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Further properties of *v*-continuity

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ABSTRACT

The object of the present paper is to study the basic properties of v-continuous functions.

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AMS-classification Numbers: *54C10; 54C08; 54C05.*

1. INTRODUCTION:

In 1963, Norman Levine introduced semi-continuous functions. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb defined pre-continuity in 1982. M.E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud defined semi-pre continuity in 1983. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb defined α-continuity in 1997. S. Balasubramanian, C. Sandhya and P. A. S. Vyjayanthi defined *v*-continuity in 2009. Inspired with these developments, we study some characterizations and properties of *v*-continuous functions.

2. PRELIMINARIES:

Definition 2.1: $A\subset X$ is called
(i) closed if its complement is open.
(ii) regular open[pre-open; semi-open; α-open; β-open] if $A = (cl\{A\})^o[A\subseteq (cl\{A\})^o$; $A\subseteq cl\{(A^o)\}$; $A\subseteq (cl\{(A^o)\})^o$; $A\subseteq cl\{((cl\ A)^o)\}]$ and regular closed[pre-closed; semi-closed; α-closed; β-closed]
if $A = cl\{A^o\}[cl\{(Ao)\} \subseteq A$; $(cl\ A)^o\subseteq A$; $cl\{((cl\ A)^o)\}\subseteq A$; $(cl\{(A^o)\})^o\subseteq A]$.
(iii) semi-θ-open if it is the union of semi-regular sets and its complement is semi-θ-closed.
(iv) *v*-open[rα-open] if there exists a regular open set U such that $U\subseteq A\subseteq cl\ U[U\subseteq A\subseteq \alpha\ (cl\ \{U\})]$.

Definition 2.2: A function f: $X \to Y$ is called continuous [resp: semi-; pre-; r-;rα-; α-; β-; ω-; *v*-] continuous if inverse image of every open set in Y is open[resp: semi-open; pre-open; regular-open; rα-open; α-open; β-open; ω-open; *v*-open] in X.

3. FURTHER RESULTS ON *V*-CONTINUOUS FUNCTIONS:

Theorem 3.1: The following statements are equivalent for a function f:
(1) f is *v*-continuous;
(2) $f^{-1}(F)\in v C(X)$ for every closed set $F\subset Y$;
(3) for each $x\in X$ and each closed set F in Y containing $f(x)$, there exists a *v*-closed set U in X containing x such that $f(U) \subset F$;
(4) for each $x\in X$ and each open set V in Y non-containing $f(x)$, there exists a *v*-open set K in X non-containing x such that $f^{-1}(V) \subset K$;
(5) $f^{-1}(cl\{(G)\}) \in v\ C(X)$ for every open subset G of Y;
(6) $f^{-1}(F^o)\in v\ O(X)$ for every closed subset F of Y.

Proof: (1)$\Leftrightarrow$ (2): Let F be closed in Y. Then Y - F is open in Y. By (1), $f^{-1}(Y - F) = X - f^{-1}(F) \in v\ O(X)$. We have $f^{-1}(F) \in v\ C(X)$. Reverse can be obtained similarly.

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(2)⇒(3): Let F be closed in Y containing f(x). By (2), $f^{-1}(F) \in v\,C(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$. Then $f(U) \subset F$.

(3) ⇒ (2): Let F be closed in Y and $x \in f^{-1}(F)$. From (3), there exists $U_x \in v\,C(X, x)$ such that $U_x \subset f^{-1}(F)$. We have $f^{-1}(F) = \cup_{x\in f^{-1}(F)} U_x$. Thus $f^{-1}(F)$ is v-closed.

(3) ⇔ (4): Let $V \in \sigma(Y)$ not containing f(x). Then, $Y - V \in C(Y, f(x))$. By (3), there exists $U \in v\,C(X,x)$ such that $f(U) \subset Y - V$. Hence, $U \subset f^{-1}(Y - V) \subset X - f^{-1}(V)$ and then $f^{-1}(V) \subset X - U$. Take $H = X - U$. Then $H \in v\,O(X)$ non-containing x. The converse can be shown easily.

