CLOSED (OR OPEN) SUB NEAR-FIELD SPACES OF COMMUTATIVE NEAR-FIELD SPACE OVER NEAR-FIELD

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ABSTRACT

Let $N$ be a commutative near-field space with $1 \neq 0$, and let $M$ be a proper sub near-field space of $N$. Recall that $M$ is an $n$-absorbing sub near-field space if whenever $x_1, x_2, \ldots, x_{n+1} \in M$ for $x_1, x_2, \ldots, x_{n+1} \in N$, then there are $n$ of the $x_i$’s whose product is in $M$. We define $M$ to be a semi-$n$-absorbing sub near-field space if $x_{n+1} \in M$ for $x \in N$ implies $x^n \in M$. More generally, for positive integers $m$ and $n$, we define $M$ to be close sub near-field space more specifically $(m, n)$-closed sub near-field space if $x^m \in M$ for $x \in N$ implies $x^n \in M$. A number of examples and results on closed (or open) sub near-field spaces of commutative near-field space over a near-field.

Key words: prime sub near-field space, radical near-field space, 2-absorbing sub near-field space, $n$-absorbing sub near-field space.


SECTION-1: INTRODUCTION

1.1 Definition: $n$-absorbing sub near-field space. Let $N$ be a commutative near-field space with $1 \neq 0$, $M$ be a Closed (or Open) sub near-field space of commutative near-field space $N$, and $n$ be a positive integer. $M$ is called $n$-absorbing sub near-field space of $N$ if whenever $x_1, x_2, \ldots, x_{n+1} \in M$ for $x_1, x_2, x_3, \ldots, x_{n+1} \in N$, then there are $n$ of the $x_i$’s whose product is in $M$.

1.2 Note: a 1-absorbing sub near-field space of $N$ is just prime sub near-field space.

1.3 Definition: semi $n$-absorbing sub near-field space. We define in this paper, $M$ to be a semi $n$-absorbing sub near-field space of $N$ if whenever $x_1, \ldots, x_{n+1} \in M$ for $x_1, x_2, x_3, \ldots, x_{n+1} \in N$, then there are $n$ of the $x_i$’s whose product is in $M$.

1.4 Note: clearly, an $n$-absorbing sub near-field space of $N$ is also semi $n$-absorbing sub near-field space of $N$, and a semi 1-absorbing sub near-field space is just a radical (semi prime near-field space) sub near-field space of $N$. Hence $n$-absorbing sub near-field space respectively semi $n$-absorbing sub near-field space of $N$ generalize prime respectively radical sub near-field space of $N$.

1.5 Definition: close (or open) sub near-field space. More generally, for positive integers $m$, $n$ we define $M$ to be an $(m, n)$-closed (or open) sub near-field space of $N$ if $x^m \in M$ for $x \in N \Rightarrow x^n \in M$.

1.6 Definition: semi $n$-absorbing sub near-field space. Thus $M$ is a semi-$n$-absorbing sub near-field space if and only if $M$ is an $(n+1, n)$ – closed (or open) sub near-field space of $N$.

1.7 Definition: radical sub near-field space. $M$ is a radical sub near-field space if and only if $M$ is a $(2, 1)$-closed (or open) sub near-field space. In fact, an $n$-absorbing sub near-field space is $(m, n)$-closed (or open) sub near-field space for every positive integer $m$.

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1.8 Note: clearly, a proper radical sub near-field space of N is (m, n)-closed (or open) radical sub near-field space for 1 ≤ m ≤ n. So we often assume that 1 ≤ n ≤ m.

The concept of 2-absorbing sub near-field space of N over a near-field introduced by Dr. N. V. Nagendram and extended to n-absorbing sub near-field space of N over a near-field with reference to A. Badawi’s study of 2-absorbing ideals of commutative rings. Several related concepts, such as 2-absorbing primary sub near-field space of N have been studied over a near-field and other generalizations of prime sub near-field space of N over a near-field are investigated.

SECTION-2: PROPERTIES OF CLOSED OR OPEN SUB NEAR-FIELD SPACES OF COMMUTATIVE NEAR-FIELD SPACE

In this section, we give the basic properties of semi n-absorbing sub near-field space of N over a near-field and (m, n)-closed (or open) sub near-field space of N over a near-field. We also determine when every proper sub near-field space of N over a near-field is (m, n)-closed (or open) sub near-field space of N over a near-field for positive integers m, n such that 1 ≤ m ≤ n.

2.1 Definition: Maximal sub near-field spaces. If K₁, K₂, ..., Kₙ are maximal sub near-field space of N, then K₁,..., Kₙ is an n-absorbing sub near-field space of N. The following analogous result holds for semi n-absorbing sub near-field space of N over a near-field.

2.2 Theorem: Let N be a commutative near-field space.

(a) A radical sub near-field space of N is (m, n)-closed (or open) sub near-field space of N over a near-field for all positive integers m and n.
(b) An n-absorbing sub near-field space of N is a semi n-absorbing sub near-field space i.e. (n+1, n)-closed (or open) sub near-field space of N over a near-field for every positive integer n.
(c) An (m, n)-closed (or open) sub near-field space of N over a near-field is (m', n')-closed (or open) sub near-field space of N over a near-field for positive integers m ≤ m' and n ≤ n'.
(d) An absorbing sub near-field space of N is (m, n)-closed (or open) sub near-field space of N over a near-field for a positive integer m.
(e) Let P₁, P₂, ..., Pₖ be radical sub near-field spaces of N. Then P₁, P₂, ..., Pₖ is (m, n)-closed (or open) sub near-field space of N over a near-field for every integer m ≥ 1 and n ≥ min {m, k}. In particular, P₁, P₂, ..., Pₖ is a semi k-absorbing sub near-field space of N over a near-field for a positive integer k.

Proof: It is obvious and directly follow (a), (b) and (c) from the definitions.

To prove (d): Let M be an n-absorbing sub near-field space of N for n is positive integer. Suppose that xⁿ ∈ M for x∈N and m > n an integer. Then xⁿ∈N. So M is (m, n)-closed (or open) sub near-field space of N over a near-field for every integer n. Clearly, M is (m, n)-closed (or open) sub near-field space of N over a near-field for every integer m. Proved (d).

To prove (e): Let xᵐ ∈ P₁...Pₖ for x ∈ N. Then xᵐ ∈ Pᵢ for every 1 ≤ i ≤ k, and thus x ∈ Pᵢ is a radical sub near-field space of N. Hence xⁿ ∈ P₁...Pₖ for some n ≥ min {m, k}. Proved (e).

This completes the proof of the theorem.

Note 2.3: It is for every integer n ≥ 2, there is a semi n-absorbing sub near-field space i.e. (n + 1, n)-closed or open sub near-field space of N over a near-field N i.e. neither a radical sub near-field space nor an m – absorbing sub near-field space i.e. (n + 1, n)-closed or open sub near-field space over a near-field N for any positive integer n.

