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# INVERSE AND DISJOINT SECURE DOMINATING SETS IN GRAPHS

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**ABSTRACT**

*Let D be a minimum secure dominating set of a graph G = (V, E). If V – D contains a secure dominating set D' of G, then D' is called an inverse secure dominating set with respect to D. The inverse secure domination number* $\gamma_s^{-1}(G)$ *of G is the minimum cardinality of an inverse secure dominating set of G. The disjoint secure domination number* $\gamma_s\gamma_s(G)$ *of a graph G is the minimum cardinality of the union of two disjoint secure dominating sets in G. In this paper, we establish some results for the inverse secure domination number. Also we initiate a study of the disjoint secure domination number and obtain some results on this new parameter.*

***Keywords:*** *Inverse domination number, inverse secure domination number, disjoint secure domination number.*

***AMS Subject Classification:*** *05C69.*

## 1. INTRODUCTION

By a graph, we mean a finite, undirected, without loops, multiple edges and isolated vertices. Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. For the general concepts, the reader may refer to [1]. A set $D$ of vertices in a graph $G$ is called a dominating set if every vertex in $V – D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. Recently several domination parameters are given in the books by Kulli in [2,3,4]. Let $D$ be a minimum dominating set of $G$. If $V – D$ contains a dominating set $D'$ of $G$, then $D'$ is called an inverse dominating set of $G$ with respect to $D$. The inverse domination number $\gamma^{-1}(G)$ of $G$ is the minimum cardinality of an inverse dominating set of $G$. The inverse domination in graphs was introduced by Kulli and Sigarkanti in [5]. Many other inverse domination parameters in domination theory were studied, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

A dominating set $D$ in $G$ is called a secure dominating set in $G$ if for every vertex $u$ in $V – D$, there exists $v$ in $D$ adjacent to $u$ such that $(D –\{v\})\cup\{u\}$ is a dominating set. The secure domination number $\gamma_s(G)$ of $G$ is the minimum cardinality of a secure dominating set of $G$. This was introduced by Cockayne *et al.* in [20].

Let $D$ be a minimum secure dominating set of $G$. If $V – D$ contains a secure dominating set $D'$ of $G$, then $D'$ is called an inverse secure dominating set with respect to $D$. The inverse secure domination number $\gamma_s^{-1}(G)$ is the minimum cardinality of an inverse secure dominating set of $G$. The inverse secure domination in graphs was found in the paper of Enriquez et al. in [21]. A $\gamma_s^{-1}$-set is a minimum inverse secure dominating set. Similarly other sets can be expected.

A dominating set $D$ of $G$ is a split dominating set if the induced subgraph $\langle D\rangle$ is disconnected. The split domination number $\gamma_{sd}(G)$ of $G$ is the minimum cardinality of a split dominating set of $G$. This concept was introduced by Kulli and Janakiram in [22].

Let $\Delta(G)$ denote the maximum degree and $\lceil x \rceil$ the least integer greater than or equal to $x$. The complement of $G$ is denoted by $\overline{G}$.

## 2. INVERSE SECURE DOMINATION

**Proposition A [21]:** Let $G$ be a connected graph with $p \geq 4$ vertices. Then

$$\gamma_s(G)\leq \gamma_s^{-1}(G).$$

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**Remark 2:** Not all graphs have an inverse secure dominating set. For example, the path $P_5$ has a secure dominating set, but no inverse secure dominating set.

By Remark 2: Proposition A is not true for $p = 5$, Thus we have

**Theorem 3:** For any graph $G$ with a $\gamma_s^{-1}$ -set,

$$\gamma_s(G) \le \gamma_s^{-1}(G) \tag{1}$$

and this bound is sharp.

**Proof:** Every inverse secure dominating set is a secure dominating set. Thus (1) holds.

The path $P_4$ achieves this bound.

**Theorem 4:** If a $\gamma_s^{-1}$-set exists in a graph $G$ with $p$ vertices, then

$$\gamma_s(G) + \gamma_s^{-1}(G) \le p \tag{2}$$

and this bound is sharp.

**Proof:** This follows from the definition of $\gamma_s^{-1}(G)$.

The path $P_4$ achieves this bound.

**Theorem 5:** If a $\gamma_s^{-1}$-set exists in a graph $G$ with $p$ vertices, then

$$\gamma(G) + \gamma_s^{-1}(G) \le p \tag{3}$$

and this bound is sharp.

