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HARMONIOUS COLORING OF CENTRAL GRAPH OF SOME TYPES OF GRAPHS<br>U. MUTHUMARI ${ }^{\mathbf{*}}$, M. UMAMAHESWARI ${ }^{\mathbf{2}}$<br>${ }^{1}$ Asst. Professor, Department of Mathematics, The Standard Fireworks Rajarathnam College for Women, Sivakasi, (T.N.), India.

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#### Abstract

Harmonious colouring is a proper vertex colouring such that no two edges share the same colour pair. The harmonious Chromatic number of a graph is the least number of colours in such a colouring. The purpose of this paper is to study the harmonious coloring of central graph of flower graph, belt graph, rose graph and steering graph and harmonious chromatic number for these graphs. Also we give some structural properties of these graphs.


Keywords: Harmonious coloring, Harmonious chromatic number, central graph of a graph, flower graph, belt graph, rose graph, steering graph.

## INTRODUCTION

## Harmonious coloring:

In graph theory, a harmonious coloring is a (proper) vertex coloring in which every pair of colors appears on at most one pair of adjacent vertices. The harmonious chromatic number $\chi_{\mathrm{H}}(G)$ of a graph $G$ is the minimum number of colors needed for any harmonious coloring of $G$.

Central graph of a graph:
The central graph of any graph $G$ is obtained by subdividing each edge of $G$ exactly once and joining all the nonadjacent vertices of G.

Definition: A Flower graph is obtained from a cycle $C=v_{1}, v_{2} \ldots \ldots v_{n}, v_{1}$ by joining the two consecutive vertices by a new vertices $u_{1}, u_{2}, \ldots . u_{n}$. A flower graph has $2 n$ vertices and $3 n$ edges, where $n$ is the length of the cycle. We denote this graph $\mathrm{F}_{\mathrm{n}}$.

Example:


Figure 1: Flower graph $\mathrm{F}_{4}$

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## Structural Properties of central graph of Flower graph:

Maximum degree $\Delta=4$
Minimum degree $\delta=2$
Number of vertices in $C\left(F_{n}\right)=5 n$
Number of edges in $C\left(F_{n}\right)=\frac{(2 n)(2 n-1)}{2}+3 n$
Maximum degree $\Delta\left[C\left(F_{n}\right)\right]=2 n-1$
Minimum degree $\delta\left[\mathrm{C}\left(\mathrm{F}_{\mathrm{n}}\right)\right]=2$
Theorem: The Harmonious chromatic number of central graph of Flower graph $C\left[F_{n}\right]$ is $\Delta\left[C\left(F_{n}\right)\right]+3$.

$$
\text { (i.e) } \chi_{H} \mathrm{C}(\mathrm{Fn})=\Delta\left[\mathrm{C}\left(\mathrm{~F}_{\mathrm{n}}\right)\right]+3 .
$$

Proof: Let $\mathrm{F}_{\mathrm{n}}$ be the Flower graph with 2 n vertices and $3 n$ edges.
Let $\left\{\mathrm{v}_{1}, \mathrm{v}_{2} \ldots \ldots . \mathrm{v}_{\mathrm{n}}, \mathrm{u}_{1}, \mathrm{u}_{2} \ldots \ldots . \mathrm{u}_{\mathrm{n}}\right\}$ be the vertices of the Flower graph $\mathrm{F}_{\mathrm{n}}$.
By the definition of Central graph, each edge is subdivided by a new vertex.
Therefore, Assume that each edge $\left(v_{i}, v_{i+1}\right)$ and the line joining $v_{i}$ and $v_{i+1}$ to a new vertex $u_{i}, i=1$ to $n$ are subdivided by the vertices $\mathrm{e}_{\mathrm{j},}, \mathrm{x}_{\mathrm{i}}, \mathrm{y}_{\mathrm{i}}, \mathrm{i} \& \mathrm{j}=1$ to n respectively.

Assign the coloring to vertices as follows:

$$
\begin{aligned}
& c\left(v_{i}\right)=\mathrm{i} \text { for } 1 \leq \mathrm{i} \leq \mathrm{n} \\
& \mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)=\mathrm{n}+\mathrm{i} \text { for } 1 \leq \mathrm{i} \leq \mathrm{n} \\
& \mathrm{c}\left(\mathrm{e}_{\mathrm{j} j}\right)=2 \mathrm{n}+1 \text { for } 1 \leq \mathrm{j} \leq \mathrm{n} \\
& \mathrm{c}\left(\mathrm{x}_{\mathrm{i}}\right)=2 \mathrm{n}+2 \text { for } 1 \leq \mathrm{i} \leq \mathrm{n} \\
& \mathrm{c}\left(\mathrm{y}_{\mathrm{i}}\right)=\mathrm{n}+\mathrm{i} \text { for } 1 \leq \mathrm{i} \leq \mathrm{n}
\end{aligned}
$$