Example 2.3(a): Let N = Z, n ≥ 2 an integer, and M = 2 3ⁿZ. Then M is a semi – n-absorbing sub near-field space i.e. (n+1, n)-closed or open sub near-field space over a near-field N. Let P₁ = 6Z and P₂ = 3Z, Pₙ = 3Z. In fact, M is a semi m-absorbing near-field space for every integer m ≥ n. However, (2 3ⁿ⁻¹)² ∈ M and 2 3ⁿ⁻¹ ∉ M. So M is not a radical sub near-field space of N. Moreover, 2 3ⁿ ∈ i, 3ⁿ ∉ M and 2 3ⁿ⁻¹ ∉ M. So I is not an n – absorbing sub near-field space of N but M is an (n+1)- absorbing sub near-field space of N. Note that for n = 1, M = 6Z is a semi 1-absorbing near-field space i.e. radical sub near-field space of N, but not a 1-absorbing sub near-field space i.e. prime sub near-field space of a near-field N over a near-field.

Example 2.3(b): Let N = Q[{Xₙ|n ∈ N}] and M = [{Xₙ}|n ∈ N]. Then Xⁿ⁺¹ ∈ M and Xⁿ⁺¹ ∉ M for every positive integer n. So not a semi n-absorbing sub near-field space i.e. (n+1, n)-closed or open sub near-field space over a near-field N for every positive integer n. Thus M is (m, n) - closed or open sub near-field space over a near-field N if and only if 1 ≤ m ≤ n.
Example 2.3(c): Let N be a commutative near-field space over a noetherian regular delta near-ring. Then every proper sub near-field space of N is an n-absorbing sub near-field space of N, and hence a semi n-absorbing sub near-field space of N, for some positive integer n. Thus by ([4] Th. 2.1), for every proper sub near-field space M of N, there exists a positive integer n such that M is (m, n) - closed or open sub near-field space over a near-field N if and only if 1 ≤ m ≤ n. Here note that the near-field space in (b) is not Noetherian near-field space.

Example 2.3(d): Clearly, an n-absorbing sub near-field space of N is also an (n+1) – absorbing sub near-field space of N. However, this need not be true for semi n-absorbing sub near-field spaces of a near-field space. For example, it is easily seen that M = 16Z is a semi 2-absorbing sub near-field space i.e. (3, 2) - closed or open sub near-field space of Z over a near-field N, but not a semi 3-absorbing sub near-field space i.e. (4, 3) - closed or open sub near-field space of Z over a near-field N.

Example 2.3(e): Let N be a valuation domain which is a commutative near-field space over a noetherian regular delta near-ring. Then a radical sub near-field space of N is also a prime sub near-field space of N i.e. a semi 1-absorbing sub near-field space of N, and hence a semi n-absorbing sub near-field space of N over a near-field, but not a 2-absorbing sub near-field space i.e. (3, 2) - closed or open sub near-field space of Z over a near-field N.

In general, a product of (m, n) - closed or open sub near-field space of N over a near-field need not be (m, n) - closed (example. A product of radical sub near-field spaces need not be a radical sub near-field space).

Theorem 2.4: Let N be a commutative near-field space over a near-field, and M1, M2, …, Mn+1 be sub near-field spaces of N such that M, is (m, n) - closed or open sub near-field spaces of N over a near-field for 1 ≤ i ≤ k. Hence Mn+1 is (m, n) - closed or open sub near-field spaces of N over a near-field for all positive integers m ≤ m1, …, mn1 and n ≤ n1, …, nn1. This completes the proof of the theorem.

Corollary 2.5: Let N be a commutative near-field space over a near-field, and M1, M2, …, Mk be (m, n) – closed or open sub near-field spaces of N over a near-field respectively semi n-absorbing sub near-field spaces of N over a near-field space.

(a) M1 ∩ … ∩ Mk is (m, n) - closed or open sub near-field space of N over a near-field for all positive integers m ≤ m1, …, mn1 and n ≤ n1, …, nn1.

(b) If M1, …, Mk are pair-wise co-maximal, then M1∩ … ∩ Mk is (m, n) - closed or open sub near-field space of N over a near-field.
Note 2.9: Every proper sub near-field space of a near-field space $N$ is strongly $(m, n)$ - closed or open sub near-field space of $N$ over a near-field for $1 \leq m \leq n$, a strongly $(m, n)$ - closed or open sub near-field space of $N$ over a near-field is a $(m, n)$ - closed or open sub near-field space of $N$ over a near-field, and a $(m, 1)$ - closed or open sub near-field space of $N$ over a near-field is also strongly $(m, 1)$ - closed or open sub near-field space of $N$ over a near-field.

Remark 2.10: However, a $(m, n)$ - closed or open sub near-field space of $N$ over a near-field need not be a strongly closed or open sub near-field space of $N$ over a near-field.

Example 2.11: Let $N = \mathbb{Z}[X, Y]$, $M = (X^2, 2XY, Y^2)$ and $P = \sqrt{M} = (X, Y)$. Suppose that $a_m \in M$ for $a \in \mathbb{N}$ and a positive integer. Then $a \in \sqrt{M}$, and thus $a = bX$ for some $b \in \mathbb{N}$, hence $a^2 = b^2X^2 + 2bcXY + c^2Y^2 \in M$, and thus $M$ is a $(m, 2)$ - closed or open sub near-field space of $N$ over a near-field for every positive integer $m \geq 3$. However, $P^2 \not\subset M$ since $XY \notin M$. So $M$ is not a strongly $(m, 2)$ - closed or open sub near-field space of $N$ over a near-field for any integer $m \geq 3$.

Theorem 2.11: Let $N$ be a commutative near-field space, $m$ a positive integer, $M$ a closed or open sub near-field space of $N$ over a near-field, and $P$ a sub near-field space of $N$ over a near-field.

(a) If $P^m \subseteq M$, then $2P^2 \subseteq M$.

(b) Suppose that $2 \in U(N)$. If $P^m \subseteq M$, then $P^2 \subseteq M$ i.e. $M$ is strongly $(m, 2)$ - closed or open sub near-field space of $N$ over a near-field.

Proof: (a) Let $x, y \in P$. Then $x^m, y^m, (x + y)^m \subseteq M$ and thus $x^2, y^2, (x + y)^2 \in M$ since $M$ is $(m, 2)$ - closed or open sub near-field space of $N$ over a near-field. Hence $2xy = (x + y)^2 - x^2 - y^2 \in M$, and thus $2P^2 \subseteq M$. Proved (a)

(b) is obvious follows from (a). Proved (b). This completes the proof of the theorem.

Example 2.12: Let $M$ be a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field if it is possible that $x^n \in M$ for every $x \in P = \sqrt{M}$, but $P^n \subseteq M$. It is also possible that $x^n \in M$ for every $x \in P = \sqrt{M}$, but $P^n \not\subset M$. Finally, it is possible to have $x^m \notin M$ for some $x \in \sqrt{M}$.

Example 2.13: Let $N = Z_3[X, Y, Z]$, $M = (X^2, Y^2, Z^2)$ and $P = \sqrt{M} = (X, Y, Z)$. Suppose that $a \in P$. Then $a = bX + cY + dZ$ for some $b, c, d \in \mathbb{N}$. Hence $a^2 = (b^2X^2 + c^2Y^2 + d^2Z^2) = b^2X^2 + c^2Y^2 + d^2Z^2 \in M$, and thus $M$ is $(3, 2)$ - closed or open sub near-field space of $N$ over a near-field. However, $P^3 \not\subset M$ since $XYZ \notin M$.

Example 2.14: Let $N = Z$ and $M = 16Z$. Then $M$ is $(3, 2)$ - closed or open sub near-field space of $N$ over a near-field. However, $2 \in \sqrt{M} = 2Z$, but $2^3 = 8 \not\in 1$.

