = p - 1$

Case-1:
$\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 5$ or $\gamma_{ctnd}(G) = p - 4$ and $\kappa(G) = p - 1$.

$\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$ on p vertices. But $\kappa(K_p) = p - 1$. Therefore, no connected graph exists with $\gamma_{ctnd}(G) = p$ and $\kappa(G) = p - 5$.

$\kappa(G) = p - 1$ if and only if $G \cong K_p$ on p vertices. But $\gamma_{ctnd}(K_p) = p$. Therefore, no graph exists with

$\gamma_{ctnd}(G) = p - 4$ and $\kappa(G) = p - 1$.

Therefore there exist no connected graphs in this case.

Case-2: $\gamma_{ctnd}(G) = p - 1$ and $\kappa(G) = p - 4$.
Since $\kappa(G) = p - 4$, $\delta(G) \geq p - 4$. If $\delta(G) = p - 1$, then G is a complete graph. If $\delta(G) = p - 2$, then G is isomorphic to $K_p - Y$, where Y is a matching in K_p. But $\kappa(G) = p - 2$. Therefore no connected graph exists.

Suppose $\delta(G) = p - 3$. Let $X = \{v_1, v_2, \ldots, v_{p-4}\}$ be a vertex cut of G and let $V - X = \{x_1, x_2, x_3, x_4\}$.

If $< V - X >$ contains an isolated vertex, then $\delta(G) \leq p - 4$.

If $< V - X > \cong 2K_2$, then $\gamma_{ctnd}(G) \leq p - 2$. Therefore no connected graph exists.

Hence $\delta(G) = p - 4$.

Subcase-2.1: $\delta(G) = 1$.
Since $\gamma_{ctnd}(G) = p - 1$, $G \cong K_{1, p-1}$ or $S_{m, n}$.

If $G \cong K_{1, p-1}$ or $S_{m, n}$, then $\kappa(K_{1, p-1}) = \kappa(S_{m, n}) = 1$ and $\kappa(G) = p - 4$ implies p = 5 and hence $G \cong K_{1, 4}$ or $S_{2, 3}$.

Subcase-2.2: $\delta(G) \geq 2$.
Then G is a graph in which each edge is a dominating edge. Let $X = \{v_1, v_2, \ldots, v_{p-4}\}$ be a vertex cut of G and let $V - X = \{x_1, x_2, x_3, x_4\}$. Then $< V - X > \cong \overline{K}_4$, $K_2 \cup \overline{K}_2$ $P_3 \cup K_1$, $2K_2$, $K_3 \cup K_1$, P_4, $K_{1,3}$, $K_3(1,0,0)$, C_4, $K_4 - e$, K_4.

If $< V - X > \cong K_2 \cup \overline{K}_2$, $P_3 \cup K_1$, $2K_2$, $K_3 \cup K_1$, P_4, $K_3(1,0,0)$, then $\gamma_{ctnd}(G) \leq p - 2$, since in these graphs each edge is not a dominating edge. Therefore no connected graph exists in this case.

Subcase-2.2.1: $< V - X > \cong \overline{K}_4$
Since each edge of G is a dominating edge, every vertex of V – X is adjacent to all the vertices in X. If $|X| = 2$, then $G \cong K_{2, 4}$ or G is a graph obtained from $K_{2, 4}$ by joining the vertices of degree 4 by an edge. Therefore $G \cong K_{2, 4}$ or G_4.

If $|X| = 3$, then $G \cong K_{3, 4}$ or G is a graph obtained from $K_{3, 4}$ by joining the vertices of degree 4 by edges.

If G is isomorphic to the graph obtained from $K_{3, 4}$ by joining any three independent vertices by exactly one edge, then $\gamma_{ctnd}(G) = p - 2$.

Therefore G is isomorphic to the graph $K_{3, 4}$ or a graph obtained from $K_{3, 4}$ by joining any three independent vertices by atleast two edges. Therefore $G \cong K_{3, 4}$ or G_4.