First we claim that, c is a proper coloring.
Since $c\left(v_{i}\right), c\left(u_{i}\right)$ and its neighbours receive distinct colors.
Further $\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right) \neq \mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)$.
Hence c is a proper coloring.
Next we claim that, c is a Harmonious coloring.
It is clear that each vertex receive a distinct color at a distance atmost 2 from all other vertices.
Thus the coloring is harmonious.
Finally we claim that c is the minimum number of colors in a harmonious coloring.
Suppose not we have the following cases.
Case (i): The color set $c\left(v_{i}\right) \& c\left(u_{i}\right)$ containing $2 n$ colors.
If we assign $2 \mathrm{n}-1$ colors, then the color pair will not distinct. which is contradict the definition of harmonious colors.
Case (ii): The neighbours of $u_{i}$ and $v_{i}$ are given the same color. Then the color pair repeats which is contradiction to the definition of harmonious coloring.

Illustration for the above theorem:


Figure 2: Central graph of Flower graph $\left[\mathrm{C}\left(\mathrm{F}_{4}\right)\right]$


Figure 3: $\chi_{H} C\left[F_{4}\right]=10$
Definition: A Belt graph is obtained from Helm graph by joining its outer vertices. It has $2 n+1$ vertices $4 n$ edges. It is denoted by $\mathrm{B}_{\mathrm{n}}$.

## Example:



## Structural Properties of central graph of graph:

Minimum degree $\delta=3$

Number of vertices in $C(G)=6 n+1$
Number of edges in $C(G)=8 n$
Maximum degree $\Delta[C(G)]=2 n$
Minimum degree $\delta[\mathrm{C}(\mathrm{G})]=2$
Theorem: The Harmonious chromatic number of central graph of Belt graph $C\left[B_{n}\right]$ is $4 n+1$. (i.e $) \chi_{H}\left[C\left(B_{n}\right)\right]=4 n+1$.
Proof: Let $\left\{\mathrm{v}_{1}, \mathrm{v}_{2} \ldots \ldots . \mathrm{v}_{\mathrm{n}}, \mathrm{u}_{1}, \mathrm{u}_{2} \ldots \ldots . \mathrm{u}_{\mathrm{n}}, \mathrm{w}_{1}\right\}$ be the vertices of the Belt graph .
By the definition of Central graph, each edge is subdivided by a new vertex.
The edges $\left(u_{i}, u_{i+1}\right) \&\left(u_{n}, u_{1}\right)$ is obtained from the outer vertices helm graph are joined. Therefore, Assume that each edges $\left(\mathrm{u}_{\mathrm{i}}, \mathrm{u}_{\mathrm{i}+1}\right) \&\left(\mathrm{u}_{\mathrm{n}}, \mathrm{u}_{1}\right)$ are subdivided by the new vertex $\mathrm{x}_{\mathrm{i}}, \mathrm{e}_{\mathrm{j}}, \mathrm{x}_{\mathrm{i}}, \mathrm{y}_{\mathrm{i}}, \mathrm{z}_{\mathrm{i}}, \mathrm{i} \& \mathrm{j}=1$ to n respectively.

Assign the coloring to vertices as follows:
$c\left(v_{i}\right)=\mathrm{i}$ for $1 \leq \mathrm{i} \leq \mathrm{n}$
$\mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)=\mathrm{n}+\mathrm{i}$ for $\mathrm{i} \leq \mathrm{i} \leq \mathrm{n}$
$c\left(w_{1}\right)=2 n+1$
$c\left(e_{j j}\right)=(2 n+2)+j$
$\mathrm{c}\left(\mathrm{x}_{\mathrm{i}}\right)=(2 \mathrm{n}+1)+\mathrm{i}$ for $1 \leq \mathrm{i} \leq \mathrm{n}$
$c\left(y_{i}\right)=(3 n+1)+i$
$\mathrm{c}\left(\mathrm{z}_{\mathrm{i}}\right)=(2 \mathrm{n}+1)+\mathrm{i}$
Clearly $4 \mathrm{n}+1$ coloring.
First we claim that, c is a proper coloring.
Since $c\left(v_{i}\right), c\left(u_{i}\right), c\left(w_{i}\right)$ and its neighbours receive distinct colors.
Further $\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right) \neq \mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right) \neq \mathrm{c}\left(\mathrm{w}_{\mathrm{i}}\right)$
Hence c is a proper coloring.
Next we claim that, c is a Harmonious coloring.
Its clear that each vertex receive a distinct color at a distance atmost 2 from all other vertices.
Thus the coloring is harmonious.
Finally we claim that c is the minimum number of colors in a harmonious coloring.
Suppose not we have the following cases.
Case (i): The color set $\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right), \mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right) \& \mathrm{c}\left(\mathrm{w}_{1}\right)$ containing $2 \mathrm{n}+1$.
If we assign 2 n colors, then the color pair will not distinct. which is contradict the definition of harmonious colors.
Case (ii): The neighbours of $u_{i}, v_{i}, w_{1}$ are given the same color. then the color pair repeats which is contradiction to the definition of harmonious coloring.