Theorem 2.15: Let $N$ be a commutative near-field space, $m$ and $n$ positive integers, $M$ a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field, and $T$ a multiplicatively closed or open sub near-field space of $N$ such that $M \cap T = \phi$.

(a) $M_T$ is a $(m, n)$ - closed or open sub near-field space of $N_T$ over a near-field of $N_T$.

(b) If $n = 2$, $2 \in T$, and $P^n \subseteq M_T$ for a sub near-field space $P$ of $N_T$, then $P^2 \subseteq M_T$ i.e. $M_T$ is a strongly $(m, 2)$ - closed or open sub near-field space of commutative near-field space $N_T$ over a near-field.

Proof: To prove (a): Let $x^m \in M_T$ for $x \in N_T$. Then $x = rt$ for some $r \in N$ and $t \in T$, and thus $x^m = r^m t^m = i/s$ for some $i \in M$ and $s \in T$. Hence $r^m s^m = i/s \in M$ for some $z \in T$, and thus $(rs)^m \in M$. Hence $(rs)^m \in M$ since $M$ is $(m, n)$ - closed or open sub near-field space of $N_T$. The “in particular” statement is clear. Proved (a).

To prove (b): Suppose that $P^n \subseteq M_T$ for a sub near-field space $P$ of $N_T$. Then $2 \in U(N_T)$ since $2 \in T$, and thus $P^2 \subseteq M_T$. Proved (b).

This completes the proof of the theorem.

Corollary 2.16: Let $N$ be a commutative near-field space, $M$ be a proper sub near-field space of $N$, and $m$ and $n$ positive integers. Then $M$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N_T$ over a near-field if and only if $M_T$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N_T$ over a near-field for every prime or maximal sub near-field space of $N$ containing $M$. In particular, $M$ is a semi n-absorbing sub near-field space if and only if $M$ is locally a semi n-absorbing sub near-field space of $N$ over a near-field.

Proof: $\Rightarrow$ is obvious.
Let $x^m \in M$ for $x \in N$, $P = \{ r \in N \mid rx^m \in M \}$ a sub near-field space of $N$ and $S$ be a prime sub near-field space of $N$ with $M \subseteq S$. Then $(x/1)^m \in M$ since $M$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field. Thus $tx^m \in M$ for some $t \in N/S$. So $P \subseteq S$. Clearly, $P \not\subseteq Q$ for every prime sub near-field space $Q$ of $N$ with $M \not\subseteq Q$. Hence $P = N$, so $x^m \in M$. Thus $M$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field. The “in particular” statement is clear. This completes the proof of the theorem.

**Corollary 2.17:** Let $N$ and $S$ be commutative near-field spaces, $m$ and $n$ positive integers, and $f \colon N \to S$ a homomorphism.

(a) If $P$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field respective semi $n$-absorbing sub near-field space of $S$, then $f^{-1}(P)$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field respective semi $n$-absorbing sub near-field space of $N$.

(b) If $f$ is surjective and $M$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field respective semi $n$-absorbing sub near-field space of $N$ containing $\ker f$, then $f(M)$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field respective semi $n$-absorbing sub near-field space of $N$.

**Corollary 2.18:** Let $m$ and $n$ be positive integers.

Let $N \subseteq S$ be an extension of commutative near-field spaces. If $P$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field respective semi $n$-absorbing sub near-field space of $S$, then $P \cap N$ is a $(m, n)$-closed or open sub near-field space of commutative near-field space $N$ over a near-field respective semi $n$-absorbing sub near-field space of $N$.

**Note 2.19:** A sub near-field space $N \times T$ has the form $M \times P$ for a sub near-field space of $N$ and $P$ is a sub near-field space of $T$.

**Remark 2.20:** A sub near-field space $S$, it will be convenient to define the improper sub near-field space $S$ to be a $(\omega, 1)$ - closed or open sub near-field space $S$ of commutative near-field space $N$ over a near-field.

**Theorem 2.21:** Let $N$ and $T$ be commutative near-field spaces, $M$ be a $(m_1, n_1)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field and $P$ a $(m_2, n_2)$ - closed or open sub near-field space of $T$. Then $M \times P$ is a $(m, n)$ - closed or open sub near-field space of $N \times T$ for all positive integers $m \leq \min\{m_1, n_1\}$ and $n \geq \max\{n_1, n_2\}$.

**Theorem 2.22:** Let $N$ be a commutative near-field space and $n$ a $+$ ve integer. Every proper sub near-field space of a commutative near-field space $N$ is a prime sub near-field space if and only if $N$ is a near-field space over a near-field.

Every proper sub near-field space of $N$ is a radical near-field space if and only if $N$ is Von Neumann regular sub near-field space. Every proper sub near-field space of $N$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(a) Every proper sub near-field space of $N$ is a prime sub near-field space if and only if $N$ is a near-field space over a near-field.

(b) Every proper sub near-field space of $N$ is a radical sub near-field space if and only if $N$ is Von Neumann regular near-field space.

(c) If every proper sub near-field space of $N$ is an $n$ – absorbing sub near-field space, then $\dim(N) = 0$ and $N$ has at most $n$ maximal sub near-field spaces.

**Proof:** is obvious.

**Theorem 2.23:** Let $N$ be a commutative near-field space and $m$ and $n$ integers with $1 \leq n \leq m$. Then the following statements are equivalent.

(a) Every proper sub near-field space of $N$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $\dim(N) = 0$ and $\omega^n = 0$ for every $\omega \in \Nil(N)$.

**Proof:** To prove (a) $\Rightarrow$ (b): Let $\omega \in \Nil(N)$. Then $\omega^n N$ ia a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field. So $\omega^n \in \omega^n N$. Thus $\omega^n = \omega^n z$ for some $z \in N$. Hence $\omega^n(1 - \omega^{m-n}z) = 0$, and thus $\omega^n = 0$ since $1 - \omega^{m-n}z \notin U(N)$ because $\omega^n - \omega^n z \notin \Nil(N)$ since $m > n$. Suppose, by way of contradiction, that $\dim(N) \geq 1$. Then there exists prime sub near-field spaces $S \subseteq Q$ of $N$. Let $x \in Q \setminus S$. As above, $x^m \in \omega^n N$. So $x^m = x^m y$ for some $y \in N$. Thus $x^m(1 - x^{m-n}y) = 0 \in S$, and hence $1 - x^{m-n}y \in S \subseteq Q$ since $x \in Q \setminus S$. But then $1 \in Q$ since $x^{m-n}y \in Q$, a contradiction. Thus $\dim(N) = 0$. Proved (a) $\Rightarrow$ (b).
To prove (b) $\Rightarrow$ (a): Let M be a proper sub near-field space of N, and assume that $x^m \in M$ for $x \in N$. Then N is a regular near-field space since $\dim(N) = 0$, and thus $x = eu + \omega$ for some idempotent $e \in N$, $u \in U(N)$, and $\omega \in Nil(N)$. If $n = 1$, then N is reduced, and thus N is Von Neumann regular near-field space since $\dim(N) = 0$. In this case, every proper sub near-field space of N is a radical sub near-field space, and hence M is (m, n) - closed or open sub near-field space of N over a near-field. Thus we may assume that $n \geq 2$. Let $k \geq n$. So $\omega^k = 0$. Then $x^k = (eu + \omega)^k = eu^k + ku^k \omega + \ldots + ku^{k-1} \omega u + \ldots + ku \omega^k + \omega^k u + \omega^k$.