(1) ⇔ (5): Let G be open subset of Y. Since (cl G) is closed, then by (1), $f^{-1}(cl\{(G)\}) \subset v\,C(X)$. The converse can be shown easily.

(2) ⇔ (6): It can be obtained similar as (1) ⇔ (5).

Example 1: Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b},{a, c}, X} and σ = {ϕ, {a}, {b}, {a, b}, X}. Then the identity function f: X→ X is v-continuous. But it is not regular set-connected.

Theorem 3.2: If f is v-continuous and $A \in RO(X)$, then $f_{|A}: A \to Y$ is v-continuous.

Remark 2: Every restriction of a v-continuous function is not necessarily v-continuous.

Example 3: Let X = {a, b, c, d}, τ = {ϕ, {a}, {b}, {a, b}, X} and σ = {ϕ, {a}, {a, b}, X}. The identity function f:X→ X is v-continuous, but, if A = {a, c, d} where A is not regular-open in (X, τ) and τ_A is the relative topology on A induced by τ, then $f_{|A}:(A, \tau_A) \to (X, \sigma)$ is not v-continuous.

Theorem 3.3: Let f be a function and $\Sigma = \{U_\alpha: \alpha \in I\}$ be a v-cover of X. If for each $\alpha \in I$, $f_{/U\alpha}$ is v-continuous, then f is an v-continuous function.

Proof: Let $F \in \sigma(Y)$. $f_{/U\alpha}$ is v-continuous for each $\alpha \in I$, $f_{/U\alpha}^{-1}(F) \in vO_{/U\alpha}$. Since $U_\alpha \in vO(X)$, by theorem 6.3[7] $f_{/U\alpha}^{-1}(F) \in vO(X)$ for each $\alpha \in I$. Then $f^{-1}(F) = \cup_{\alpha\in I} f_{/U\alpha}^{-1}(F) \in vO(X)$. This gives f is v-continuous.

Theorem 3.4: Let f be a function and let g: X→ X × Y be the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is v-continuous, then f is v-continuous.

Proof: Let $V \in \sigma(Y)$, then $X \times V \in \sigma(X \times Y)$. Since g is v-continuous, then $f^{-1}(V) = g^{-1}(X \times V) \in v\,O(X)$. Thus, f is v-continuous.

Theorem 3.5: Let f and g be functions. Then, the following properties hold:
(1) If f is v. c. and g is regular set-connected, then $g \bullet f$ is v. c.
(2) If f is v. c. and g is perfectly continuous, then $g \bullet f$ is v. c.

Proof: (1) Let $V \in \eta(Z)$. Since g is regular set-connected, $g^{-1}(V)$ is clopen. Since f is v-continuous, $f^{-1}(g^{-1}(V)) = (g \bullet f)^{-1}(V)$ is v-clopen. Therefore, $g \bullet f$ is v. c.
(2) can be obtained similarly.

Definition 3.2: A function f is called M-v-open if image of v-open is v-open.

Theorem 3.6: If f is surjective M-v-open [resp: M-v-closed] and g is a function such that $g \bullet f$: is v-continuous, then g is v-continuous.

Proof: Let $V \in \sigma(Z)$. Since $g \bullet f$ is v-continuous, $(g \bullet f)^{-1}(V) = f^{-1} \bullet g^{-1}(V)$ is v-open. Since f is surjective M-v-open, $f(f^{-1} \bullet g^{-1}(V)) = g^{-1}(V)$ is v-open. Therefore, g is v-continuous.

Theorem 3.7:
(i) If f is r-irresolute and contra continuous, then f is regular set-connected.
(ii) If f is contra-r-irresolute and almost continuous, then f is regular set-connected.

Theorem 3.8: If f is v-continuous, then for each point $x \in X$ and each filter base Λ in X v-converging to x, the filter base $f(\Lambda)$ is rc-convergent to f(x).

Proof: Let $x \in X$ and Λ be any filter base in X v-converging to x. Since f is v-continuous, then for any $V \in \sigma(Y)$ containing f(x), there exists $U \in v\,O(X)$ containing x such that $f(U) \subset V$. Since Λ is v-converging to x, there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is rc-convergent to f(x).

