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HAMMING INDEX GENERATED BY THE INCIDENCE MATRIX OF SOME THORN GRAPHS

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ABSTRACT

Let B(G) be the incidence matrix of a graph G. The row in B(G) corresponding to a vertex v, denoted by s(v) is the string which belongs to Z_n^2, a set of n-tuples over a field of order two. The Hamming distance between the strings s(u) and s(v) is the number of positions in which s(u) and s(v) differ. The Hamming index is the sum of Hamming distances between all pair of vertices of G. In this paper we obtain the Hamming index of some thorn graph i.e. $G^*(p_k)$, generated by the incidence matrix.

Keywords: *Hamming distance, Hamming index, incidence matrix.*

2000 Mathematics Subject Classification: *05C99.*

1. INTRODUCTION

In modern world of communications, data items are constantly being transmitted from point to point. The basic problem in transmission of data is that of receiving the data as sent and not receiving a distorted piece of data. Coding theory has developed techniques for introducing redundant information in transmitted data that help in detecting, and sometimes in correcting errors. The basic unit of information, called message, is a finite sequence of characters. The every character or symbol that is to be transmitted is represented as a sequence of n elements from the set $Z_2 = \{0, 1\}$. The set Z_2 is a group under binary operation $\oplus$ with addition modulo 2.

Therefore for any positive integer n, $Z_n^2 = Z_2 \times Z_2 \times Z_2 \quad \times Z_2$ *(n factors)* is a group under the operation $\oplus$ defined by $(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Element of Z_n^2 is an n -tuple $(x_1, x_2, \ldots, x_n)$ written as $x = x_1x_2, \ldots, x_n$ where every x_i is either 0 or 1 and is called a string or word. The number of 1 *'s* in $x = x_1 x_2, \ldots, x_n$ is called the weight of x and is denoted by $wt(x)$.

Let $x = x_1 x_2, \ldots, x_n$ and $y = y_1 y_2 \ \ldots, y_n$ be the elements of Z_n^2. Then the sum x $\oplus$ y is computed by adding the corresponding components of x and y under addition modulo 2. That is, $x_i + y_i = 0$ if $x_i = y_i$ and $x_i + y_i = 1$ if $x_i \neq y_i$, $i = 1, 2, \ldots, n$. The Hamming distance $H_d \ (x, y)$ between the strings $x = x_1, x_2, \ldots x_n$ and $y = y_1 y_2 \ \ldots, y_n$ is the number of i *'s* such that $x_i \neq y_i$, $1 \le i \le n$ [9, 10].

Thus $H_d(x, y)$ = Number of positions in which x and y differ = $wt(x \oplus y)$.

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \ldots v_n\}$ be the vertex set of G and $E(G) = \{e_1, e_2, \ldots, e_m\}$ be the edge set of G. The degree of a vertex v, denoted by $deg_G(v)$ is the number of edges incident to it. A graph is said to be r -regular graph if all its vertices have same degree equal to r. The edges which are incident to the vertices u and v simultaneously are called the common incident edges of u and v and the edges which are not incident to u and v simultaneously are called non-common incident edges of u and v. The distance between two vertices u and v in G is the length of shortest path joining u and v and is denoted by $d_G(u, v)$.

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A graph G with vertex set $V(G)$ is called a Hamming graph if each vertex $v \in V(G)$ can be labeled by a string $s(v)$ of a fixed length such that $H_d\big(s(u), s(v)\big) = d_G(u,v)$ for all $u, v \in V(G)$, where $d_G(u,v)$ is the length of shortest path joining u and v in G [3]. Hamming graphs are known as an interesting graph family in connection with the error-correcting codes and association schemes [1,2,4,7]. Hamming distance between the strings generated by the adjacency matrix of a graph are recently studied by [9, 10].

2. PRELIMINARIES

The degree of a vertex v, denoted by $deg_G(v)$ is the number of edges incident to it. A graph is said to be r -regular graph if all its vertices have same degree equal to r.

