PRIME ACCESSIBLE RINGS

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ABSTRACT

In this paper, we prove that a 2- and 3- divisible prime accessible ring is either associative or commutative and a 2- and 3- divisible semiprime accessible ring is associative.

Keywords: Prime ring, semiprime ring, accessible ring, n-divisible ring, nucleus.

1. INTRODUCTION

Kleinfeld [1] studied the structure of standard and accessible rings. He proved that simple accessible rings are either associative or commutative. Thedy [3] studied the rings, in which the associator commutes with all elements of the ring i.e., ((w, x, y), z) = 0. He proved that simple nonassociative rings satisfying this identity are either associative or commutative.

In this paper we see that accessible rings satisfy the identity ((w, x, y), z) = 0. We prove that the associator and multiple of the associator are in the nucleus of an accessible ring. Using these properties, we show that a 2- and 3- divisible prime accessible ring is either associative or commutative and a 2- and 3- divisible semiprime accessible ring is associative.

2. PRELIMINARIES

A ring is defined to be accessible if the following two identities hold:

\[(x, y, z) + (z, x, y) - (x, z, y) = 0. \quad (1)\]

\[((w, x), y, z) = 0. \quad (2)\]

for all x, y, z in R, where the associator is defined as (x, y, z) = (xy)z - x(yz) for all x, y, z in R, the commutator is defined as (x, y) = xy - yx for all x, y in R.

Throughout this paper R represents an accessible ring. R is said to be prime whenever A and B are ideals of R such that AB=0, then either A=0 or B=0. R is said to be semiprime if for any ideal A of R, A^2 = 0 implies A=0. R is said to be n-divisible if nx = 0 implies x = 0 for all x in R and n, a natural number. The nucleus N of R is defined as the set of all elements n in R such that (n, R, R) = (R, n, R) = (R, R, n) = 0.

By substituting z = y in (1), we obtain the flexible law

\[(y, x, z) = 0. \quad (3)\]

A linearization of the above identity yeilds

\[(y, x, z) = -(z, x, y). \quad (4)\]

Then

\[(x, y, z) + (y, z, x) + (z, x, y) = 0. \quad (5)\]

In any arbitrary ring the identity

\[(xy, z) = x(y, z) + (x, z)y + (x, y, z) + (z, x, y) - (x, z, y)\]

holds.

From (1) this identity becomes (xy, z) = x(y, z) + (x, z)y.

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Another identity which holds in an arbitrary ring is
\[(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z.\] (7)

If \(n\) is an element of the nucleus \(N\) of \(R\), then because of the flexible law \((R, R, n) = 0\). Finally because of (1) it follows that \((R, n, R) = 0\).

If \(n\) is substituted for \(w\) in (7), we obtain
\[(nx, y, z) = n(x, y, z), n \in N.\] (8)

We now proceed to develop further identities that hold in arbitrary accessible rings. The elements \(u, v, w, x, y, z\) will denote arbitrary elements of such rings.

By repeated use of (6), we break up \(((w, x, y), z)\) as
\[
((w, x, y), z) = (wx \cdot y - w \cdot xy, z) = wx \cdot (y, z) + w(x, z) \cdot y + (w, z)x \cdot y - w \cdot x(y, z) - w \cdot (x, z)y - (w, z)xy.
\]

Since (2) implies that every commutator is in the nucleus,
\[((w, x, y), z) = 0.\] (9)

Hence every associator commutes with every element of \(R\).

The associator ideal \(A\) of \(R\) is defined as \(A=\Sigma(R, R, R) + (R, R, R)R\)

Let \(S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y).\)

In every ring we have the identities
\[\begin{align*}
(xy, z) + (yz, x) + (zx, y) &= S(x, y, z), \\
((x, y), z) + ((y, z), x) + ((z, x), y) &= S(x, y, z) - S(x, z, y).
\end{align*}\] (10) (11)

3. MAIN RESULTS

First we prove some properties of the nucleus in \(R\).

