CONNECTED TOTAL DOMINATING SETS AND ITS POLYNOMIAL OF $K_n \times P_r$

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(Received On: 06-08-16; Revised & Accepted On: 30-08-16)

ABSTRACT

In this paper, we are going to study the connected total domination polynomial of $K_n \times P_r$. The connected total domination polynomial of a graph $G$ of order $n$ is defined as

$$D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)} d_{ct}(G, i) x^i,$$

where $d_{ct}(G, i)$ is the number of connected total dominating sets of $G$ with size $i$ and $\gamma_{ct}(G)$ is the connected total domination number of $G$.

Keywords: Connected total dominating set, connected total domination number, connected total domination polynomial.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph of order $n$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{ u \in V / uv \in E \}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{ v \}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a connected total dominating set of $G$ if every vertex $v \in V$ is adjacent to at least one element of $S$ and the induced sub graph $G[S]$ is connected. The connected total domination number $\gamma_{ct}(G)$ is a $\gamma$-set.

The polynomial, $D_{ct}(G, x) = \sum_{i=\gamma_{ct}(G)} d_{ct}(G, i) x^i$ is defined as connected total domination polynomial of $G$.

where $d_{ct}(G, i)$ is the number of connected total dominating sets with size $i$.

2. CONNECTED TOTAL DOMINATION POLYNOMIALS

Definition: 2.1 A graph $G$ consists of a pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set whose elements are called points (or) vertices and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges of the graph $G$.

Definition: 2.2 If $e = \{u, v\}$ is an edge of a graph $G$, written $e = uv$, we say that $e$ joins the vertices $u$ and $v$. Also we say that $u$ and $v$ are adjacent vertices, $u$ and $v$ are incident with $e$. If two vertices or not joined, then we say that they are not-adjacent.

Definition: 2.3 The graph $G$ is complete if every two distinct vertices of $G$ are adjacent. A complete graph with $n$ vertices is denoted by $K_n$.

Definition: 2.4 A walk of a graph $G$ is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, ..., v_{n-1}, x_n, v_n$ beginning and ending with points such that each line $x_i$ is incident with $v_{i-1}$ and $v_i$. A walk is called a path if all its points are distinct.

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Definition: 2.5 A subset $S$ of vertices in a graph $G$ is said to be a dominating set, if every vertex $v \in V - S$ is adjacent to atleast one element of $S$. A dominating set of $G$ is said to be a total dominating set, if every vertex $v \in V$ is adjacent to atleast one element of $S$.

Definition: 2.6 A total dominating set of $G$ is said to be a connected total dominating set, if the induced sub graph $G[S]$ of $G$ is connected.

Theorem: 2.7 Let $G = K_n \times P_2$, then the total connected domination polynomial of $G$ is,

$$D_{ct}(G, x) = nx^2[1 + x]^2(n-1) - n\left[\frac{2(n-1)}{C_1}\right]x^{2n-1} - (n-1)x^{2n} + 2x^n[1 + x]^n$$

Proof: $G_1 = K_n$ be the complete graph with $n$ vertices, $G_2 = P_2$, its product $G = G_1 \times G_2$ is given in figure 1.1.

The vertices of $G$ are denoted by $\{v_{11}, v_{12}, ..., v_{1n}, v_{21}, v_{22}, ..., v_{2n}\}$

Let, $S_i = \{V_{ki} / k = 1,2; i = 1,2, ..., n \}$ and $T_j = \{V_{jk} / j = 1,2; k = 1,2, ..., n \}$

The total connected dominating set with cardinality 2 are,

$$D_{ct}(G, 2) = \{[S_i] / i = 1,2, ..., n\}$$

Therefore, $d_{ct}(G, 2) = n$

The total connected dominating set with cardinality 3 are,

$$D_{ct}(G, 3) = \{S_i \cup \{x_j\} / i, j = 1,2, ..., n; \quad i \neq j\}$$

Therefore, $d_{ct}(G, 3) = n\left[\frac{2(n-1)}{C_1}\right]$ 

The total connected dominating set with cardinality 4 are,

$$D_{ct}(G, 4) = \{S_i \cup \{x_j, x_k\} / i, j, k = 1,2, ..., n; \quad i \neq j, k\}$$