The edges which are incident to the vertices u and v simultaneously are called the common incident edges of u and v and the edges which are not incident to u and v simultaneously are called non-common incident edges of u and v.

The incidence matrix of G is a matrix $B(G) = [b_{ij}]$ of order $n \times m$, in which $b_{ij} = 1$. if the vertex v_i is incident to the edge e_j and $b_{ij} = 0$, otherwise. Denote by $s(v)$, the row of the incidence matrix corresponding to the vertex v. It is a string in the set of Z_m^2 all m-tuples over the field of order two.

Hamming index is the sum of Hamming distances between all pairs of strings generated by the incidence matrix of a graph G is denoted by $H_B(G)$. Thus,

$$H_B(G) = \sum_{1 \le i < j \le n} H_d(s(v_i), s(v_j)).$$

Example 1:

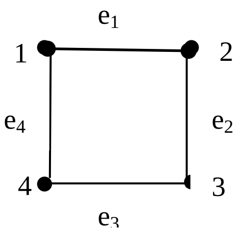

Fig. 1

Incidence matrix of a graph G of Fig. 1 is

$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

and the strings are $s(v_1) = 0101$, $s(v_2) = 1010$, $s(v_3) = 0101$, $s(v_4) = 1010$

$H_d\big(s(v_1), s(v_2)\big) = 2$, $\quad H_d\big(s(v_1), s(v_3)\big) = 2$, $\quad H_d\big(s(v_1), s(v_4)\big) = 2$
$H_d\big(s(v_2), s(v_3)\big) = 2$, $\quad H_d\big(s(v_2), s(v_4)\big) = 4$, $\quad H_d\big(s(v_3), s(v_4)\big) = 2$.

Therefore $H_B(G) = 2+2+2+2+2+4+2 = 14$

The *thorn graph* of the graph G with parameters $p_1, p_2 \ldots p_n$ is obtained by attaching p_i new vertices of degree one to the vertex u_i of the graph G, $i = 1, 2, \ldots n$.

The thorn graph of the graph G will be denoted by $G^*(P_k)$, or if the respective parameter need to be specified, by $G^*(p_1, p_{2,\ldots} p_n)$.

We need the following result:

Theorem [10]: Let G be a graph with n vertices and m edges.

(i) If u and v are adjacent vertices, then, $H_d(s(u), s(v)) = d(u) + d(v) - 2$.

(ii) If u and v are nonadjacent vertices, then, $H_d(s(u), s(v)) = d(u) + d(v)$.

3. RESULTS

Theorem 3.1: Let G be a r- regular on n vertices. Then Hamming Index generated by the incidence matrix of thorn graph $G^*(p_k)$ is given by

$$H_B[G^*(p_k)] = n(n-1)(k+r)-2m+nk[n(2k+r+1)-3].$$

Proof: Let G be a r -regular graph on n vertices, then the incidence matrix of $G^*(p_k)$ is given by

$$B\left(G^*(p_k)\right)=\begin{bmatrix} B(G) & I & I & \cdots & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}$$

Where, $B(G)$ is the incidence matrix of *G*, *I* is the identity matrix of order *n* and *0* is the null matrix. Now we consider the following cases for $G^*(p_k)$ graph.

Case-i: There are *m* - pairs of (v_i, v_j) vertices in $G^*(p_k)$ where, $1 \le i < j \le n$, which are adjacent. Therefore,

$$\begin{aligned} H_d\left(s(v_i), s(v_j)\right) &= \deg(v_i) + \deg(v_j) - 2 \\ &= \mathrm{k + r + k + r - 2} \\ &= 2(\mathrm{k + r - 1}) \end{aligned}$$

Therefore for *m*-pairs, $H_d\left(s(v_i), s(v_j)\right) = 2\mathrm{m}\,(\mathrm{k + r - 1})$.