**Theorem 2.24**: Let N be a commutative near-field space over a near-field and n a positive integer. Then the following statements are equivalent.

(a) Every proper sub near-field space of N is (m, n) - closed or open sub near-field space of commutative near-field space over a near-field.

(b) There is an integer $m > n$ such that every proper sub near-field space of N is (m, n) - closed or open sub near-field space of commutative near-field space of N over a near-field.

(c) for every proper sub near-field space of N there is an integer $m_1 > n$ such that M is (m, n) - closed or open sub near-field space of commutative near-field space of N over a near-field.

(d) Every proper sub near-field space of N is a semi n-absorbing sub near-field space i.e. (n+1, n) - closed or open sub near-field space of commutative near-field space of N over a near-field.

(e) $\dim(N) = 0$ and $\omega^n = 0$ for every $\omega \in Nil(N)$.

**Proof**: Is obvious that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (e) and from theorem 2.15 (e) $\Rightarrow$ (a) for $m > n$ and the fact that every proper sub near-field space is (m, n) - closed or open sub near-field space of commutative near-field space of N over a near-field for 1 $\leq m \leq n$. This completes the proof of the theorem.

**Corollary 2.25**: Let N be a commutative near-field space and n a positive integer. Then the following statements are equivalent.

(a) Every proper sub near-field space of N is radical sub near-field space.

(b) Every proper sub near-field space of N is (m, n) - closed or open sub near-field space of commutative near-field space over a near-field for all positive integers m, n.

(c) There is a positive integer n such that every proper sub near-field space M of N is (m, n) - closed or open sub near-field space of commutative near-field space of N over a near-field for $m \geq n$.

(d) There is a positive integer n such that every proper sub near-field space M of N is (m, n) - closed or open sub near-field space of commutative near-field space of N over a near-field for $m_1 > n$.

(e) There is a positive integer n such that every proper sub near-field space M of N is a semi n – absorbing sub near-field space i.e. (n+1, n) - closed or open sub near-field space of commutative near-field space of N over a near-field.

(f) N is a Von Neumann regular near-field space.

**Proof**: Is obvious that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (f) and since a reduced commutative near-field space N with $\dim(N) = 0$ is Von Neumann regular near-field space. Also (f) $\Rightarrow$ (a) by theorem 2.22. The "moreover"statement holds since an integral domain is Von Neumann regular near-field space if and only if it is a near-field over a near-field. This completes the proof of the theorem.

**Corollary 2.26**: Let N be a reduced commutative near-field space and n a positive integer. Then every proper sub near-field space of N is an n-absorbing sub near-field space of N if and only if N is isomorphic to the direct product of at most n near-field spaces over a near-field.

**Note 2.27**: Let N be a commutative Noetherian near-field space. Then every proper sub near-field space of N is an n-absorbing sub near-field space, and thus a semi n-absorbing sub near-field space i.e. (n+1, n) - closed or open sub near-field space of commutative near-field space of N over a near-field for positive integer n. However, if there is a fixed positive integer n such that every proper sub near-field space of N is a semi n-absorbing sub near-field space of N, then $\dim(N) = 0$.

**SECTION 3. PRINCIPAL SUB NEAR-FIELD SPACES OF COMMUTATIVE NEAR-FIELD SPACE**

In this section, we specialize to the case of principal sub near-field space of N over a near-field in integral domains. For an integral domain N, we determine $N(M) = ((m, n) \in N \times N / M)$ is (m, n)-closed or open sub near-field space of N over a near-field for $M = p_{1}^{k_{1}} \ldots p_{l}^{k_{l}} N$, where $p_{1}, \ldots, p_{l}$ are non-associate prime sub near-field space of N over a near-field and $k_{1}, k_{2}, \ldots, k_{l}$ are positive integers.

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Theorem 3.1: Let $N$ be an integral domain, $m$ and $n$ integers with $1 \leq n \leq m$, and $M = p^kN$, where $p$ is a prime element of $N$ and $k$ is a $\pm$ve integer. Then the following statements are equivalent.

(a) $M$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $k = ma + r$, where $a$ and $r$ are integers such that $a \geq 0$, $1 \leq r \leq n$, $a(m \mod n) + r \leq n$, and if $a \neq 0$, then $m = n + c$ for some integer $c$ with $1 \leq c \leq n - 1$.

(c) If $m = bn + c$ for integers $b$ and $c$ with $b \geq 2$ and $0 \leq c \leq n - 1$, then $k \in \{1, 2, ..., n\}$. If $m = n + c$ for an integer $c$ with $1 \leq c \leq n - 1$, then $k \in \bigcup_{h=1}^{n} \{mi + h \mid i \in Z \text{ and } 0 \leq ic \leq n - h\}$.

Proof: is obvious.

Theorem 3.2: Let $N$ be an integral domain, $n +ve$ integer, and $M = p^kN$, where $p$ is a prime element of $N$ and $k$ is a $+ve$ integer. Then the following statements are equivalent.

(a) $M$ is a semi $n$-absorbing sub near-field space of commutative near-field space $N$ over a near-field i.e. $(n+1, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $k = (n + 1)a + r$, where $a$ and $r$ are integers such that $a \geq 0$, $1 \leq r \leq n$, and $a + r \leq n$.

(c) $k \in \bigcup_{j=1}^{n} \{(n + 1)i + h \mid i \in Z \text{ and } 0 \leq i \leq n - h\}$ for every $1 \leq j \leq i$ moreover, $\{k \in N| p^kN \text{ is } (n+1, n)\}$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field $\} = n(n+1)/2$.

Proof: is obvious.

Corollary 3.3: Let $N$ be an integral domain, $M = p^kN$, where $p$ is a prime element of $N$ and $k$ is a positive integer. Then $M$ is a semi 2-absorbing sub near-field space of commutative near-field space $N$ over a near-field if and only if $k \in \{1, 2, 4\}$.

Note 3.3(a): This can be extended to product of prime powers of sub near-field spaces of $N$. If $p_1, p_2, ..., p_n$ are non associate prime elements of $N$ and $k_1, k_2, ..., k_n$ are positive integers, and $n$ a positive integer. Then $p_1^{k_1} \cap p_2^{k_2} \cap ... \cap p_n^{k_n} = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_n^{k_n}$ for all positive integers $k_1, k_2, ..., k_n$.

Note 3.3(b): $p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_n^{k_n}$ is an $m$-absorbing sub near-field space of $N$ if and only if $m \geq k_1 + k_2 + ... + k_n$.

Theorem 3.4: Let $N$ be an integral domain, $m$ and $n$ a positive integers with $1 \leq n \leq m$, and $M = p^kN = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_n^{k_n}$, $p_1, p_2, ..., p_n$ are non associate prime elements of $N$ and $k_1, k_2, ..., k_n$ are positive integers. Then the following statements are equivalent.

(a) Let $M$ be $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $p^kN$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field for every $1 \leq j \leq i$.