Lemma 1: If \(R\) is a prime accessible ring, then the associator is in the nucleus \(N\) of \(R\).

Proof: From (8) and the fact that every commutator is in the nucleus, we get \((v, x) (x, y, z) = ((v, x)x, y, z).\)

It follows from (6) that \((vx, x) = v(x, x) + (v, x)x\) (or) \((vx, x) = (v, x)x.\)

Consequently \(((v, x)x, y, z) = ((vx, x)y, z) = 0.\)

Therefore \((v, x) (x, y, z) = 0.\) (12)

A linearization of this identity becomes
\[\begin{align*}
(v, w) (x, y, z) &= - (v, x) (w, y, z).
\end{align*}\] (13)

Using (6), (9), (13) and (2), we obtain
\[\begin{align*}
((v, w) (x, y, z), u) &= (v, w) ((x, y, z), u) + ((v, w), u) (x, y, z) \\
&= ((v, w), u) (x, y, z) \\
&= - (u, (v, w)) (x, y, z) \\
&= (u, x) ((v, w), y, z) \\
&= 0.
\end{align*}\]

Therefore \(((v, w) (x, y, z), u) = 0.\)

Now using (8), (13) and (9), we obtain
\[\begin{align*}
((v, w) (x, y, z), t, u) &= (v, w) ((x, y, z), t, u) \\
&= - (v, (x, y, z)) (w, t, u) \\
&= 0.
\end{align*}\]
Thus \((v, w) (x, y, z), t, u) = 0\).

Since \((v, w) \in N\) we have \((v, w) ((x, y, z), t, u) = 0\). \hspace{1cm} (14)

Since \(R\) is prime and not commutative, (14) implies that \(((x, y, z), t, u) = 0\).

By using the linearization of flexible property (3), we obtain \((u, t, (x, y, z)) = 0\). Finally because of (1), it follows that \((t, (x, y, z), u) = 0\). Therefore the associator \((x, y, z)\) is in the nucleus \(N\).

This completes the proof of the lemma.

**Lemma 2:** In an accessible ring \(R\), \((R(R, R, R), R) = 0\).

**Proof:** By commuting (7) with \(r\), we get
\[
((wx, y, z), r) - ((w, xy, z), r) + ((w, x, yz), r) = (w(x, y, z), r) + ((w, x, y)z, r)
\]

By using (9), we obtain
\[
(w(x, y, z), r) = -((w, x, y)z, r).
\hspace{1cm} (15)

Equation (15) with \(w=y\) and using (3) gives
\[
(y(x, y, z), r) = -((y, x, y)z, r) = 0.
\hspace{1cm} (16)

Linearization of (16) with \(y=w+y\) yeilds
\[
(w(x, y, z), r) = -((y, w, z), r).
\hspace{1cm} (17)

By substituting \(z=y\) in (15) and using (17) repeatedly, we get
\[
(w(x, y, y), r) = -((w, x, y)y, r)
= ((w, y, y)x, r)
= -((y, w, y)y, r)
= ((y, y, y)x, r)
= 0.
\hspace{1cm} (18)

i.e. \((w(x, y, y), r) = 0\).

Linearization of (18) with \(y=y+z\) yeilds
\[
(w(x, y, z), r) = -((w, x, z), r).
\hspace{1cm} (19)

Using the linearization of flexible identity, the above equation yeilds
\[
(w(x, y, z), r) = (w(y, z, x), r).
\]

Similarly \((w(y, z, x), r) = (w(z, x, y), r)\).

Therefore \((w(x, y, z), r) = (w(y, z, x), r) = (w(z, x, y), r)\). \hspace{1cm} (20)

Using (19) and (5) gives
\[
0 = (w((x, y, z) + (y, z, x) + (z, x, y)), r)
0 = 3(w(x, y, z), r).
\]

Since \(R\) is 3- divisible, we have \((w(x, y, z), r) = 0\).

i.e. \((R(R, R, R), R) = 0\).