Case-ii: There are $\binom{n}{2} - m$ pairs of (v_i, v_j) vertices in $G^*(p_k)$ where, $1 \le i < j \le n$, which are non- adjacent.

Therefore,

$$\begin{aligned} H_d\left(s(v_i), s(v_j)\right) &= \deg(v_i) + \deg(v_j) \\ &= \mathrm{k + r + k + r} \\ &= 2(\mathrm{k + r}) \end{aligned}$$

Therefore for $\binom{n}{2} - m$ pairs, $H_d\left(s(v_i), s(v_j)\right) = (n^2 - n - 2m)\,(\mathrm{k + r})$.

Case-iii: There are *nk* - pairs of $\left(v_i, v_i^j\right)$ vertices in $G^*(p_k)$ where, $1 \le i < j \le n$, which are adjacent. Therefore,

$$\begin{aligned} H_d\left(s(v_i), s(v_i^j)\right) &= \deg(v_i) + \deg(v_i^j) - 2 \\ &= \mathrm{k + r + 1 - 2} \\ &= \mathrm{k + r - 1} \end{aligned}$$

Therefore for *nk* -pairs, $H_d\left(s(v_i), s(v_i^j)\right) = \mathrm{nk}\,(\mathrm{k + r - 1})$.

Case-iv: There are $\binom{nk}{2}$ - pairs of $\left(v_i^j, v_i^j\right)$ vertices in $G^*(p_k)$ where, $1 \le i < j \le n$, which are non- adjacent. Therefore,

$$\begin{aligned} H_d\left(s(v_i^j), s(v_i^j)\right) &= \deg(v_i^j) + \deg(v_i^j) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Therefore for $\binom{nk}{2}$ -pairs, $H_d\left(s(v_i^j), s(v_i^j)\right) = 2\binom{nk}{2}$.

Case-v: There are $nk(n-1)$ - pairs of $\left(v_i \ , v_i^j\right)$ vertices in $G^*(p_k)$ where, $1 \le i < j \le n$, which are non- adjacent.

Therefore,

$$H_d\left(s(v_i), s(v_i^j)\right) = \deg\left(v_i\right) + \deg\left(v_i^j\right)$$
$$= \mathrm{k} + \mathrm{r} + 1$$

Therefore for $nk(n-1)$ -pairs, $H_d\left(s(v_i), s(v_i^j)\right) = \mathrm{nk(n-1)(k+r+1)}$.

Hence summing up all the five cases, we obtain

$$H_B[G^*(p_k)] = n(n-1)(k+r) - 2m + nk[n(2k+r+1) - 3].$$

Corollary 3.2: Let C_n be a cycle on *n* vertices, then Hamming index of $C_n^*(p_k)$ is given by

$$H_B[C_n^*(p_k)] = 2(k+2)\binom{n}{2} + 2\binom{nk}{2} - 2(m+nk) + n^2k(k+3).$$

Corollary 3.3: Let K_n be a complete graph on n vertices, then Hamming index of $K_n^*(p_k)$ is given by

$$H_B[K_n^*(p_k)] = 2(k+n-1)\binom{n}{2} + 2\binom{nk}{2} - 2(m+nk) + n^2k(k+n).$$

Theorem 3.4: Let P_n be a path on n vertices, then Hamming index of $P_n^*(p_k)$ is given by

$$H_B[P_n^*(p_k)] = 2k^2n^2 + 4kn^2 - 6kn + 2n^2 - 5n + 1.$$

Proof: Let P_n be a path on *n*- vertices. The graph $P_n^*(p_k)$ is obtained by inserting new vertices v_j^i where, i = 1, 2, . . . , n and j = 1, 2, . . . , k and joining v_j^i to v_i by an edge.