(c) if $m = bn + c$ for integers $b$ and $c$ with $b \geq 2$ and $0 \leq c \leq n - 1$, then $k_j \in \{1, 2, 3, ..., n\}$ for every $1 \leq j \leq i$. If $m = n + c$ for an integer $c$, $1 \leq c \leq n - 1$, then $k_j \in \bigcup_{h=1}^{n} \{mv + h \mid v \in Z \text{ and } 0 \leq vc \leq n - h\}$ for every $1 \leq j \leq i$.

Proof: To prove (a) $\Rightarrow$ (b): Let $M_j = p^kN$ for $x \in N$. Let $y = x(p_1^{k_1} \cdot \cdots \cdot p_i^{k_i})/p_j^{k_j} \in N$. They $y^m \in M$, and hence $y^m \in M$, since $M$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field for every $1 \leq j \leq i$. Proved (1) $\Rightarrow$ (2).

To prove (b) $\Rightarrow$ (a): obvious and clear since $p_1^{k_1}N \cap \cdots \cap p_n^{k_n}$. Proved (b) $\Rightarrow$ (a). And is clear and obvious (b) $\Rightarrow$ (c). This completes the proof of the theorem.

Corollary 3.5: Let $N$ be an principal sub near-field space, $M$ be a proper sub near-field space of $N$, and $m$ and $n$ integers with $1 \leq n \leq m$, Then the following statements are equivalent.

(a) Let $M$ be $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $M = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_n^{k_n} = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_n^{k_n}$, $p_1, p_2, ..., p_n$ are non associate prime elements of $N$ and $k_1, k_2, ..., k_n$ are positive integers. One of the following holds good.

(i) if $m = bn + c$ for integers $b$ and $c$ with $b \geq 2$ and $0 \leq c \leq n - 1$, then $k_j \in \{1, 2, 3, ..., n\}$ for every $1 \leq j \leq i$.

(ii) If $m = n + c$ for an integer $c$, $1 \leq c \leq n - 1$, then $k_j \in \bigcup_{h=1}^{n} \{mv + h \mid v \in Z \text{ and } 0 \leq vc \leq n - h\}$ for every $1 \leq j \leq i$. 

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Corollary 3.6: Let $N$ be an integral domain, $M = p_1^{k_1}, p_2^{k_2}, \ldots, p_k^{k_k}$, where $p_1, p_2, \ldots, p_k$ are non-associate prime elements of $N$ and $k_1, k_2, \ldots, k_k$ are positive integers, and $n$ a positive integer. Then the following statements are equivalent.

(a) Let $M$ be semi $n$-absorbing sub near-field space i.e. $(n+1, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $k_j \in \bigcup_{h=1}^n \{(n+1)v + h \mid v \in Z \text{ and } 0 \leq v < n-h \}$ for every $1 \leq j \leq i$.

Corollary 3.7: Let $N$ be a principal sub near-field space, $M$ a proper sub near-field space of $N$, and $n$ a positive integer. Then the following statements are equivalent.

(a) Let $M$ be semi $n$-absorbing sub near-field space i.e. $(n+1, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) $M = p_1^{k_1}, p_2^{k_2}, \ldots, p_k^{k_k}N$, where $p_1, p_2, \ldots, p_k$ are non-associate prime elements of $N$ and $k_1, k_2, \ldots, k_k$ are positive integers, and $k_j \in \bigcup_{h=1}^n \{(n+1)v + h \mid v \in Z \text{ and } 0 \leq v < n-h \}$ for every $1 \leq j \leq i$.

Theorem 3.8: Let $N$ be an integral domain, $m$ and $n$ a positive integers with $1 \leq n \leq m$, and $M = p^kN$, where $p$ is prime element of $N$ and $k$ is a positive integer. Then the following statements are equivalent.

(a) Let $M$ be $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(b) Exactly one of the following statements holds good.

(i) If $1 \leq k \leq n$.

(ii) there is a $+ve$ integer $a$ such that $k = ma + r = ma + r = na + d$ for integers $r$ and $d$ with $1 \leq r, d \leq n-1$.

(iii) There is a $+ve$ integer $a$ such that $k = ma + r = n(a + 1)$ for integer $r$ with $1 \leq r \leq n-1$.

Proof: To prove (a) $\Rightarrow$ (b): Suppose that $M$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field. Then $k = ma + r$, where $a$ and $r$ are integers such that $a \geq 0$, $1 \leq r \leq n$, $a \mod n + r \leq n$ and if $a \neq 0$, then $m = n + c$ for an integer $c$ with $1 \leq c \leq n - 1$. Thus if $a = 0$, then $1 \leq k \leq n$. Hence assume that $a \neq 0$. Note that $m \mod n = c$. Since $c < 0$ and $ac + r \leq n$, we conclude that $1 \leq r \leq n$. Since $k = ma + r$ and $m = n + c$, we have $k = (n + c) + r = ma + ac + r$. Let $d = ac + r$. Then $d \leq n$. If $d < n$, then $k = ma + r = ma + d$, where $1 \leq r, d \leq n-1$. Then $k = ma + r = n(a + 1)$, where $1 \leq r \leq n - 1$. Proved (a) $\Rightarrow$ (b).

To prove (b) $\Rightarrow$ (a): Suppose that $1 \leq k \leq n$. It is clear that $M$ is a $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field. Next, suppose that there is an integer $a \geq 1$ such that $k = ma + r = na + d$, where $1 \leq r, d \leq n - 1$. Then $m = n + (d - r)/a$, and thus $m = n + c$ for an integer $c$ with $1 \leq c \leq n - 1$. Hence $M$ is $(m, n)$ - closed or open sub near-field space $N$ over a near-field. Finally, suppose that there is an integer $a \geq 1$ such that $k = ma + r = n(a + 1)$, where $1 \leq r \leq n - 1$. Then $m = n + (n - r)/a = n + c$ for an integer $c$ with $1 \leq c \leq n - 1$, and thus $M$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field. This completes the proof of the theorem.

Theorem 3.9: Let $a, d, m, n, r$ and $w$ be positive integers $1 \leq r \leq m$, $1 \leq w \leq n < m$, and $1 \leq d \leq a$.

(a) If $ma + r = na + w$, then $1 \leq r \leq w < n$ and $1 \leq a < n$

(b) If $ma + r = n(a + 1)$, then $1 \leq r \leq n$ and $1 \leq a < n$

(c) If $ma + r = n(a + 1) + d$, then either $m = n + 1$ or $1 \leq a < n$.

Proof: To prove (a): Suppose that $ma + r = na + w$. Then $w - r = a(m - n) > 0$ and $1 \leq w \leq n$. Thus $1 \leq r \leq w < n$, and hence $0 < w - r < n$. Thus $a = (w - r)/(m - n)$ since $0 < w - r < n$ and $m - n \geq 1$. Proved (a).

To prove (b): Suppose that $ma + r = n(a + 1)$. Then $n - r = a(m - n) > 0$. Thus $1 \leq r < n$, and $a = (n - r)/(m - n)$ since $0 < n - r < n$ and $m - n \geq 1$. Proved (b).

To prove (c): Suppose that $ma + r = n(a + 1) + d$ and $a \geq n$. Then $0 < m - n = a(m - n)/a = (n + d - r)/a = n/a + d/a - r/a < 2$ since $1 < n \leq a$. Thus $m - n = 1$. Proved (c).

This completes the proof of the theorem.