**Proof:** By definition, $\gamma(G) \le \gamma_s(G)$. By Theorem 4, $\gamma_s(G)+ \gamma_s^{-1}(G) \le p$. Thus (3) holds.

The path $P_4$ and the cycle $C_4$ achieve this bound.

**Theorem B [22]:** For any graph $G$ with an endvertex,

$$\gamma(G) = \gamma_{sd}(G).$$

We obtain a relation between $\gamma_{sd}(G)$ and $\gamma_s^{-1}(G)$.

**Theorem 6:** Let $G$ be a graph with an endvertex. If a $\gamma_s^{-1}$-set exists in $G$ with $p$ vertices, then

$$\gamma_{sd}(G) + \gamma_s^{-1}(G) \le p \tag{4}$$

and this bound is sharp.

**Proof:** From Theorem 5, we have $\gamma(G) + \gamma_s^{-1}(G) \le p$. From Theorem B, we have $\gamma(G) = \gamma_{sd}(G)$. Thus (4) holds.

The path $P_4$ achieves this bound.

**Theorem 7:** For any graph $G$ without isolated vertices and with an endvertex,

$$\gamma_{sd}(G) \le \gamma_s(G) \tag{5}$$

and this bound is sharp.

**Proof:** From Theorem B, $\gamma(G) = \gamma_{sd}(G)$ and by definition $\gamma(G) \le \gamma_s(G)$. Hence (5) holds.

The path $P_4$ achieves this bound.

**Corollary 8:** Let $G$ be a graph with an endvertex. If a $\gamma_s^{-1}$ -set exists in $G$, then

$$\gamma_{sd}(G) \le \gamma_s^{-1}(G) \tag{6}$$

We obtain lower and upper bounds on $\gamma_s^{-1}(G)$.

**Theorem 9:** For any graph $G$ with $p$ vertices and with a $\gamma_s^{-1}$-set,

$$\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \le \gamma_s^{-1}(G) \le \left\lceil \frac{p\Delta(G)}{\Delta(G)+1} \right\rceil. \tag{7}$$

**Proof:** It is known that $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \le \gamma(G)$ and since $\gamma(G) \le \gamma_s^{-1}(G)$, we see that the lower bound in (7) holds.

By Theorem 4, we have

$$\gamma_s^{-1}(G) \leq p - \gamma_s(G).$$

Since $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma(G) \leq \gamma_s(G)$ and the above inequality,

$$\gamma_s^{-1}(G) \leq p - \left\lceil \frac{p}{\Delta(G)+1} \right\rceil.$$

Thus the upper bound in (7) holds.

**Theorem C [21]:** Let $G$ be a connected graph with $p \geq 2$ vertices. Then $\gamma_s(G) = 1$ if and only if $G = K_p$.

We obtain the following bounds for $\gamma_s^{-1}(G)$.

**Theorem 10:** Let $G$ be a connected graph with $p \geq 4$ vertices. If $G$ has a $\gamma_s^{-1}$-set and $G \neq K_p$, then

$$2 \leq \gamma_s^{-1}(G) \leq p - 2 \tag{8}$$

and these bounds are sharp.

**Proof:** Suppose $G$ is connected and $G \neq K_p$. By Theorem C, $\gamma_s(G) \geq 2$. Since $2 \leq \gamma_s(G)$ and by Theorem 3, $\gamma_s(G) \leq \gamma_s^{-1}(G)$, we see that the lower bound of (8) follows.

By Theorem 4, we have $\gamma_s^{-1}(G) \leq p - \gamma_s(G)$ and since $2 \leq \gamma_s(G)$

$$\gamma_s^{-1}(G) \leq p - 2.$$

Thus $\quad 2 \leq \gamma_s^{-1}(G) < p - 2.$

The path $P_4$ achieves both lower and upper bounds.

Now we obtain a Nordhaus - Gaddum type result for secure domination number.

**Theorem 11:** Let $G$ be a graph with $p \geq 4$ vertices and $G \neq K_p$. If a $\gamma_s^{-1}$ -set exists and $G$ and $\overline{G}$ have no isolated vertices, then

$$4 \leq \gamma_s^{-1}(G) + \gamma_s^{-1}(\overline{G}) \leq 2(p-2)$$

$$4 \leq \gamma_s^{-1}(G)\gamma_s^{-1}(\overline{G}) \leq (p-2)^2$$

and these bounds are sharp.