Therefore, $d_{ct}(G, 4) = n\left[\frac{2(n-1)}{C_2}\right]$ 

The total connected dominating set with cardinality 5 are,

$$D_{ct}(G, 5) = \{S_i \cup \{x_j, x_k, x_l\} / i, j, k, l = 1,2, ..., n; \quad i \neq j, k, l\}$$

Therefore, $d_{ct}(G, 5) = n\left[\frac{2(n-1)}{C_3}\right]$ 

Proceeding in this way, we get

The total connected dominating set with cardinality $n$ are,

$$D_{ct}(G, n) = \{S_i \cup \{x_j, x_k, ..., x_t\} / i, j, k, ..., t = 1,2, ..., n; \quad i \neq j, k, ..., t\} \cup \{T_j / j = 1,2\}$$
Therefore, \( d_{ct}(G, n) = n \left[ 2(n-1)C_{n-2} \right] + 2 \)

The total connected dominating set with cardinality \( n + 1 \) are,

\[
D_{ct}(G, n + 1) = \{ \cup \{ x_j, x_k, \ldots, x_t \} / i, j, k, \ldots, t + 1 = 1, 2, \ldots, n ; \ i \neq j, k, \ldots, t + 1 \} \cup \{ T_j \cup \{ x_k \} / \{ x_k \} \notin T_j, \ j = 1, 2, \ldots, n; \ j \neq k \}
\]

Therefore, \( d_{ct}(G, n + 1) = n \left[ 2(n-1)C_{n-1} \right] + 2(nC_1) \)

The total connected dominating set with cardinality \( n + 2 \) are,

\[
D_{ct}(G, n + 2) = \{ \cup \{ x_j, x_k, \ldots, x_t+1 \} / i, j, k, \ldots, t + 2 = 1, 2, \ldots, n ; \ i \neq j, k, \ldots, t + 2 \} \cup \{ T_j \cup \{ x_k, x_l \} / \{ x_k, x_l \} \notin T_j, \ j = 1, 2, \ldots, n; \ j \neq k, l \}
\]

Therefore, \( d_{ct}(G, n + 2) = n \left[ 2(n-1)C_n \right] + 2(nC_2) \)

Proceeding in this way, we get

The total connected dominating set with cardinality \( 2n - 2 \) are,

\[
D_{ct}(G, 2n - 2) = \{ \cup \{ x_j, x_k, \ldots, x_t \} / i, j, k, \ldots, \alpha = 1, 2, \ldots, n ; \ i \neq j, \ldots, \alpha \} \cup \{ T_j \cup \{ x_k, x_l, \ldots, x_\alpha \} / \{ x_k, x_l, \ldots, x_\alpha \} \notin T_j, \ j = 1, 2, \ldots, n; \ j \neq k, l, \ldots, \alpha \}
\]

Therefore, \( d_{ct}(G, 2n - 2) = n \left[ 2(n-1)C_{2(n-2)-2} \right] + 2(nC_{n-2}) \)

The total connected dominating set with cardinality \( 2n - 1 \) are,

\[
D_{ct}(G, 2n - 1) = \{ S_i \cup \{ x_j \} / \{ x_j \} \notin S_i ; \ i = 1, 2, \ldots, n \}
\]

Therefore, \( d_{ct}(G, 2n - 1) = 2n \)

The total connected dominating set with cardinality \( 2n \) are,

\[
D_{ct}(G, 2n) = 1
\]

Therefore, \( d_{ct}(G, 2n) = 1 \)

Hence, the total connected domination polynomial of \( G \) is,

\[
D_{ct}(G, x) = nx^2 + n \left[ 2(n-1)C_1 \right] x^3 + n \left[ 2(n-1)C_2 \right] x^4 + \ldots + n \left[ 2(n-1)C_{n-2} \right] x^n + n \left[ 2(n-1)C_{n-1} \right] x^{n+1} + n \left[ 2(n-1)C_n \right] x^{n+2} + \ldots + n \left[ 2(n-1)C_{2(n-2)-2} \right] x^{2n-2} + 2nx^{2n-1} + x^{2n}
\]

\[
\Rightarrow D_{ct}(G, x) = nx^2 + n \left[ 2(n-1)C_1 \right] x^3 + n \left[ 2(n-1)C_2 \right] x^4 + \ldots + n \left[ 2(n-1)C_{n-2} \right] x^n + n \left[ 2(n-1)C_{n-1} \right] x^{n+1} + \ldots + n \left[ 2(n-1)C_n \right] x^{n+2} + 2nx^{2n-1} + x^{2n}
\]