Theorem 3.10: Let $N$ be an integral domain, $n$ a positive integer, and $M = p^kN$, where $p$ is prime element of $N$ and $k$ is a positive integer. Let $m$ be a positive integer and $n$ be the smallest $+ve$ integer such that $M$ is $(m, n)$ - closed or open sub near-field space of commutative near-field space $N$ over a near-field.

(a) If $m \geq k$, then $m = k$.

(b) Let $m < k$ and write $k = ma + r$, where $a$ is a $+ve$ integer and $0 \leq r \leq m$.

(i) If $r = 0$, then $n = m$. 

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(ii) If \( r \neq 0 \) and \( a \geq m \) then \( n = m. \)

(iii) If \( r \neq 0 \) and \( a < m \) and \((a + 1)k\), then \( n = k\(a + 1\). \)

(iv) If \( r \neq 0 \) and \( a < m \) and \((a + 1)k\), then \( n = [k/(a + 1)] + 1. \)

**Proof:** To prove (a): If \( m \geq k \), then \( p^n \in M \). So \( n \geq k \). Clearly, \( M \in (m, k) \) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. So \( n = k \) is the smallest integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field when \( m \geq k \). Proved (a).

To prove (b): Assume that \( m > 1 \) and \( n \leq m \) by the above (a) comments.

To prove (i): Suppose that \( r = 0 \). Then \( M \) is not \((m, m - 1)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field since \((p^a)^m = p^k \in M \) and \((p^a)^{m - 1} = p^{ma - a} \notin M \). Thus \( n = m \) since \( M \) is \((m, m)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (i).

To prove (ii): Suppose that \( r \neq 0 \) and \( a \geq m \). If \( n \neq m \) then \( n < m < k \). Thus either \( k = ma + r = na + d \) or \( k = ma + r = n(a + 1) \), where \( 1 \leq r, d < n \). Hence \( a < n < m \) which is a contradiction to \( n \neq m \). So \( n = m \). Proved (ii).

To prove (iii): Suppose that \( r \neq 0 \), \( a < m \) and \((a + 1)k\). Let \( i = k/(a + 1) \). Then \( k = ma + r = i(a + 1) \) with \( 1 \leq i < m \). So \( i \leq r < i \). \( M \) is \((a, i)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field if it is clear that \( i \) is the smallest such positive integer. Thus \( n = i = k/(a + 1) \). Proved (iii).

To prove (iv): Suppose that \( r \neq 0 \), \( a < m \), and \((a+1)\) does not divide \( k \). Let \( i = [k/(a + 1)] \). Then \( k = ma + r = i(a + 1) + d \), where \( 1 \leq d \leq a \) and \( 1 \leq i \leq m \). Thus either \( m = i + 1 \) or \( 1 \leq d \leq a < i \). Let us first suppose that \( m = i + 1 \). Since \((a + 1)k\), \( k \neq i + 1 \) and thus \( M \) is not \((m, i)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Hence \( n = m = i + 1 = [k/(a + 1)] + 1 \) is the smallest positive integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Further suppose that \( 1 \leq d \leq a < i \) and \( m \neq i + 1 \). So, \( i + 1 < m \). Since \( k = i(a + 1) + d \), we have \( k = (i + 1)a + i + d - a \). Let \( j = i + d - a \in Z \). Then \( 1 \leq j \leq i \) since \( 1 \leq d \leq a < i \). Thus \([k/(a + 1)] \) = \( a \). Since \( k = ma + r = (i + 1)a + j \) with \( 1 \leq j \leq i \), we have \( 1 \leq r < j \leq i \). Hence \( M \) is \((m, i + 1)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Since \((a + 1)\) does not divide \( k \), we have \( k \neq i \) \((a + 1)\), and thus \( M \) is not \((m, i)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Hence \( n = i + 1 = [k/(a + 1)] + 1 \) is the smallest positive integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (iv).

This completes the proof of the theorem.

**Note 3.10 (a):** For fixed positive integers \( n \) and \( k \), we determine the largest positive integer \( m \) \((\infty)\) such that \( M = p^kN \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. If \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field for every positive integer \( m \), we will say that \( M \) is \((\infty, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field.

**Theorem 3.11:** Let \( N \) be an integral domain, \( n \) a positive integer, and \( M = p^kN \), where \( p \) is prime element of \( N \) and \( k \) is a positive integer.

(a) If \( n \geq k \), then \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field.

(b) Let \( n < k \) and write \( k = na + r \), where \( a \) is a positive integer and \( 0 \leq r \leq n \). let \( m \) be the largest positive integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field.

(i) If \( a > n \), then \( m = n \)

(ii) If \( a = n \) and \( r = 0 \), then \( m = n + 1. \)

(iii) If \( a = n \) and \( r \neq 0 \), then \( m = n. \)

(iv) If \( a < n, r = 0 \) and \((a - 1)k\), then \( m = k\(a - 1\) - 1. \)

(v) If \( a < n \) and \( r = 0 \), and \((a - 1)k\), then \( m = [k\(a - 1\)]. \)

(vi) If \( a < n \) and \( r \neq 0 \), and \( a\k\), then \( m = [k\(a - 1\)] - 1. \)

(vii) If \( a < n, r \neq 0 \), and \( a\k\), then \( m = [k\a - 1]. \)

**Proof:** To prove (a): Let \( x^n \in M \) for \( x \in N \) and \( m \) a positive integer. Then \( p|x^n \). So \( p|x \) since \( p \) is prime. Thus \( p^a|x^a \). So \( x^a \in M \) since \( n \geq k \). Hence \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (a).
To prove (b): by the above comments, \( m \geq n \). Suppose that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field and \( m > n \). If \( r = 0 \), then \( k = m(a – 1) + w = na \), where \( 1 \leq w < n \) and \( a – 1 < n \). If \( r \neq 0 \), then \( k = ma + d = na + r \), where \( 1 \leq d < r < n \) and \( a < n \). Proved (b).

To prove (i): Suppose that \( a > n \). If \( m \neq n \), then \( m > n \). So either \( a – 1 < n \) or \( a < n \) by the above comments. In either case, \( a \leq n \), a contradiction. Thus \( m = n \). proved (i).

To prove (ii): Suppose that \( a = n \) and \( r = 0 \). So \( k = n^2 \) and \( n \geq 2 \) since \( n < k \). Note that \( (p^n)^{n+1} \in M \Rightarrow a (n + 1) \geq k = n^2 \Rightarrow a \geq n \Rightarrow an \geq n^2 = k \Rightarrow (p^n)^{n+1} \in M \). So \( M \) is \((n+1, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. However, \( M \) is not \((n+2, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field since \((p^n)^{n+2} \notin M \). Thus \( m = n + 1 \). Proved (ii).

To prove (iii): Suppose that \( a = n \) and \( r \neq 0 \). If \( m > n \), then \( a < n \) by the above comments, is a contradiction \( \therefore m = n \). Proved (iii).

To prove (iv): Suppose that \( a < n \), \( r = 0 \), and \( (a – 1) \) does not divides \( k \). Let \( f = k / (a – 1) \). So \( k = f(a – 1) \) and \( a < n < f \). Hence \( M \) is \((f, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Note that by a contradiction of \( f \), if \( k \) is the largest +ve integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (iv).

