INVERSE AND DISJOINT SECURE DOMINATING SETS IN GRAPHS

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ABSTRACT

Let \( D \) be a minimum secure dominating set of a graph \( G = (V, E) \). If \( V - D \) contains a secure dominating set \( D' \) of \( G \), then \( D' \) is called an inverse secure dominating set with respect to \( D \). The inverse secure domination number \( \gamma_s^{-1}(G) \) of \( G \) is the minimum cardinality of an inverse secure dominating set of \( G \). The disjoint secure domination number \( \gamma_s \cap \gamma_s(G) \) of a graph \( G \) is the minimum cardinality of the union of two disjoint secure dominating sets in \( G \). In this paper, we establish some results for the inverse secure domination number. Also we initiate a study of the disjoint secure domination number and obtain some results on this new parameter.

Keywords: Inverse domination number, inverse secure domination number, disjoint secure domination number.

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1. INTRODUCTION

By a graph, we mean a finite, undirected, without loops, multiple edges and isolated vertices. Let \( G = (V, E) \) be a graph with \( p \) vertices and \( q \) edges. For the general concepts, the reader may refer to [1]. A set \( D \) of vertices in a graph \( G \) is called a dominating set if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set of \( G \). Recently several domination parameters are given in the books by Kulli in [2,3,4]. Let \( D \) be a minimum dominating set of \( G \). If \( V - D \) contains a dominating set \( D' \) of \( G \), then \( D' \) is called an inverse dominating set of \( G \) with respect to \( D \). The inverse domination number \( \gamma^{-1}(G) \) of \( G \) is the minimum cardinality of an inverse dominating set of \( G \). The inverse domination in graphs was introduced by Kulli and Sigarkanti in [5]. Many other inverse domination parameters in domination theory were studied, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

A dominating set \( D \) in \( G \) is called a secure dominating set in \( G \) if for every vertex \( u \) in \( V - D \), there exists \( v \) in \( D \) adjacent to \( u \) such that \( (D - \{v\}) \cup \{u\} \) is a dominating set. The secure domination number \( \gamma_s(G) \) of \( G \) is the minimum cardinality of a secure dominating set of \( G \). This was introduced by Cockayne et al. in [20].

Let \( D \) be a minimum secure dominating set of \( G \). If \( V - D \) contains a secure dominating set \( D' \) of \( G \), then \( D' \) is called an inverse secure dominating set with respect to \( D \). The inverse secure domination number \( \gamma_s^{-1}(G) \) of \( G \) is the minimum cardinality of an inverse secure dominating set of \( G \). The inverse secure domination in graphs was found in the paper of Enriquez et al. in [21]. A \( \gamma_s^{-1} \)-set is a minimum inverse secure dominating set. Similarly other sets can be expected.

A dominating set \( D \) of \( G \) is a split dominating set if the induced subgraph \( \langle D \rangle \) is disconnected. The split domination number \( \gamma_{sd}(G) \) of \( G \) is the minimum cardinality of a split dominating set of \( G \). This concept was introduced by Kulli and Janakiram in [22].

Let \( \Delta(G) \) denote the maximum degree and \( \lceil x \rceil \) the least integer greater than or equal to \( x \). The complement of \( G \) is denoted by \( \overline{G} \).

2. INVERSE SECURE DOMINATION

Proposition A [21]: Let \( G \) be a connected graph with \( p \geq 4 \) vertices. Then

\[ \gamma_s(G) \leq \gamma_s^{-1}(G). \]
Remark 2: Not all graphs have an inverse secure dominating set. For example, the path $P_5$ has a secure dominating set, but no inverse secure dominating set.

By Remark 2: Proposition A is not true for $p = 5$, Thus we have

**Theorem 3:** For any graph $G$ with a $\gamma_{s^{-1}}$-set,

$$\gamma_s(G) \leq \gamma_{s^{-1}}(G)$$

and this bound is sharp.

**Proof:** Every inverse secure dominating set is a secure dominating set. Thus (1) holds.

The path $P_4$ achieves this bound.

**Theorem 4:** If a $\gamma_{s^{-1}}$-set exists in a graph $G$ with $p$ vertices, then

$$\gamma(G) + \gamma_{s^{-1}}(G) \leq p$$

and this bound is sharp.

