UNCOUNTABLY MANY POSITIVE SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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ABSTRACT
We consider the difference equation
\[ \Delta (x_n - P_n x_{n-\tau}) + q_n f(x_{n-\sigma}) = 0 \]
where \( n \geq n_0, \tau > 0, \sigma \geq 0 \) are integers.

INTRODUCTION
We are concerned with the first order neutral Delay nonlinear Difference equation
\[ \Delta (x_n - P_n x_{n-\tau}) + q_n f(x_{n-\sigma}) = 0 \]  \hspace{1cm} (1)

(H1) \( r_n \in c'((n_0, \infty), (0, \infty)), \sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty \)
(H2) \( p_n \in c((n_0, \infty), (0, \infty)), p \equiv 0 \)
(H3) \( \emptyset(\lambda) \in c'((-\infty, \infty), (0, \infty)), \emptyset(\lambda) \neq 0, x \neq 0 \)
(H4) \( f(x) \in c'((-\infty, \infty), (-\infty, \infty)), \lambda f(x) > 0, x < 0 \)
(H5) \( G(x) = \frac{F(x)}{\emptyset(\lambda)} > 0 (x \neq 0); G(x) \) is non decreasing \((0, \infty)\) and non increasing \((-\infty, 0)\)
(H6) \( g(n) \in c([n_0, \infty) \rightarrow (0, \infty)), g(n) \geq n \)
A non trivial solution \( \{x_n\} \) is said to be oscillatory if it has arbitrarily large zeros otherwise \( \{x_n\} \) is said to be non oscillatory The proof is an adaptation of that given (1) where the special case \( g(n) = n \) was consider

Lemma 1.1: (Krasnoselskii’s fixed point theorem)
Let \( X \) be a Banach space, Let \( \Omega \) be a bounded close convex subset of \( X \) and let \( s_1, s_2 \) be maps of \( \Omega \) into \( X \) such that \( s_1x + s_2y \in \Omega \) for every \( x, y \in \Omega \).

If \( s_1 \) is contractive and \( s_2 \) is completely continuous. Then the equation \( s_1x + s_2x = x \) solution in \( \Omega \)

Theorem: Suppose that there exist bounded from below and from above by the function \( u_n, v_n \in c'([n_0, \infty), (0, \infty)) \) constant \( c > 0, k_2 \geq k_1 \geq 0, n \geq n_0 + m \) such that
\[ u_n \leq v_n, n \geq n_0 \]
\[ v_n - u_n \geq k_1, n_0 \leq n \leq n_1 \]
\[ \frac{1}{u(n-\tau)} \left( u_n - k_1 \sum_{\sigma=n}^{\infty} p_n f(v_{n-\sigma}) \right) \leq u_n \leq \frac{1}{v(n-\tau)} \left( v_n - k_2 \sum_{\sigma=n}^{\infty} p_n f(u_{n-\sigma}) \right) \leq v_n \leq \frac{c}{n} \leq \frac{c}{n_1} \]
Then eq. (1) has uncountable many positive solution which are bounded by the Functions \( u, v \).
Proof: Let \( c \ (k_0, \infty), \mathbb{R} \) be the set of all continuous bounded functions with The norm \( ||x|| = \sup_{t \geq 0} |x_t| \). Then \( c([n_0, \infty),\mathbb{R}) \) is Banach space.

We define a close bounded an convex subset of \( c ([n_0, \infty), \mathbb{R}) \) as
\[
\Omega = \{ x_n \epsilon c ([n_0, \infty), \mathbb{R}) : u_0 \leq x_n \leq v_n, n \geq n_0 \}
\]

For \( k \epsilon [k_1, k_2] \) we define two maps \( s_1 \) & \( s_2: \Omega \rightarrow c ([n_0, \infty)) \) as follows
\[
s_1 x_n = \begin{cases} k + a_n x_n - \tau & n \geq n_1 \\ s_2 x_n + v_n - v_{n_1} & n_0 \leq t \leq n_1 \end{cases}
\]

We will show that for any \( x, y \epsilon \Omega \) we have \( s_1 x + s_2 y \epsilon \Omega \) for every \( x, y \epsilon \Omega \) and \( t \geq t_1 \) with regard to (4) we obtain
\[
s_1 x_n + s_2 y_n = k + a_n x_n - \tau - \sum_{n=0}^{\infty} p_s f (x_{n-s}) \geq k + u_n(t) - k \geq u_n
\]

For \( n \epsilon [n_0, n_1] \) we have
\[
s_1 x_n + s_2 y_n = s_1 x_n + s_2 y_n + v_n - v_{n_1} \leq v_n + v_n - v_{n_1} = v_n
\]

Further more for \( n \geq n_1 \) we get
\[
s_1 x_n + s_2 y_n = k + a_n x_n - \tau - \sum_{n=0}^{\infty} p_s f (x_{n-s}) \geq k + u(t) - k \geq u_n
\]

Let \( n \epsilon [n_0, n_1] \) with regards to (3) we get
\[
v_n - v_{n_1} \geq u_n, \ n_0 \leq t \leq n_1
\]

Then \( n \epsilon [n_0, n_1] \) and any \( x, y \epsilon \Omega \) we obtain
\[
s_1 x_n + s_2 y_n = s_1 x_n + s_2 y_n + v_n - v_{n_1} \leq v_n + v_n - v_{n_1} = v_n
\]

Then we have prove that \( s_1 x + s_2 y \epsilon \Omega \) for any \( x, y \epsilon \Omega \)

We will show that \( s_1 \) is a contraction mapping on \( \Omega \) for \( x, y \epsilon \Omega \) & \( n \geq n_1 \) we have
\[
||s_1 x - s_1 y|| = ||a_n||x_n + y_n|| \leq c ||x-y||
\]

This implies
\[
||s_1 x + s_2 y|| \leq c ||x+y||
\]

Also for \( n \epsilon (n_0, n_1) \) the above inequalities is valid.