Theorem 3.9: Let f be a function and $x\in X$. If there exists $U\in RO(X)$ such that $x\in U$ and the restriction of $f_{|U}$ is v-continuous at x, then f is v-continuous at x.

Proof: Suppose that $F\in\sigma(Y)$ containing $f(x)$. Since $f_{|U}$ is v-continuous at x, there exists $V\in vO(U, x)$ such that $f(V) = (f_{|U})(V)\subset F$. Since $U\in RO(X, x)$, $V\in vO(X, x)$. This shows clearly that f is v-continuous at x.

Lemma 3.1:
(i) If V is an open set, then $sCl_\theta(V) = sCl(V)$.
(ii)If V is an regular-open set, then $sCl(V) = Int(Cl(V))$.

Theorem 3.10: For a v-continuous function f, the following conditions are equivalent:
(i) $v\mathrm{cl}\{(f^{-1}(V))\} \subseteq f^{-1}(sCl_\theta (V))$ for every open subset V of Y;
(ii) $v\mathrm{cl}\{(f^{-1}(V))\} \subseteq f^{-1}(\mathrm{scl}\{(V)\})$ for every open subset V of Y;
(iii) $v\mathrm{cl}\{(f^{-1}(V))\} \subseteq f^{-1}((\mathrm{cl}\ V)^o)$ for every open subset V of Y;
(iv) $\mathrm{cl}\{(f^{-1}(V))^o\}\subseteq f^{-1}((\mathrm{cl}\ V)^o)$ for every open subset V of Y.

Proof: (i) $\Rightarrow$ (ii) follows from Lemma 3.1(i).

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) follows from Lemma 3.1(ii).

(iii) $\Rightarrow$ (iv) Since $v\mathrm{cl}\{(f^{-1}(V))\} = f^{-1}(V) \cup \mathrm{cl}\{(f^{-1}(V))^o)\}$, it follows from (iii) that $\mathrm{cl}\{(f^{-1}(V))^o)\}\subseteq f^{-1}((\mathrm{cl}\ V)^o)$.

The next result is an immediate consequence of Theorems 3.1 and 3.4.

Theorem 3.11: Let f be a function and let S be any collection of subsets of Y containing the open sets. Then f is v-continuous iff $v\mathrm{cl}\{(f^{-1}(S))\} \subseteq f^{-1}(sCl_\theta (S))$ for every $S\in S$.

Definition 3.2: A function f is called (v, s)-continuous if for each $x\in X$ and each $V\in SO(Y, f(x))$, there exists $U\in v O(X, x)$ such that $f(U) \subset \mathrm{cl}\{V\}$.

Theorem 3.12: For a function f, the following properties are equivalent:
(1) f is (v, s)-continuous;
(2) f is v-continuous;
(3) $f^{-1}(V)$ is v-open in X for each θ-semi-open set V of Y;
(4) $f^{-1}(F)$ is v-open in X for each θ-semi-closed set F of Y.

Proof: (1) $\Rightarrow$ (2): Let $F\in\sigma(Y)$ and $x\in f^{-1}(F)$. Then $f(x) \in F$ and F is semi-open. Since f is (v, s)-continuous, there exists $U\in vO(X, x)$ such that $f(U)\subset \mathrm{cl}(F) = F$. Hence $x\in U\subset f^{-1}(F)$ which implies that $x\in v(f^{-1}(F))^o$. Therefore, $f^{-1}(F)\subset v(f^{-1}(F))^o$ and hence $f^{-1}(F) = v(f^{-1}(F))^o$. This shows that $f^{-1}(F)\in vO(X)$. It follows from Theorem 3.1, f is v-continuous.

(2) $\Rightarrow$ (3): Follows from the fact that every θ-semi-open set is the union of regular closed sets.

(3) $\Leftrightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (1): Let $x\in X$ and $V\in SO(Y, f(x))$. Since cl V is regular closed, it is θ-semi-open.

Now, put $U = f^{-1}(\mathrm{cl}\ V)$. Then $U\in v O(X, x)$ and $f(U) \subset \mathrm{cl}\ V$. This shows that f is ($v$, s)-continuous.