**Proof:** This follows from the definition of $\gamma_{s^{-1}}(G)$.

The path $P_4$ and the cycle $C_4$ achieve this bound.

**Theorem B [22]:** For any graph $G$ with an endvertex,

$$\gamma(G) = \gamma_{sd}(G).$$

We obtain a relation between $\gamma_{sd}(G)$ and $\gamma_{s^{-1}}(G)$.

**Theorem 6:** Let $G$ be a graph with an endvertex. If a $\gamma_{s^{-1}}$-set exists in $G$ with $p$ vertices, then

$$\gamma_{sd}(G) + \gamma_{s^{-1}}(G) \leq p$$

and this bound is sharp.

**Proof:** From Theorem 5, we have $\gamma(G) + \gamma_{s^{-1}}(G) \leq p$. From Theorem B, we have $\gamma(G) = \gamma_{sd}(G)$. Thus (4) holds.

The path $P_4$ achieves this bound.

**Theorem 7:** For any graph $G$ without isolated vertices and with an endvertex,

$$\gamma_{sd}(G) \leq \gamma_s(G)$$

and this bound is sharp.

**Proof:** From Theorem B, $\gamma(G) = \gamma_{sd}(G)$ and by definition $\gamma(G) \leq \gamma_s(G)$. Hence (5) holds.

The path $P_4$ achieves this bound.

**Corollary 8:** Let $G$ be a graph with an endvertex. If a $\gamma_{s^{-1}}$-set exists in $G$, then

$$\gamma_{sd}(G) \leq \gamma_{s^{-1}}(G)$$

We obtain lower and upper bounds on $\gamma_{s^{-1}}(G)$.

**Theorem 9:** For any graph $G$ with $p$ vertices and with a $\gamma_{s^{-1}}$-set,

$$\left[ \frac{p}{\Delta(G)+1} \right] \leq \gamma_{s^{-1}}(G) \leq \left[ \frac{p\Delta(G)}{\Delta(G)+1} \right].$$

**Proof:** It is known that $\left[ \frac{p}{\Delta(G)+1} \right] \leq \gamma(G)$ and since $\gamma(G) \leq \gamma_{s^{-1}}(G)$, we see that the lower bound in (7) holds.
By Theorem 4, we have
\[ \gamma^{-1}_s(G) \leq p - \gamma_s(G). \]

Since \[ \left\lceil \frac{p}{\Delta(G) + 1} \right\rceil \leq \gamma(G) \leq \gamma_s(G) \] and the above inequality,
\[ \gamma^{-1}_s(G) \leq p - \left\lceil \frac{p}{\Delta(G) + 1} \right\rceil. \]

Thus the upper bound in (7) holds.

**Theorem C [21]:** Let \( G \) be a connected graph with \( p \geq 2 \) vertices. Then \( \gamma(G) = 1 \) if and only if \( G = K_p \).

We obtain the following bounds for \( \gamma^{-1}_s(G) \).

**Theorem 10:** Let \( G \) be a connected graph with \( p \geq 4 \) vertices. If \( G \) has a \( \gamma^{-1}_s \)-set and \( G \neq K_p \), then
\[ 2 \leq \gamma^{-1}_s(G) \leq p - 2 \] (8)

and these bounds are sharp.

**Proof:** Suppose \( G \) is connected and \( G \neq K_p \). By Theorem C, \( \gamma(G) \geq 2 \). Since \( 2 \leq \gamma(G) \) and by Theorem 3, \( \gamma(G) \leq \gamma^{-1}_s(G) \), we see that the lower bound of (8) follows.

By Theorem 4, we have \( \gamma^{-1}_s(G) \leq p - \gamma_s(G) \) and since \( 2 \leq \gamma_s(G) \)
\[ \gamma^{-1}_s(G) \leq p - 2. \]

Thus
\[ 2 \leq \gamma^{-1}_s(G) < p - 2. \]

The path \( P_4 \) achieves both lower and upper bounds.

Now we obtain a Nordhaus - Gaddum type result for secure domination number.

**Theorem 11:** Let \( G \) be a graph with \( p \geq 4 \) vertices and \( G \neq K_p \). If a \( \gamma^{-1}_s \)-set exists and \( G \) and \( \overline{G} \) have no isolated vertices, then
\[ 4 \leq \gamma^{-1}_s(G) + \gamma^{-1}_s(\overline{G}) \leq 2(p - 2) \]
\[ 4 \leq \gamma^{-1}_s(G)\gamma^{-1}_s(\overline{G}) \leq (p - 2)^2 \]

and these bounds are sharp.