If $|X| = 4$, then $G \cong K_{4, 4}$ or G is a graph obtained from $K_{4, 4}$ by joining the independent vertices by edges.

If G is isomorphic to a graph obtained from $K_{4, 4}$ by joining any four independent vertices by atmost four edges then $\gamma_{ctnd}(G) \leq p - 2$.

Therefore G is isomorphic to $K_{4, 4}$ or the graph obtained from $K_{4, 4}$ by joining any four independent vertices by atleast five edges. Therefore $G \cong K_{4, 4}$ or G_4.

If $|X| \geq 5$, then $p - 4 \geq 5$ implies $p \geq 9$.

If X is independent then κ (G) = 4 $\neq$ p – 4. If X is not independent and if there exists atleast one edge in < X > which is not a dominating edge, then $\gamma_{ctnd}(G) = p - 2$.

Therefore each edge of < X > is a dominating edge. Therefore G $\cong G_4$.

Subcase-2.2.2: < V – X > $\cong$ $C_4, K_{1,3}, K_4 - e$.

Since each edge of G is a dominating edge, every vertex of V – X is adjacent to all the vertices in X.

If each edge of < X > is a dominating edge, then $\kappa(G) \neq p - 4$.

If X is independent and $|X| \neq 4$, then $\kappa(G) = 4 \neq p - 4$.

If X is independent and |X| = 4, then G $\cong G_5$.

If X is not independent and if there exists atleast one edge in < X > which is not a dominating edge, then $\gamma_{ctnd}(G) \leq p - 2$.

If $|X| \geq 4$, then $p - 4 \geq 4$ implies $p \geq 8$. Let G is a graph such that X can be partitioned into two sets S and V – S. Let S = $\{v_1, v_2, v_3, v_4\}$ be a set of four independent vertices in X and V – S = (X – S) $\cup$ (V – X). As in the subcase 2.2.1, if each edge in V – S is a dominating edge, then G $\cong G_4$.

Subcase-2.2.3: < V – X > $\cong$ K_4.

Since each edge of G is a dominating edge, every vertex of V – X is adjacent to all the vertices in X. If < X > is complete, then G $\cong K_p$. But $\gamma_{ctnd}(K_p) = p$.

As in the subcase 2.2.2, if X is independent and |X| = 4, then G $\cong G_5$.

As in the subcase 2.2.1, if each edge in V – S is a dominating edge, then G $\cong G_4$.

Hence G is isomorphic to the graph $K_{1,4}, S_{2,3}, K_{2,4}, K_{3,4}, K_{4,4}, G_4, G_5$.

Case-3: $\gamma_{ctnd}(G) = p - 2$ and $\kappa(G) = p - 3$.
Since $\kappa(G) = p - 3$, δ G) $\geq$ p – 3. If $\delta(G) = p - 1$, then G is a complete graph. If $\delta(G) = p - 2$, then G is isomorphic to $K_p - Y$, where Y is a matching in K_p and $\kappa(G) = p - 2$. But $\gamma_{ctnd}(G) = p - 1$. Therefore no connected graph exists.

By Theorem 2.7. Notation 2.8. Theorem 2.9. and Theorem 2.10. $\gamma_{ctnd}(G) = p - 2$ if and only if

1. G $\cong$ T, where T is a tree either obtained from a path P_n ($n \geq 4$ and $n < p$) by attaching pendant edges at atleast one of the end vertices of P_n.

   Or

   obtained from P_3 by attaching pendant edges at either both the end vertices or all the vertices of P_3.
2. $G \notin \mathcal{G}$, if $\delta(G) = 1$
3. If $\delta(G) \geq 2$, then G is one of the following graphs.
   (i) A cycle on atleast five vertices.
   (ii) A wheel with six vertices.
   (iii) G is the one point union of complete graphs.
   (iv) G is obtained by joining two complete graphs by an edge.
   (v) G is a graph such that there exists a vertex $v \in V(G)$ such that G – v is a complete graph on (p – 1) vertices.
   (vi) G is a graph such that there exists a vertex $v \in V(G)$ such that G – v is $K_{p-1} - e$, ($e \in E(K_{p-1})$) and N(v) contains atleast one vertex of degree (p – 3) in $K_{p-1} - e$.