The incidence matrix of $P_n^*(p_k)$ is given by

$$B\left(P_n^*(p_k)\right) = \begin{bmatrix} B(P_n) & I & I & \cdots & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}$$

where $B(P_n)$ is the incidence matrix of P_n, *I* is the identity matrix of order *n* and 0 is the null matrix. To prove the result we consider the following cases:

Case-i: The Hamming distance for a pair of adjacent vertices (v_i, v_j) is given by

$$H_d(v_i, v_j) = d(v_i) + d(v_j) - 2$$

There are two pairs of adjacent vertices which are having exactly one vertex as end vertex, whose distance is given by [2k+1] and remaining [n-3] pairs having distance (2k+2).

Therefore total Hamming distance is $2[2k+1] + (2k+2)(n-3)$.

Case-ii: The Hamming distance for a pair of non adjacent vertices (v_i, v_j) is $H_d(v_i, v_j) = d(v_i) + d(v_j)$

There are *(n - 3)* pairs of vertices in which one vertex is end vertex having distance *(2k+3)* and two vertices are end vertices which are non adjacent having distance *(2k+2)* remaining $\binom{n}{2} - 2n + 3$ vertices are at a distance *(2k+4).*

Therefore total Hamming distance is $(2k+3)(n-3) + 2(k+1) + 2(k+2)\binom{n}{2} - 2n + 3.$

Case-iii: The Hamming distance for a pair of adjacent vertices (v_i, v_i^j) is $H_d(v_i, v_i^j) = d(v_i) + d(v_i^j) - 2.$

There are *2k* pairs of vertices which consists of exactly one vertex as end vertices which are at a distance *k* and remaining $(n-2)k$ pairs are at distance *(k+1)* .

Therefore total Hamming distance is $2k^2 + (k+1)(n-2)k.$.

Case-iv: The Hamming distance for pair of non adjacent vertices $(v_i^j\ v_j^j)$ is $d(v_i^j) + d(v_j^j)$. There are $\binom{nk}{2}$ pairs which are at a distance 2. Therefore total Hamming distance is $2 . \binom{nk}{2}$

Case-v: For a pair of non adjacent $(v_i\ \ v_i^j)$ vertices the hamming distance is $d(v_i) + d(v_i^j)$.

There are $2(n-1)k$ pairs of vertices including the pendent vertex which are at a distance *(k + 2)* and remaining distance $\left[\binom{n}{2} - (n-1)\right]2k$ pairs are at a distance (k + 3) .

Therefore total Hamming distance is $2k(k+2)(n-1) + (k+3)\left[\binom{n}{2} - (n-1)\right]2k$

Combining all the above five cases, we obtain

$$H_B[P_n^*(p_k)] = 2k^2n^2 + 4kn^2 - 6kn + 2n^2 - 5n + 1.$$

Let *G + ke* be a graph obtained from *G* by adding *k* independent edges to *G*. Thus if *G* has *n* vertices and *m* edges then G + *ke* has *n* vertices and *m + k* edges.

Theorem 3.5: Let *G* be a connected graph with n vertices. Then $\mathrm{H_B(G + ke) = H_B(G) + k(2n - 4)}$.

Proof: Let *G* be a connected graph with *n* vertices, let *u* and *v* be any two non adjacent vertices of G. Hamming index of the graph *G + ke* can be generated as $\mathrm{H_B(G + ke) = H_B(G)} +$ Hamming distance between *u* and *k(n - 2)* vertices other than *v* + Hamming distance between *v* and *k(n - 2)* vertices other than *u*. Where, *u* and *v* are the corresponding vertices of the k^{th} pair. Therefore,

$$\begin{aligned} \mathrm{H_B(G + ke)} &= \mathrm{H_B(G)} + \sum_{x=1}^{k(n-2)} \mathrm{H_d}[s(u), s(x)] + \sum_{x=1}^{k(n-2)} \mathrm{H_d}[s(v), s(x)] \\ &= H_B(G) + k(n-2) + k(n-2) \\ &= H_B(G) + k(2n-4). \end{aligned}$$

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