**Proof:** Since \( G \) and \( \overline{G} \) have no isolated vertices and \( G \neq K_p \),
\[ 2 \leq \gamma^{-1}_s(G) \] and \( 2 \leq \gamma^{-1}_s(\overline{G}) \).

Thus both lower bounds follow.

By Theorem 10, we have
\[ \gamma^{-1}_s(G) \leq p - 2 \] and \( \gamma^{-1}_s(\overline{G}) \leq p - 2. \)

Thus both upper bounds follow.

The path \( P_4, 2K_2 \) and cycle \( C_4 \) achieve these bounds.

### 3. DISJOINT SECURE DOMINATION

We introduce the concept of disjoint secure domination number.

**Definition 12:** The disjoint secure domination \( \gamma_s(G) \) of a graph \( G \) is the minimum cardinality of the union of two disjoint secure dominating sets in \( G \). We say that two disjoint secure dominating sets, whose union has cardinality \( \gamma_s(G) \), is a \( \gamma_s\)-pair of \( G \).
Remark 13: Not all graphs have a disjoint secure domination number. For example, the cycle $C_5$ does not have two disjoint secure dominating sets.

Theorem 14: For any graph $G$ with $\gamma_s^{-1}(G)$,
$$2\gamma_s(G) \leq \gamma_s^d(G) \leq \gamma_s(G) + \gamma_s^{-1}(G) \leq p.$$

Definition 15: A graph $G$ is called $\gamma_s^d$-minimum if it has two disjoint $\gamma_s$-sets, that is, $\gamma_s^d(G) = 2\gamma_s(G)$.

Definition 16: A graph $G$ is called $\gamma_s^d$-maximum if $\gamma_s^d(G) = p$.

The disjoint domination number $\gamma_s(G)$ of a graph $G$ is the minimum cardinality of the union of two disjoint dominating sets in $G$, see [23]. Many other disjoint domination numbers were studied, for example, in [7, 8, 9, 14, 24].

When the disjoint secure domination number exists the following inequality holds.

Proposition 17: For any graph $G$ with two disjoint secure dominating sets,
$$\gamma_s(G) \leq \gamma_s^d(G).$$

The cycle $C_4$, the paths $P_5, P_4$ achieve this bound.

The following results indicate the disjoint secure domination numbers of some standard graphs.

Proposition 18: For the complete graph $K_p$, $p \geq 2$,
$$\gamma_s^d(K_p) = \gamma_s^d(K_p) = 2\gamma_s(K_p) = 2.$$

Proposition 19: For the complete bipartite graph $K_{m,n}$, $4 \leq m \leq n$,
$$\gamma_s^d(K_{m,n}) = 2\gamma_s(K_{m,n}) = 8.$$

The complete graphs $K_p$, $p \geq 2$ and the complete bipartite graphs $K_{m,n}$, $4 \leq m \leq n$ are $\gamma_s^d$-minimum.

The graphs $K_2$ and $K_4, 4$ are $\gamma_s^d$-maximum.

4. SOME OPEN PROBLEMS

Many questions are suggested by this research among them are the following:

Problem 1: Characterize graphs $G$ for which $\gamma_s(G) = \gamma_s^{-1}(G)$.

Problem 2: Characterize graphs $G$ for which $\gamma_s(G) + \gamma_s^{-1}(G) = p$.

Problem 3: Characterize graphs $G$ for which $\gamma_s^d(G) = \gamma_s^d(G)$.

Problem 4: Characterize graphs $G$ for which $\gamma_s^d(G) = 2\gamma_s(G)$.

Problem 5: Characterize the class of $\gamma_s^d$-minimum graphs.

Problem 6: Characterize the class of $\gamma_s^d$-maximum graphs.

Problem 7: Obtain bounds for $\gamma_s\gamma_s(G) + \gamma_s\gamma_s(G)$.

Problem 8: What is the complexity of the decision problem corresponding to $\gamma_s^d(G)$?

Problem 9: Is DISJOINT SECURE DOMINATION NP-complete for a class of graphs?

REFERENCES


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