Illustration for the above theorem:


Definition: A Rose graph is obtained from a wheel graph $w_{n}$ with $n$ vertices ( $n \geq 4$ ) by joining the two consecutive vertices $v_{1}, v_{2}, \ldots \ldots . . v_{n}$ by a new vertices $u_{1}, u_{2}, \ldots . u_{n}$. A Rose graph has $2 n+1$ vertices and $4 n$ edges. Where $n$ is length of the cycle .We denote this graph by $\mathrm{R}_{\mathrm{n}}$.

## Example:



Figure 1: Rose graph $\mathrm{R}_{4}$

## Structural Properties of central graph of Rose graph:

Minimum degree $\delta=2$
Number of vertices in $C\left(R_{n}\right)=6 n+1$
Number of edges in $C\left(R_{n}\right)=\frac{(2 n)(2 n+1)}{2}+4 n$
Maximum degree $\Delta\left[C\left(R_{n}\right)\right]=p-1$ where $p$ is the point on the orignal graph
(or) 2 n .
Minimum degree $\delta\left[\mathrm{C}\left(\mathrm{R}_{\mathrm{n}}\right)\right]=2$
Theorem: The Harmonious chromatic number of central graph of Rose graph $C\left[R_{n}\right]$ is $4 n-1$. (i.e) $\chi_{H} \mathrm{C}\left(\mathrm{R}_{\mathrm{n}}\right)=4 \mathrm{n}-1$.

Proof: Let $\mathrm{F}_{\mathrm{n}}$ be the Rose graph with $2 \mathrm{n}+1$ vertices and 4 n edges.
Let $\left\{\mathrm{v}_{1}, \mathrm{v}_{2} \ldots \ldots \mathrm{v}_{\mathrm{n}}, \mathrm{u}_{1}, \mathrm{u}_{2} \ldots \ldots \mathrm{u}_{\mathrm{n}}, \mathrm{w}_{1}\right\}$ be the vertices of the Rose graph $\mathrm{R}_{\mathrm{n}}$.
By the definition of Central graph, each edge is subdivided by a new vertex.
Therefore, Assume that each edge $\left(v_{i}, v_{i+1}\right)$ and the line joining $v_{i}$ and $v_{i+1}$ to a new vertex $u_{i}, i=1$ to $n$ are subdivided by the vertices $\mathrm{e}_{\mathrm{j},}, \mathrm{z}_{\mathrm{i}}, \mathrm{x}_{\mathrm{i}}, \mathrm{y}_{\mathrm{i}}, \mathrm{i} \& \mathrm{j}=1$ to n respectively.

Assign the coloring to vertices as follows:
$c\left(v_{i}\right)=\mathrm{i}$ for $1 \leq \mathrm{i} \leq \mathrm{n}$
$\mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)=\mathrm{n}+\mathrm{i}$ for $\mathrm{i} \leq \mathrm{i} \leq \mathrm{n}$
$\mathrm{c}\left(\mathrm{w}_{1}\right)=2 \mathrm{n}+1$
$\mathrm{c}\left(\mathrm{e}_{\mathrm{j} j}\right)=2 \mathrm{n}+2$ for $1 \leq \mathrm{j} \leq \mathrm{n}$
$\mathrm{c}\left(\mathrm{z}_{\mathrm{i}}\right)=2 \mathrm{n}+3$ for $1 \leq \mathrm{j} \leq \mathrm{n}$
$c\left(x_{i}\right)=(2 n+3)+i$ for $1 \leq i \leq n$
$\mathrm{c}\left(\mathrm{y}_{\mathrm{i}}\right)=\mathrm{n}+\mathrm{i}$ for $1 \leq \mathrm{i} \leq \mathrm{n}$
Clearly 4n-1 coloring.
First we claim that, c is a proper coloring.
Since $c\left(v_{i}\right), c\left(u_{i}\right)$ and $c\left(w_{1}\right)$ and its neighbors receive distinct colors. Further $c\left(v_{i}\right) \neq c\left(u_{i}\right) \neq c\left(w_{1}\right)$.
Hence c is a proper coloring.
Next we claim that, c is a Harmonious coloring.
Its clear that each vertex receive a distinct color at a distance atmost 2 from all other vertices.
Thus the coloring is harmonious.
Finally we claim that c is the minimum number of colors in a harmonious coloring.
Suppose not we have the following cases.
Case (i): The color set $\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right), \mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right) \& \mathrm{c}\left(\mathrm{w}_{1}\right)$ containing $2 \mathrm{n}+1$ colors.
If we assign 2 n colors, then the color pair will not distinct. which is contradict the definition of harmonious colors.