\[
\Rightarrow D_{ct}(G, x) = nx^2 + n \left[ 2(n-1)C_1 \right] x^3 + n \left[ 2(n-1)C_2 \right] x^4 + \ldots + n \left[ 2(n-1)C_{n-2} \right] x^n + \ldots + n \left[ 2(n-1)C_{n-1} \right] x^{n+1} + \ldots + n \left[ 2(n-1)C_n \right] x^{n+2} + \ldots + n \left[ 2(n-1)C_{2(n-2)-2} \right] x^{2n-2}
\]

\[
\Rightarrow D_{ct}(G, x) = nx^2 \left[ 1 + x \right]^{2(n-1)} - n \left[ 2(n-1)C_1 \right] x^{2n-1} - (n-1)x^{2n} + 2x^n \left[ 1 + x \right]^n
\]
Theorem: 2.8 Let $G = K_n \times P_r$, then the total connected domination polynomial of $G$ is,
\[ D_{tc}(G, x) = nx'[1 + x]^{r(n-1)} - n\left[\frac{r(n-1)C_1 - r}{(n-1)x^{r-1}} - (n-1)x^r + (r-2)(n(n-1)x^r + x^{r-2}) \right]^{n-2} + x^{r-2} \] for some $r > 2$.

Proof: $K_n$ be the complete graph with $n$ vertices, $P_r$ is a path of length $r$. Then its product $G = K_n \times P_r$ is given in figure 1.2

The vertices of $G$ is denoted by $V_{ij} / i = 1,2,\ldots, r; j = 1,2,\ldots, n$)
Let, $S_i = \{V_{ki} / k = 1,2,\ldots, r; i = 1,2,\ldots, n\}$ and
$T_j = \{V_{ki} / j = 1,2,\ldots, r; k = 1,2,\ldots, n\}$

The total connected dominating set with cardinality $r$ are,
$D_{tc}(G, r) = \{S_i / i = 1,2,\ldots, n\}$

Therefore, $d_{tc}(G, r) = n$

The total connected dominating set with cardinality $r + 1$ are,
$D_{tc}(G, r + 1) = \{S_i \cup \{x_j\} / i, j = 1,2,\ldots, n; \ i \neq j\} \cup \{S_i - \{v_{ki}\} \cup \{v_{kj}\} / k = 1,2,\ldots, r; \ i, j = 1,2,\ldots, n; \ i \neq j\}$

Therefore, $d_{tc}(G, r + 1) = n\left[\frac{r(n-1)C_1}{r-1} + (r-2)[n(n-1)]\right]$

The total connected dominating set with cardinality $r + 2$ are,
$D_{tc}(G, r + 2) = \{S_i \cup \{v_{ki}, v_{kj}\} / i, j, k = 1,2,\ldots, n; \ i \neq j, k\} \cup \{S_i \cup \{v_{ki}\} \cup \{v_{kj}\} / k = 1,2,\ldots, r; \ i, j = 1,2,\ldots, n; \ i \neq j\}$

Therefore, $d_{tc}(G, r + 2) = n\left[\frac{r(n-1)C_2}{r-2} + (r-2)[n(n-1)] [(n-2)C_1]\right]$

Proceeding in this way, we get
The total connected dominating set with cardinality \((r-2)n\) are,
\[
D_{ct}(G, (r-2)n) = \left\{ S_i \cup \{x_i, x_{j_k}, ..., x_t\}/ i, j_k, ..., t = 1,2, ..., n; \ i \neq j_k, ..., t \right\}
\cup \left\{ S_j \cup \{v_{k_i}, v_j\} / k = 1,2, ..., r; \ i, j, ..., t = 1,2, ..., n; \ i \neq j, ..., t \right\}
\cup \left\{ T_j \cup \{x_k\}/ \{x_k\} \in T_j; j = 1,2, ..., r; k = 1,2, ..., n; j \neq k \right\}
\]

Therefore,
\[
d_{ct}(G, (r-2)n) = n \left[ \frac{r(n-1)C_{r(2(n-2))}}{+(r-2)[n(n-1)](n-2)rC_{(r-2)(n-r)}]} + (2n)C_1 \right]
\]