This completes the proof of the lemma.

**Theorem 1:** A 2- and 3- divisible prime accessible ring is either associative or commutative.

**Proof:** By using (6) we get
\[
(w(x, y, z), r) = w((x, y, z), r) + (w, r) (x, y, z).
\]
By using (9) and lemma 2, we obtain

\[(w, r)(x, y, z) = 0.\]  \hspace{1cm} (20)

We know that \(A\) is an ideal consisting of all finite sums of elements of the form \((x, y, z)\) or of the form \(w(x, y, z)\) and \(B\) is an ideal consisting of all finite sums of elements of the form \((x, y)\) or of the form \(w(x, y)\).

From (20), it follows that \(BA = 0\)

Since \(R\) is prime, we have either \(B = 0\) or \(A = 0\).

If \(B = 0\), then \(R\) is commutative. If \(A = 0\), then \(R\) is associative.

Hence \(R\) is either commutative or associative.

**Theorem 2:** A 2- and 3- divisible semiprime accessible ring \(R\) is associative.

**Proof:** From lemma 1, we have

\[((x, y, z), r, s) = 0.\]  \hspace{1cm} (21)

By taking associators of (7) and using (9) we get

\[(w(x, y, z), r, s) + ((w, x, y)z, r, s) = 0.\]  \hspace{1cm} (22)

By substituting \(w = y\) in (22) and using (3)

\[(y(x, y, z), r, s) + ((y, x, y)z, r, s) = 0.\]  \hspace{1cm} (23)

So that \((y(x, y, z), r, s) = 0.\)

By linearizing (23) with \(y = w + y\) yields

\[(w(x, y, z), r, s) + (y(x, w, z), r, s) = 0.\]  \hspace{1cm} (24)

By substituting \(z = y\) in (22) and (24) we have

\[(w(x, y, y), r, s) + ((w, x, y)y, r, s) = 0.\]  \hspace{1cm} (25)

and

\[(w(x, y, y), r, s) + (y(x, w, y), r, s) = 0.\]  \hspace{1cm} (26)

By subtracting the equation (26) from (25), we get

\[((w, x, y)y, r, s) - (y(x, w, y), r, s) = 0.\]

i.e.

\[((w, x, y)y, r, s) = (y(x, w, y), r, s).\]  \hspace{1cm} (27)

Since \((y(x, y, z), r, s) = 0\), we have \((y(y, x, z), r, s) = 0\) and – \((z(y, y, x), r, s) = 0.\)

This implies that \((z(x, y, y), r, s) = 0.\)

i.e.

\[((x, y, y)z, r, s) = 0.\]  \hspace{1cm} (28)

Linearization of (28) with \(y = w + y\) yeilds

\[((x, w, y)z, r, s) + ((x, y, w)z, r, s) = 0.\]

Using linearization of flexible identity, the above equation yeilds

\[((x, w, y)z, r, s) = ((w, y, x)z, r, s).\]

Similarly \(((w, y, x)z, r, s) = ((y, x, w)z, r, s).\)

Therefore \(((x, w, y)z, r, s) = ((w, y, x)z, r, s) = ((y, x, w)z, r, s).\)  \hspace{1cm} (29)

Using (29) and (5) gives

\[0 = (((w, x, y) + (x, y, w) + (y, w, x))z, r, s)\]

\[0 = 3((w, x, y)z, r, s).\]

Since \(R\) is 3- divisible, we have \(((w, x, y)z, r, s) = 0.\)

Using (8) and lemma 1, we have \((w, x, y)(z, r, s) = 0.\)  \hspace{1cm} (30)
We know that $A$ is an associator ideal consisting of all elements of the form $(x, y, z)$ and $w(x, y, z)$. From (30) it follows that $A^2=0$. Since $R$ is semiprime, we obtain $A=0$. Hence $R$ is associative.

This completes the proof of the theorem.

REFERENCES


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