Theorem 3.13: For a function f, the following properties are equivalent:
(1) f is v-continuous;
(2) $f(v(\mathrm{cl}\ A)) \subset sCl_\theta (f(A))$ for every subset A of X;
(3) $v\mathrm{cl}\{(f^{-1}(B))\} \subset f^{-1}(sCl_\theta (B))$ for every subset B of Y.

Proof: (1) $\Rightarrow$ (2): Let $A \subset X$. Suppose that $x\in v\mathrm{cl}\{(A)\}$ and $G\in SO(Y, f(x))$. Since f is v-continuous, by Theorem 3.12, there exists $U\in v O(X, x)$ such that $f(U)\subset \mathrm{cl}\ G$. Since $x\in v\mathrm{cl}\{(A)\}$, $U\cap A \neq \phi$; and hence $\phi \neq f(U) \cap f(A) \subset \mathrm{cl}\ G \cap f(A)$. Therefore, $f(x) \in sCl_\theta (f(A))$ and hence $f(v\mathrm{cl}\{(A)\}) \subset sCl_\theta (f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. Then $f(v\mathrm{cl}\{(f^{-1}(B))\})\subset sCl_\theta(f(f^{-1}(B)))\subset sCl_\theta(B)$ and hence $v\mathrm{cl}\{(f^{-1}(B))\}\subset f^{-1}(sCl_\theta (B))$.

(3) $\Rightarrow$ (1): Let $V\in SO(Y, f(x))$. Since $\mathrm{cl}\{V\}\cap (Y - \mathrm{cl}\ V) = \phi$,

we have $f(x) \notin sCl_\theta$ (Y - cl V) and hence $x \notin f^{-1}(sCl_\theta(Y - cl\{V\}))$. By (3), $x \notin v cl\{(f^{-1}(Y-cl\{V\}))\}$. There exists $U \in vO(X, x)$ such that $U \cap f^{-1}(Y-cl\{V\}) = \phi$; hence $f(U) \cap (Y-cl\{V\}) = \phi$. This shows that $f(U) \subset cl\{V\}$. Therefore, f is v-continuous.

4. THE PRESERVATION THEOREMS:

Theorem 4.1: Let f be an v-continuous surjection. Then the following statements hold:
(1) if X is v-compact, then Y is S-closed[resp: nearly compact].
(2) if X is v-Lindelof, then Y is S-Lindelof[resp: nearly Lindelof].
(3) if X is countably v-compact, then Y is countably S-closed[resp: nearly countably compact].

Theorem 4.2: If f is an r-continuous and contra-continuous surjection and X is mildly compact (resp. mildly countably compact, mildly Lindelof), then Y is nearly compact (resp. nearly countably compact, nearly Lindelof) and S-closed (resp. countably S-closed, S-Lindelof).

Proof: Since f is r-continuous and contra-continuous, for $\{V_\alpha : \alpha \in I\}$ be any regular closed (respectively regular open) cover of Y, we have $\{f^{-1}(V_\alpha : \alpha \in I\}$ is a clopen cover of X and since X is mildly compact, there exists a finite subset I_o of I such that $X = \cup\{f^{-1}(V_\alpha : \alpha \in I_o\}$. Since f is surjective, we obtain $Y = \cup \{V_\alpha : \alpha \in I_o\}$. This shows that Y is S-closed (respectively nearly compact). The other proofs can be obtained similarly.

Theorem 4.3: If X is v-ultra-connected and f is v-continuous and surjective, then Y is hyperconnected.

Proof: Assume that Y is not hyperconnected. Then there exists an open set V such that V is not dense in Y. Then there exist disjoint non-empty regular open subsets B_1 and B_2 in Y, namely $(cl\ V)^o$ and Y - cl V. Since f is v-continuous and onto, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint non-empty v-open subsets of X. By assumption, the v-ultra-connectedness of X implies that A_1 and A_2 must intersect, which is a contradiction. Therefore Y is hyperconnected.

Theorem 4.4: If f is v-continuous surjection and X is v-connected, then Y is connected.

Proof: Suppose that Y is not connected space. There exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is v-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint v-open sets in X and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$, which is a contradiction for v-connectedness of X. Hence, Y is connected.

Corollary 4.1: If f is v-continuous surjection and X is r-connected, then Y is connected.