To prove (v): Suppose that \( a \leq n \), \( r = 0 \), and \( (a – 1) \) does not divides \( k \). Let \( f = k / (a – 1) \). So \( k = f(a – 1) + d \) and \( 1 \leq d < a – 1 \). Since \( a \leq n < f \) we have \( 1 \leq d < a – 1 < f \). Since \( k = f(a – 1) + d = na \) with \( 1 \leq d < f \), we have \( d < n \). Hence \( M \) is \((f, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Note that by a contradiction of \( f \), if \( k = f(a – 1) + c \) for some \( 1 \leq c < a – 1 \), then \( i < f \). Thus \( m = f = [k(a – 1)] \) is the largest +ve integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (v).

To prove (vi): Suppose that \( a < n \), \( r \neq 0 \), and \( a \) does not divides \( k \). Let \( f = k / a \). So \( k = fa \) and \( f \geq n + 1 \). Then \( M \) is not \((f, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Let us assume that \( f – 1 > n \). Thus \( k = fa = f(a – 1 + 1) = a + a \). Since \( a < n < f – 1 \) and \( k = f(a – 1) + d = na + r \). We conclude that \( M \) is \((f, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Note that by construction of \( f \), if \( k = ia + c \) for some \( 1 \leq c < a \), then \( i < f \). Thus \( m = f = [k(a – 1)] \) is the largest +ve integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (vi).

To prove (vii): Suppose that \( a < n \), \( r \neq 0 \), and \( a \) does not divides \( k \). Let \( f = k / a \). So \( k = fa + d \), where \( 1 \leq d < a \). Since \( a < n < f \), we have \( 1 \leq d < a < f \). Since \( k = fa + d = na + r \) and \( 1 \leq d < f \), we have \( d < n \). Thus \( M \) is \((f, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Note that by construction of \( f \), if \( k = ia + c \) for some \( 1 \leq c < a \), then \( i < f \). Thus \( m = f = [k/a] \) is the largest positive integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field. Proved (vii). This completes the proof of the theorem.

**Theorem 3.12:** Let \( N \) be an integral domain and \( M = p_1^{k_1}, p_2^{k_2}, \ldots, p_k^{k_k}N \), where \( p_1, p_2, \ldots, p_k \) are non associate prime elements of \( N \) and \( k_1, k_2, \ldots, k_k \) are positive integers.

(a) Let \( m \) be a positive integer. If \( n_j \) is the smallest positive integer such that \( p_j^{n_j}N \) is \((m, n_j)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field for \( 1 \leq j \leq i \), then \( n = \max \{n_1, n_2, \ldots, n_i\} \) is the smallest positive integer such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field.

(b) Let \( n \) be a positive integer. If \( m_j \) is the largest positive integer (or \( \infty \)) such that \( p_j^{m_j}N \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field for \( 1 \leq j \leq i \), then \( m = \min \{m_1, m_2, \ldots, m_i\} \) is the largest positive integer (or \( \infty \)) such that \( M \) is \((m, n)\) - closed or open sub near-field space of commutative near-field space \( N \) over a near-field.

**Proof:** Is obvious.

**SECTION-4: GENERAL RESULTS ON CLOSED OR OPEN SUB NEAR-FIELD SPACES OF COMMUTATIVE NEAR-FIELD SPACE.**

In this section, we continue the study of \((m, n)\)-closed or open sub near-field space of \( N \) over a near-field and give several examples to illustrate earlier results. For a proper sub near-field space \( M \) of \( N \) over a near-field we investigate
the two functions $f_t$ and $g_t$ defined by $f_t(m) = \min \{ n/M \text{ is } (m, n)\text{-closed or open sub near-field space of } N \}$ and $g_t(n) = \sup \{ m/M \text{ is } (m, n)\text{-closed or open sub near-field space of } N \}$.

We assume throughout that all closed or open sub near-field space of $N$ are commutative with $1 \neq 0$ and that $f_t(1) = 1$ for all near-field homomorphism $f_t : N \to S$. For such a near-field space $N$, $\dim(N)$ denotes the Krull dimension of $N$, $\sqrt{\forall M}$ denotes the radical sub near-field space of a near-field space $M$ of $N$, and $\text{nil}(N)$, $\text{Z}(N)$, and $U(N)$ denote the set sub near-field space nilpotent elements, zero divisors, and units of $N$, respectively; and $N$ is reduced $\text{nil}(N) = \{0\}$.

Recall that $N$ is von Neumann regular if for every $x \in N$, there is $y \in N$ such that $x^2y = x$, and that $N$ is $\pi$-regular if for every $x \in \mathbb{N}$, there is $y \in \mathbb{N}$ a positive integer $n$ such that $x^2y = x^n$. Moreover, $N$ is $\pi$-regular respectively von Neumann regular if and only if $\dim(N) = 0$ respectively $N$ is reduced $\text{nil}(N) = \{0\}$.

Thus $N$ is $\pi$-regular sub near-field space if and only if $N/\text{nil}(N)$ is von Neumann regular sub near-field space of $N$ over a near-field. As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ will denote the positive integers, integers, integers modulo $n$, and rational numbers respectively.

Let $M$ be a proper sub near-field space of a commutative near-field space $N$ over a near-field. We define $\text{N}(M) = \{(m, n) \in N \times N/M \text{ is } (m, n)\text{-closed or open sub near-field space of } N \text{ over a near-field}\}$. Thus $\{(m, n) \in N \times N/1 \leq m \leq n\} \subseteq \text{N}(M) \subseteq N \times N$ and $\text{N}(M) = N \times N$ if and only if $\sqrt{\forall M} = M$. We start with some elementary properties of $\text{N}(M)$. If we define $\text{N}(N) = N \times N$, then the results in this section hold for all sub near-field spaces of $N$ over a near-field.

**Theorem 4.1:** Let $N$ be a commutative near-field space over a near-field. $M$ and $P$ be proper sub near-field spaces of a near-field space $N$ over a near-field, and $m$, $n$ and $k$ positive integers.

(a) $(m, n) \in \text{N}(M)$ for all positive integers $m$ and $n$ with $m \leq n$.
(b) If $(m, n) \in \text{N}(M)$, then $(m, n) \in \text{N}(M)$ for all positive integers $m$ and $n$ with $1 \leq m' \leq m$ and $n' \geq n$.
(c) $(m, n) \in \text{N}(M)$, then $(kn, kn) \in \text{N}(M)$.
(d) $(m, n), (n, k) \in \text{N}(M)$, then $(m, k) \in \text{N}(M)$.
(e) If $(m, n), (m+1, n+1) \in \text{N}(M)$, for $m > n$, then $(m+1, n) \in \text{N}(M)$.
(f) If $(n, 2), (n+1, 2) \in \text{N}(M)$, for an integer $n \geq 3$, then $(n+2, 2) \in \text{N}(M)$, and
(g) $(m, 2) \in \text{N}(M)$ for every positive integer $m$.
(h) $(m, n) \in \text{N}(M)$, for positive integers $m$ and $n$ with $n \leq m/2$, then $(m+1, n) \in \text{N}(M)$ and $(k, n) \in \text{N}(M)$, for every positive integer $k$.
(i) $(m, n) \in \text{N}(M)$, for every positive integers $m$ and $n$ with $n \leq m/2$, then $N(M \times P) = N(M) \cap N(P) \subseteq N(M \cap P)$.