**Proof:** Since $G$ and $\overline{G}$ have no isolated vertices and $G \neq K_p$,

$$2 \leq \gamma_s^{-1}(G) \text{ and } 2 \leq \gamma_s^{-1}(\overline{G}).$$

Thus both lower bounds follow.

By Theorem 10, we have

$$\gamma_s^{-1}(G) \leq p - 2 \text{ and } \gamma_s^{-1}(\overline{G}) \leq p - 2.$$

Thus both upper bounds follow.

The path $P_4$, $2K_2$ and cycle $C_4$ achieve these bounds.

## 3. DISJOINT SECURE DOMINATION

We introduce the concept of disjoint secure domination number.

**Definition 12:** The disjoint secure domination $\gamma_s\gamma_s(G)$ of a graph $G$ is the minimum cardinality of the union of two disjoint secure dominating sets in $G$. We say that two disjoint secure dominating sets, whose union has cardinality $\gamma_s\gamma_s(G)$, is a $\gamma_s\gamma_s$-pair of $G$.

**Remark 13:** Not all graphs have a disjoint secure domination number. For Example, the cycle $C_5$ does not have two disjoint secure dominating sets.

**Theorem 14:** For any graph $G$ with $\gamma_s^{-1}(G)$,

$$2\gamma_s(G) \le \gamma_s\gamma_s(G) \le \gamma_s(G) + \gamma_s^{-1}(G) \le p.$$

**Definition 15:** A graph $G$ is called $\gamma_s\gamma_s$-minimum if it has two disjoint $\gamma_s$-sets, that is, $\gamma_s\gamma_s(G) = 2\gamma_s(G)$.

**Definition 16:** A graph $G$ is called $\gamma_s\gamma_s$-maximum if $\gamma_s\gamma_s\ (G) = p$.

The disjoint domination number $\gamma\gamma(G)$ of a graph $G$ is the minimum cardinality of the union of two disjoint dominating sets in $G$, see [23]. Many other disjoint domination numbers were studied, for example, in [7, 8, 9, 14, 24].

When the disjoint secure domination number exists the following inequality holds.

**Proposition 17:** For any graph $G$ with two disjoint secure dominating sets,

$$\gamma\gamma(G) \le \gamma_s\gamma_s(G).$$

The cycle $C_4$, the paths $P_2$, $P_4$ achieve this bound.

The following results indicate the disjoint secure domination numbers of some standard graphs.

**Proposition 18:** For the complete graph $K_p$, $p\ge 2$,

$$\gamma\gamma(K_p) = \gamma_s\gamma_s\ (K_p) = 2\gamma_s(K_p) = 2.$$

**Proposition 19:** For the complete bipartite graph $K_{m,\,n}$, $4 \le m \le n$,

$$\gamma_s\gamma_s(K_{m,\,n}) = 2\gamma_s(K_{m,\,n}) = 8.$$

The complete graphs $K_p$, $p\ge 2$ and the complete bipartite graphs $K_{m,\,n,}$ $4 \le m \le n$ are $\gamma_s\gamma_s$ -minimum.

The graphs $K_2$ and $K_{4,\,4}$ are $\gamma_s\gamma_s$ -maximum.

## 4. SOME OPEN PROBLEMS

Many questions are suggested by this research among them are the following:

**Problem 1:** Characterize graphs $G$ for which $\gamma_s(G) = \gamma_s^{-1}(G)$.

**Problem 2:** Characterize graphs $G$ for which $\gamma_s(G) + \gamma_s^{-1}(G) = p$.

**Problem 3:** Characterize graphs $G$ for which $\gamma\gamma(G) = \gamma_s\gamma_s(G)$.

**Problem 4:** Characterize graphs $G$ for which $\gamma_s\gamma_s(G) = 2\gamma(G)$.

**Problem 5:** Characterize the class of $\gamma_s\gamma_s$ -minimum graphs.

**Problem 6:** Characterize the class of $\gamma_s\gamma_s$ -maximum graphs.

**Problem 7:** Obtain bounds for $\gamma_s\gamma_s\left(G\right)+\gamma_s\gamma_s\left(\overline{G}\right)$.

**Problem 8:** What is the complexity of the decision problem corresponding to $\gamma_s\gamma_s(G)$ ?

**Problem 9:** Is DISJOINT SECURE DOMINATION NP-complete for a class of graphs?

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