We conclude that \( s_1 \) is a contraction mapping on \( \Omega \)

We now show that \( s_2 \) is completely continuous. First we show that \( s_2 \) is continuous. Let \( x^i = \{x^i_n\} \epsilon \Omega \) be such that \( x^i_n \rightarrow x_n \) as \( n \rightarrow \infty \). Because \( x \epsilon \Omega \) and \( x^i_n \epsilon \Omega \) for \( n \geq n_1 \) we have
\[
(s_2 x^i_n - s_2 x_n) \leq \sum_{n=0}^{\infty} p_s f(x_{n-s}) \leq \sum_{n=0}^{\infty} p_s f(x_{n-s})
\]

Since \( |f (x_{n-s}) - f (x_{n-s})| \rightarrow 0 \) as \( i \rightarrow \infty \) be applying the lebesgue dominant Convergence their we obtain
\[
\lim_{i \rightarrow \infty} |s_2 x^i - s_2 x| = 0
\]

This means the \( s_2 \) is continuous.

we now show that \( s_2 \) is relatively compact in \( \Omega \), it is sufficient to share

By Arzela ascolic theorem that the family of functions \( \{s_2 x: x \epsilon \Omega \} \) is uniformly
\[
\sum_{n=0}^{\infty} p_s f(x_{n-s}) < c/2
\]

The \( x \epsilon \Omega \), \( N_2 > N_1 \geq n \)
where
\[
(s_2 x) (N_2) - (s_2 x) (N_1) \leq \sum_{n=N_2}^{\infty} p_s f(x_{n-s}) + \sum_{n=N_1}^{\infty} p_s f(x_{n-s}) \leq \epsilon/2 + \epsilon/2 = \epsilon
\]
\[(s_2 x) (N_2) - (s_2 x) (N_1) \leq \sum_{n=1}^{N_2} \{ p, f(x_{n-\sigma}) \} (N_2 - N_1), \ n_1 \leq \& \leq n \]

Then there exist \( s_i = c/M \) when \( M = \max p, f(x_{n-\sigma}) \) there exist \( n_1 \leq \& \leq n \)
\[(s_2 x) (N_2) - (s_2 x) (n_1) < c \text{ if } 0 < N_2 - N_1 < s_i \]

Next we show that equation (1) has uncountable many bounded positive solution \( \Omega \).

Let \( \tilde{k} \in [k_1, k_2] \) be such that \( \tilde{k} \neq k \).

We assume that \( x, y \in \Omega \)
\[
x_n = k + an \ x_{(n-\sigma)} + \sum_{n=1}^{n} p_1 f(x_{n-\sigma}), \ n \geq n_1
\]
\[
y_n = \tilde{k} + an \ y_{(n-\sigma)} + \sum_{n=1}^{n} p_1 f(y_{n-\sigma}), \ n \geq n_1
\]

Is follow that there exist a \( n_2 > n_1 \) satisfy
\[
\sum_{n=1}^{n} p_1 [ f(x_{n-\sigma}) + f(y_{n-\sigma})] \leq |k - \tilde{k}|
\]

In order to prove that the set of bounded positive solution of equation (1) is Constant it is sufficient to very that \( x \neq y \) for \( n \geq n_2 \)

We get |\( x_n - y_n(1 + x) \) ||\( x-y || \geq |k - \tilde{k}| - \sum_{n=1}^{n} p_1 [ f(x_{n-\sigma}) + f(y_{n-\sigma})]

**Corollary:** Suppose that their exist bounded from below and from above by Function u and v \( \epsilon C ([n_0, \infty) (0, \infty)) \) that \( c > 0 \ k_2 > k_1 \geq 0, n_1 \geq n_0 + m \) such that (2) (4) holds
\[
\Delta u_n - \Delta u_n \leq 0, \ n_1 \leq n \leq n_1
\]
\[
H(t) = v_{n+1} - u_n - u_{n-1} + u_n
\]
\[
H'(t) = \Delta v_{n-1} - \Delta u_n \leq 0
\]
\[
H_u(t) = 0
\]

**Example:** \( \Delta (x_n - x_{n-1}) + \frac{1}{n} x_{n-1} = 0 \)

**Input:**
\[
x(n+1) - 2x(n) + x(n-1) + \frac{1}{n} x(n-1) = 0
\]

**Plot:**

Alternate forms:
\[
\left( \frac{1}{n} + 1 \right) x(n-1) + x(n+1) = 2x(n)
\]
\[
nx(n+1) = (-n-1) x(n-1) + 2nx(n)
\]
\[
\frac{n}{n+1} x(n-1) - 2n \ k(n) + n \ k(n+1) = 0
\]

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Value plot and recurrence plot:

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REFERENCES


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