**Proof:** To prove (a) to (d): It easily follows from the basic definitions. Hence Proved (a) to (d).

For (e) to (f): If $m < n$, then $(m+1, n) \in \text{N}(M)$ by (a). For $m \geq n$, suppose that $x^{m+1} \in M$ for $x \in N$, then $x^{m+1} \in M$ since $M$ is $(m+1, n+1)$ - closed or open sub near-field space of $N$ over a near-field. Thus $x^m \in M$ since $m \geq n + 1$, and hence $x^n \in M$ since $M$ is $(m, n)$ - closed or open sub near-field space of $N$ over a near-field. Thus $M$ is $(m+1, n)$ - closed or open sub near-field space of $N$ over a near-field. Proved (e).

For (f): Suppose that $x^{m+2} \in M$ for $x \in N$. Then $(x^2)^n = x^{2n} \in M$ since $2n \geq n + 2$ because $n \geq 2$. Hence $x^4 = (x^2)^2 \in M$ since $(n, 2)$ - closed or open sub near-field space of $N$ over a near-field. But then $x^n \in M$ since $n \geq 3$. Thus $x^2 \in M$ since $M$ is $(n+1, 2)$ - closed or open sub near-field space of $N$ over a near-field. Hence $M$ is $(n+2, 2)$ - closed or open sub near-field space of $N$ over a near-field. Similarly, $(k, 2) \in \text{N}(M)$ for every integer $k \geq n + 3$. So by (b), $M$ is $(k, 2)$ - closed or open sub near-field space of $N$ over a near-field for every positive integer $k$. Proved (f).

For (g): Let $x^{m+1} \in M$ for $x \in N$. Then $(x^2)^m = x^{2m} \in M$, and hence $x^{2m} = (x^2)^m \in M$ since $M$ is $(m, n)$ - closed or open sub near-field space of $N$ over a near-field. Thus $x^m \in M$ since $2m \leq m$, and hence $x^m \in M$ since $M$ is $(m, n)$ - closed or open sub near-field space of $N$ over a near-field. Thus $M$ is $(m+1, n)$ - closed or open sub near-field space of $N$ over a near-field. Similarly, $(k, n) \in \text{N}(M)$ for every integer $k \geq n$, and hence $(k, n) \in \text{N}(M)$ for every positive integer $k$ by (b). Proved (g).

To prove (h): obvious with the help of proof of (g). Proved (g).

For (i): Clearly $M \times P$ is $(m, n)$ - closed or open sub near-field space of $N$ over a near-field if and only if $M$ and $P$ are both $(m, n)$ - closed or open sub near-field space of $N$ over a near-field. Thus $N(M \times P) = N(M) \cap N(P)$. Thus $N(M) \cap N(P) \subseteq N(M \cap P)$ follows that $N(M \times P) = N(M) \cap N(P) \subseteq N(M \cap P)$. Hence proved (i).

This completes the proof of the theorem.
Note 4.2: The m ≠ n hypothesis is needed and since (n, n) ∈ N(M) for every positive integer n.

Note 4.3: The n ≥ 3 hypothesis is needed and for n = 1, we have (1, 2), (2, 2) ∈ N(M) for every proper sub near-field space M of N, but in general, (3, 2) ∉ N(M). For n = 2, we have (2, 2), (3, 2) ∈ N(M) does not imply (4, 2) ∈ N(M). For example, let N = Z and M = 16Z. Then (2,2), (3,2) ∉ N(M), but (4,2) ∈ N(M).

Note 4.4: The inclusion may be strict. For example, let N = Z, M = 8Z and P = 16Z. Then (3, 2) ∉ N(P) = N(M ∩ P). However, (3, 2) ∈ N(M). So N(M) ∩ N(P) ⊆ N(M ∩ P).

Note 4.5: More generally, N(M × P) = N(M) ∩ N(P) for all sub near-field spaces M and P of a commutative near-field space of N and T, respectively.

Let M be a proper sub near-field space of a commutative near-field space N over a near-field and m and n be +ve integers. We define $f_1(m) = \min \{n/M is (m, n) - closed or open sub near-field space of N over a near-field\} \in \{1,2,...,m\}$ and $g_1(n) = \sup \{m/M is (m, n) - closed or open sub near-field space of N over a near-field\} \in \{n, n+1,....\} \cup \{\infty\}$. So $f_1 : N \rightarrow N$ and $g_1 : N \rightarrow N \cup \{\infty\}$. The columns respectively rows of N(M) determine $f_1$ (or $g_1$). Then either function $f_1$ or $g_1$ is determined the other, and either function determines N(M). It is sometimes useful to view $f_1$ (or $g_1$) as an N-valued respectively N valued non-decreasing sequence $f_1 = (f_1(m))$ (or $g_1 = (g_1(n))$). Note that $f_1 = (1,1,1,...)$ if and only if $f_1 = (\infty,\infty,\infty,...)$, if and only if $\forall M = M$. if we define N(N) = N × N, then $f_N = (1,1,1,...)$ and $g_N = (\infty,\infty,\infty,...)$. Also $f_1$ is eventually constant if and only if $g_1$ is eventually constant, if and only if $g_1$ is eventually $\infty$. We next give some elementary properties of the two functions $f_1$ and $g_1$.

Theorem 4.6: Let N be a commutative near-field space, M be a proper sub near-field space of N and m and n are +ve integers. Let $f_1(m) = \min \{n/M is (m, n) - closed or open sub near-field space of N over a near-field\}$ and $g_1(n) = \sup \{m/M is (m, n) - closed or open sub near-field space of N over a near-field\}$.

(a) $1 \leq f_1(m) \leq m$
(b) $f_1(m) \leq f_1(m+1)$
(c) If $f_1(m) < m$, then either $f_1(m+1) = f_1(m)$ or $f_1(m+1) \geq f_1(m) + 2$.
(d) $n \geq g_1(n) \leq \infty$.
(e) $g_1(n) \leq g_1(n+1)$
(f) If $g_1(n) > n$, then either $g_1(n+1) = g_1(n)$ or $g_1(n+1) \geq g_1(n) + 2$.

Proof: Obvious.

Theorem 4.7: Let N be a commutative near-field space and M and P proper sub near-field spaces of N. Let $f_1(m) = \min \{n/M is (m, n) - closed or open sub near-field space of N over a near-field\}$ and $g_1(n) = \sup \{m/M is (m, n) - closed or open sub near-field space of N over a near-field\}$.

(a) $f_{M \cap P} \leq f_M \lor f_P$
(b) $g_{M \cap P} \leq g_M \lor g_P$
(c) $N(M \cap P) = N(M) \cap N(P)$.

Proof: Obvious.

Theorem 4.8: Let N be a sub near-field space and x, y ∈ N co-prime elements. Then N(xyN) = N(xN ∩ yN) = N(xN) ∩ N(yN). Moreover, $f_{xyN} = f_{xN} \lor f_{yN}$ and $g_{xyN} = g_{xN} \land g_{yN}$.

Proof: Obvious.

Theorem 4.9: Let N be a commutative near-field space, n a positive integer, and M an n-absorbing sub near-field space of N. Then $f_1(m) \leq n$ for every positive integer m. Thus $f_1$ and $g_1$ are eventually constant. In particular, if N is Noetherian, then $f_1$ and $g_1$ are eventually constant for every proper sub near-field space M of N.

Proof: Obvious.

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