Case (ii): The neighbors of $u_{i}, v_{i}$ and $w_{1}$ are given the same color. then the color pair repeats which is contradiction to the definition of harmonious coloring.

## Illustration for the above theorem:



Central graph of Rose graph $C\left[\mathrm{R}_{4}\right]$


Definition: A Steering graph is a graph formed by connecting a any two vertices of a cycle by a single vertex. A Steering graph with $n$ vertices $(n>3)$. A stearing graph has $n+1$ vertex and $n+2$ edges. It is denoted by $S_{n}$.

## Example:



Stearing graph $\mathrm{S}_{4}$

## Structural properties of central graph of stearing graph:

Maximum degree $\Delta\left[\mathrm{S}_{\mathrm{n}}\right]=3$
Minimum degree $\delta\left[\mathrm{S}_{\mathrm{n}}\right]=2$
Number of vertices in $C\left[S_{n}\right]=2 n+3$
Number of edges in $\mathrm{C}\left[\mathrm{S}_{\mathrm{n}}\right]=\frac{(n+1) n}{2}+(\mathrm{n}+2)$
Maximumdegree $\Delta[\mathrm{C}(\mathrm{Sn})]=\mathrm{n}$ (length of the cycle in original graph)
Minimum degree $\delta\left[\mathrm{C}\left(\mathrm{S}_{\mathrm{n}}\right)\right]=2$
Theorem: The harmonious chromatic number of central graph of stearing graph $C\left[S_{n}\right]$ is $2 n$, where $n$ is the length of the cycle.

$$
\text { (i.e) } \chi_{H}\left[\mathrm{C}\left(\mathrm{~S}_{\mathrm{n}}\right)\right]=2 \mathrm{n} .
$$

Proof: Let $S_{n}$ be the stearing graph with $n+1$ vertices and $n+2$ edges.
Let $\left\{\mathrm{v}_{1}, \mathrm{v}_{2}, \ldots . . \mathrm{v}_{\mathrm{n}+1}\right\}$ be the vertices of $\mathrm{S}_{\mathrm{n}}$. let the vertices joining $\mathrm{v}_{1}$ and $\mathrm{v}_{2}$ we obtained a new vertex $\mathrm{u}_{1}$. By the definition of Central graph, each edge is subdivided by a new vertex. Assume that each edge $\left(\mathrm{v}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i}+1}\right)$ and the line joining $v_{i}$ and $v_{i+1}, v_{n}$ and $v_{1}$ are subdivided by the vertices $f_{i}, i=1,2 \ldots . n$ respectively.And the line joining $\mathrm{v}_{1}$ and $\mathrm{u}_{1}, \mathrm{v}_{2}$ and $\mathrm{u}_{1}$ are subdivided the vertices $\mathrm{x}_{1}, \mathrm{y}_{1}$.

Assign coloring to the vertices as follows:
$\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right)=\mathrm{i}$ for $\mathrm{L} \leq \mathrm{i} \leq \mathrm{n}+1$
$\mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)=\mathrm{n}+1$
$\mathrm{c}\left(\mathrm{f}_{\mathrm{i}}\right)=\mathrm{n}+\mathrm{i}$ for $1 \leq \mathrm{i} \leq \mathrm{n}$
$\mathrm{c}\left(\mathrm{x}_{1}\right)=2 \mathrm{n}-1$
$c\left(y_{1}\right)=2 n$
Clearly c is 2 n coloring.
First we claim that c is proper coloring.
Since each $c\left(v_{i}\right), c\left(u_{i}\right)$ and its neighbours receive distinct colors.
Further $\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right) \neq \mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)$.
Hence c is a proper coloring.
Next we claim that c is harmonious coloring.
It is clear that each vertex receive a distinct color at a distance atmost 2 from all other vertices.

Thus the coloring is harmonious.
Finally we claim that c is minimum number of colors in a harmonious coloring.
Suppose not we have following cases.
Case (i): The color set $\mathrm{c}\left(\mathrm{v}_{\mathrm{i}}\right)$ and $\mathrm{c}\left(\mathrm{u}_{\mathrm{i}}\right)$ contains $\mathrm{n}+1$ colors. If we assign n colors, then the will not be distinct which contradict the definition of harmonious coloring.

Case (ii): The neighbour of $v_{i}$ and $u_{i}$ are given the same color,then the color pair repeats which is contradiction to the definition of harmonious coloring. Hence the minimum number of colors in a harmonious coloring for central graph of stearing graph is 2 n .

Illustration of the above theorem:


$$
\chi_{H}\left[\mathrm{C}\left(\mathrm{~S}_{\mathrm{n}}\right)\right]=2(4)=8
$$

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