Theorem 4.5: If f is a v-continuous injection and Y is weakly Hausdorff, then X is v-T_1.

Proof: Assume Y is weakly Hausdorff. For any $x \neq y \in X$, there exists $V, W \in \sigma(Y)$ such that $f(x) \in V, f(y) \notin V, f(x) \notin W$ and $f(y) \in W$. Since f is v-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are v-open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. Hence X is v-T_1.

Corollary 4.2: If f is a r-continuous injection and Y is weakly Hausdorff, then X is v-T_1.

Corollary 4.3: If f is a v-continuous injection and Y is weakly Hausdorff, then X is semi-T_1.

Corollary 4.4: If f is a v-continuous injection and Y is weakly Hausdorff, then X is β-T_1.

5. *v*-REGULAR GRAPHS:

Recall that for a function f, $G(f) = \{(x, f(x)): x \in X\} \subset X \times Y$ is called the graph of f.

Definition 5.1: A graph $G(f)$ of a function f is said to be v-regular if for each $(x, y) \in (X \times Y) - G(f)$, $U \in vO(X, x)$ and $V \in RO(Y, y)$such that $(U \times V) \cap G(f) = \phi$.

Lemma 5.1: The following properties are equivalent for a graph $G(f)$ of a function:
(1) $G(f)$ is v-regular;
(2) for each point $(x, y) \in (X \times Y) - G(f)$, there exists $U \in vO(X, x)$ and $V \in RO(Y, y)$ such that $f(U) \cap V = \phi$.

Proof: Follows from definition 5.1 and for any $A \subset X$ and $B \subset Y$, $(A \times B) \cap G(f) = \phi$ iff $f(A) \cap B = \phi$.

Theorem 5.2: If f is v-continuous and Y is T_2, then $G(f)$ is v-regular graph in $X \times Y$.

Proof: Assume Y is T_2. Let $(x, y) \in (X\times Y) - G(f)$. It follows that $f(x) \neq y$. Since Y is T_2, there exist disjoint open sets V and W containing $f(x)$ and y, respectively. We have $((cl\ V)^o) \cap ((cl\ W)^o) = \phi$. Since f is v-continuous,

$f^{-1}((cl\ V)^o) \in vO(X, x)$. Take $U = f^{-1}((clV)^o)$. Then $f(U) \subset ((clV)^o)$. Therefore, $f(U) \cap ((clW)^o) = \phi$ and $G(f)$ is v-regular in $X\times Y$.

Corollary 5.1: If f is v-continuous and Y is r-T_2, then $G(f)$ is v-regular graph in $X\times Y$.

Corollary 5.2: If f is r-continuous and Y is T_2, then $G(f)$ is v-regular graph in $X\times Y$.

Corollary 5.3: If f is r-continuous and Y is r-T_2, then $G(f)$ is v-regular graph in $X\times Y$.

Theorem 5.3: Let f have a v-regular graph $G(f)$. If f is injective, then X is v-T_1.

Proof: Let $x\neq y\in X$. Then, we have $(x, f(y)) \in (X\times Y) - G(f)\}$. By definition 5.1, there exists $U\in vO(X)$ and $V\in RO(Y)$ such that $(x, f(y)) \in U\times V$ and $f(U) \cap V = \phi$; hence $U\cap f^{-1}(V) = \phi$. Therefore, we have $y\notin U$.
Thus, $y\in X - U$ and $x\notin X - U$. We obtain that $X - U\in v\,O(X)$. Hence X is v-T_1.

Theorem 5.4: Let f have a v-regular graph $G(f)$. If f is surjective, then Y is weakly T_2.

Proof: Let $y_1 \neq y_2 \in Y$. Since f is surjective $f(x) = y_1$ for some $x\in X$ and $(x, y_2)\in (X\times Y) - G(f)$. By definition 5.1, there exists $U\in v\,O(X)$ and $F\in RO(Y)$ such that $(x, y_2)\in U\times F$ and $f(U) \cap F = \phi$; hence $y_1\notin F$. Then $y_2\notin Y - F\in \sigma(Y)$ and $y_1\in Y - F$. Thus Y is weakly T_2.

CONCLUSION

Author studied some properties of v-continuity.

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