The total connected dominating set with cardinality \((r-2)n + 1\) are,
\[
D_{ct}(G, (r-2)n + 1) = \left\{ S_i \cup \{x_i, x_{j_k}, ..., x_t, x_{t+1}\}/ i, j_k, ..., t + 1 = 1,2, ..., n; \ i \neq j_k, ..., t + 1 \right\}
\cup \left\{ S_j \cup \{v_{k_i}, v_j\} / k = 1,2, ..., r; \ i, j, ..., t + 1 = 1,2, ..., n; \ i \neq j, ..., t + 1 \right\}
\cup \left\{ T_j \cup \{x_k\}/ \{x_k\} \in T_j; j = 1,2, ..., r; k = 1,2, ..., n; j \neq k \right\}
\]

Therefore,
\[
d_{ct}(G, (r-2)n + 1) = n \left[ \frac{r(n-1)C_{r(2(n-2))}}{+(r-2)[n(n-1)](n-2)rC_{(r-2)(n-r)}]} + (2n)C_1 \right]
\]

Proceeding in this way, we get

The total connected dominating set with cardinality \(rn - 2\) are,
\[
D_{ct}(G, rn - 2) = \left\{ S_i \cup \{x_i, x_{j_k}, ..., x_t, x_{t+1}\}/ i, j_k, ..., \alpha = 1,2, ..., n; \ i \neq j_k, ..., \alpha \right\}
\cup \left\{ S_j \cup \{v_{k_i}, v_j\} / k = 1,2, ..., r; \ i, j, ..., \alpha = 1,2, ..., n; \ i \neq j, ..., \alpha \right\}
\cup \left\{ T_j \cup \{x_k, x_{t_1}, ..., x_{t_2}\}/ \{x_k, x_{t_1}, ..., x_{t_2}\} \in T_j; j = 1,2, ..., r; \ k, t_1 = 1,2, ..., n; \ j \neq k, t, ..., \alpha \right\}
\]

Therefore,
\[
d_{ct}(G, rn - 2) = n \left[ \frac{r(n-1)C_{r(\alpha-2)}}{+(r-2)[n(n-1)](n-2)rC_{(r-2)(\alpha-2)}]} + (2n)C_{2n-2} \right]
\]

The total connected dominating set with cardinality \(rn - 1\) are,
\[
D_{ct}(G, rn - 1) = \left\{ S_i - \{x_i\} / \{x_i\} \in S_i, i = 1,2, ..., n \right\}
\]

Therefore, \(d_{ct}(G, rn - 1) = rn\)

The total connected dominating set with cardinality \(rn\) are,
\[
D_{ct}(G, rn) = 1
\]

Therefore, \(d_{ct}(G, rn) = 1\)

Hence the total connected polynomial of \(G\) is
\[
D_{ct}(G, x) = nx^r + \left[ n \left[ \frac{r(n-1)C_1}{+(r-2)[n(n-1)]} \right] x^{r+1} \right.
\]
\[
+ \left[ n \left[ \frac{r(n-1)C_2}{+(r-2)[n(n-1)]} \right] (n-2)C_1 \right] x^{r+2} +
\]
\[
... + \left[ n \left[ \frac{r(n-1)C_{r(2(n-2))}}{+(r-2)[n(n-1)]} \right] (n-2)C_{(r-2)(n-r)} \right] x^{r+2n} +
\]
\[
+ \left[ n \left[ \frac{r(n-1)C_{(r-2)(n-r)}]}{+(r-2)[n(n-1)]} \right] (n-2)C_{(r-2)(n-r)} \right] x^{r+2n+1} +
\]
\[
... + \left[ n \left[ \frac{r(n-1)C_{r(2^{m+1})}}{+(r-2)[n(n-1)]} \right] (n-2)C_{(r-2)(n-r)} \right] x^{r+2n+2} +
\]
\[
\left. + \left[ n \left[ \frac{r(n-1)C_{r(2^{m+2})}}{+(r-2)[n(n-1)]} \right] (n-2)C_{(r-2)(n-r)} \right] x^{r+2n+3} \right] \right] = nx^{rn-2} + nx^{rn-1} + x^{rn}
\]
\[ D_{ct}(G, x) = \left[ nx^r + n(x-1)C_{r-2} \right] x^{r+1} + n(x-1)C_{r-2} x^{r+2} + \ldots + n(x-1)C_{r-2} x^{r+n} \]
\[ + n(x-1)C_{r-2} x^{r+n+1} + \ldots + n(x-1)C_{r-2} x^{r+n+2} + nx^{r+1} + x^n \]

Source of support: Nil, Conflict of interest: None Declared

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