Case-3.1: $G \cong T$,
κ (T) = 1 and κ (G) = p – 3 implies p = 4. But this case is not possible, since p ≥ 5.

Therefore no connected graph exists in this case.

Case-3.2: $G \notin \mathcal{G}$ and $\delta(G) = 1$
κ (G) = p – 3 implies p = 4. Therefore $G \cong K_3(1,0,0)$.

Case-3.3: $\delta(G) \geq 2$.

Subcase-3.3.1: A cycle on atleast five vertices.
κ (C_p) = 2 and κ (G) = p – 3 implies p = 5. Hence $G \cong C_5$.

Subcase-3.3.2: A wheel with six vertices.

In this case, κ (G) = p – 3. Hence $G \cong W_6$.

Subcase-3.3.3: G is the one point union of complete graphs.

In this case, κ (G) = 1 = p – 3 implies p = 4. Therefore no connected graph exists in this case, since p ≥ 5.

Subcase-3.3.4: G is obtained by joining two complete graphs by an edge.

In this case, κ (G) = 2 = p – 3 implies p = 5. Therefore no connected graph exists in this case, since p ≥ 6.

Subcase-3.3.5: G is a graph such that there exists a vertex $v \in V(G)$ such that G – v is a complete graph on (p – 1) vertices.

In this case, κ (G) = deg(v) = p – 3. Therefore G is a graph obtained from K_{p-1} by joining a vertex v $\notin V(K_{p-1})$ to exactly (p – 3) vertices of K_{p-1}.

Subcase-3.3.6: G is a graph such that there exists a vertex v $\in V(G)$ such that G – v is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and N(v) contains atleast one vertex of degree (p – 3) in $K_{p-1} - e$.

Since κ (G) = p – 3, G is a graph obtained from $K_{p-1} - e$ by joining a vertex v $\notin V(K_{p-1} - e)$ to exactly (p – 3) vertices of $K_{p-1} - e$.

Case-4: $\gamma_{ctnd}(G)$ = p – 3 and κ (G) = p – 2.

Since κ (G) = p – 2, δ (G) ≥ p – 2. If δ (G) = p – 1, then G is a complete graph. If δ (G) = p – 2, then G is isomorphic to $K_p - Y$, where Y is a matching in K_p and κ (G) = p – 2. But $\gamma_{ctnd}(G)$ = p – 1.

Therefore no connected graph exists in this case.

Hence G is isomorphic to one of the following graphs
(a) $K_{1,4}$, $S_{2,3}$, $K_{2,4}$, $K_{3,4}$, $K_{4,4}$, G_4, G_5, $K_3(1,0,0)$, C_5, W_6.
(b) G is a graph obtained from K_{p-1} by joining a vertex v $\notin V(K_{p-1})$ to exactly (p – 3) vertices of K_{p-1}.
(c) G is a graph obtained from K_{p-1} - e by joining a vertex v $\notin V(K_{p-1} - e)$ to exactly (p – 3) vertices of K_{p-1} - e.

Conversely, if $G \cong K_{1,4}$, $S_{2,3}$, $K_{2,4}$, $K_{3,4}$, $K_{4,4}$, G_4, G_5, then $\gamma_{ctnd}(G)$ = p – 1 and κ (G) = p – 4.

If $G \cong K_3(1,0,0)$, C_5, W_6, G is a graph obtained from K_{p-1} by joining a vertex $v \notin V(K_{p-1})$ to exactly (p – 3) vertices of K_{p-1}, or G is a graph obtained from K_{p-1} - e by joining a vertex v $\notin V(K_{p-1} - e)$ to exactly (p – 3) vertices of K_{p-1} –e, then $\gamma_{ctnd}(G)$ = p – 2 and κ (G) = p – 3.

Hence $\gamma_{ctnd}(G)$ + κ (G) = 2p – 5.

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