EXISTENCE RESULTS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS
WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT
In this paper, we investigate the existence results for nonlinear impulsive fractional differential equations with integral boundary conditions by using fixed point theorem and Green's function.

Keywords: Integral boundary; impulsive; fractional differential equation; Green's function; fixed point theorem.

1. INTRODUCTION
For the last decades, fractional calculus has received a great attention because fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various processes of science and engineering. The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. For an introduction of the basic theory of impulsive differential equations, see Lakshmikantham et al. [11], Bainov and Simeonov [4], and Samoilenko and Perestyuk [14] and the references therein. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [1, 2, 5, 6, 9, 10, 15-17, 19] and the references therein.

The Green's functions for boundary-value problems for ordinary and fractional differential equations have been investigated in detail in [8, 20].

In [3], the authors have studied the existence of solutions for fractional integro-differential equations with impulsive and integral conditions. In [7] the authors have studied a class of nonlinear fractional differential equations with impulsive and fractional integral boundary conditions.

In [18] the authors investigated the expression and properties of Green's function for a second-order singular boundary value problem with integral boundary conditions and delayed argument.

In [12] the authors discussed nonlinear impulsive fractional differential equations involving the two-point and integral boundary conditions. The authors in [13] proved the existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions.

Inspired by the above works, we consider the existence of solutions for impulsive fractional differential equations with integral boundary conditions

\[
\begin{align*}
\mathcal{D}_0^q y(t) &= \omega(t)f(t, y(t), y'(t)), \quad 1 < q < 2, \quad t \in J = [0,1], \quad t \neq t_k, \\
\Delta y|_{t=t_k} &= I_k(y(t_k)), \quad \Delta y'|_{t=t_k} = J_k(y(t_k)), \quad t_k \in (0,1), \quad k = 1, 2, \ldots, n, \\
y(0) &= y(1) = \int_0^1 g(s)y(s)ds,
\end{align*}
\]

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In this section, we give some preliminaries for discussing the solvability of problem (1.1). In section 4, we get some existence results for problem (1.1) by means of standard fixed point theorem. Finally, the rest of this paper is organized as follows. In section 2, we give some preliminaries about the fractional integral and derivative operator. In section 3, we present the expression and properties of Green's function associated with problem (1.1). In section 4, we get some existence results for problem (1.1) by means of standard fixed point theorem. Finally, in section 5, the example is also illustrated.

2. PRELIMINARIES

In this section, we give some preliminaries for discussing the solvability of problem (1.1).

**Definition 2.1:** The fractional (arbitrary) order integral of the function \( h \in L^1(J, R_+) \) of order \( q \in R_+ \) is defined by
\[
I_0^q h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds,
\]
where \( \Gamma(\cdot) \) is the Euler Gamma function.

**Definition 2.2:** For a function \( h \) given on the interval \( J \), the Caputo-type fractional derivative of order \( q > 0 \) is defined by
\[
^cD_0^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{d^n}{ds^n} (t-s)^{n-q} h(s)ds, \quad n = [q] + 1,
\]
where the function \( h(t) \) has absolutely continuous derivatives up to order \( (n-1) \).

**Lemma 2.1:** Let \( h(t) > 0 \), then the differential equation \( ^cD_0^q h(t) = 0 \) has the following solution:
\[
h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{(n-1)} t^{n-1}, \quad c_i \in R, i = 0, 1, 2, ..., n-1, n = [q] + 1.
\]

**Lemma 2.2:** Let \( q > 0 \), then \( I^q_i ^cD_0^q h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{(n-1)} t^{n-1} \), for some \( c_i \in R, i = 0, 1, 2, ..., n-1, n = [q] + 1 \).

We define the set of functions as follows
\[
J' = [0,1] \times \{t_1, t_2, ..., t_n\} \quad \text{and} \quad \text{PC}[J', R] = \{y: J' \rightarrow R; y \in C((t_k, t_{k+1}), R), y(t_k^+) \text{ and } y(t_k^-) \text{ exist and } y(t_k^-) = y(t_k), k = 1, 2, 3, ..., n\}
\]
\[
\text{PC}^1[J, R] = \{y \in \text{PC}[J', R]; y(t_k^+) \text{ and } y(t_k^-) \text{ exist and } y \text{ is left continuous at } t_k, k = 1, 2, 3, ..., n\}
\]
Then \( PC[J, R] \) is a Banach space with the norm
\[
\|y\|_{\text{PC}} = \sup_{t \in J}\|y(t)\|, \quad \text{PC}^1[J, R] \text{ is a Banach space with the norm}
\]
\[
\|y\|_{\text{PC}^1} = \max\{\|y\|_{\text{PC}}, \|y\|_{\text{PC}} \}
\]

3. EXPRESSION AND PROPERTIES OF GREEN’S FUNCTION

**Definition 3.1:** A function \( y \in \text{PC}^1[J, R] \cap C^2[J', R] \) with its Caputo derivative of order \( q \) existing on \( J \) is a solution of problem (1.1) if it satisfies (1.1). We give the following hypotheses:

\( H_1 \) \( \omega : J \rightarrow [0, +\infty) \) is a continuous function, and there exist \( t_0 \in J \) such that \( \omega(t_0) > 0; \)

\( H_2 \) \( f : J \times R \times R \rightarrow R \) is a continuous function.

\( H_3 \) \( I_{k_0}, k_1 : R \rightarrow R \) are continuous functions.

\( H_4 \) \( g \in L_1[0,1] \) is nonnegative and \( \mu \in [0,1], \)
\[
\mu = \int_0^1 g(t)dt \tag{3.1}
\]
Consider the following fractional impulsive boundary value problem :
\[
\begin{align*}
^cD_0^q y(t) &= \sigma(t), 1 < q < 2, \quad t \in J_1 = \{t_1, t_2, ..., t_n\}, \\
\Delta y|_{t=t_k} &= I_k(y(t_k)), \quad \Delta y|_{t=t_k} = J_k(y(t_k)), \quad t_k \in (0,1), \quad k = 1, 2, ..., n, \\
y(0) &= y(1) = \int_0^1 g(s)y(s)ds
\end{align*}
\tag{3.2}
\]
Proposition 3.1: The solution of problem (3.2) can be expressed by

\[ y(t) = \sum_{i=1}^{n+1} H_2(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \sum_{i=1}^{n} H_1(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} H_2(t, t_i)I_i(y(t_i)) \]

\[ - \sum_{i=1}^{n} H_1(t, t_i)J_i(y(t_i)), \quad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots, n, t_0 = 0, t_{n+1} = 1. \]

where

\[ H_1(t, s) = G_1(t, s) + \frac{1}{1-\mu} \int_{0}^{1} g(u)G_1(s, u)du \]

\[ H_2(t, s) = G_1'(t, s) + \frac{1}{1-\mu} \int_{0}^{1} g(u)G_1'(s, u)du \]

\[ G_1(t, s) = \begin{cases} t(1-s), & t \leq s, \\ s(1-t), & t \geq s, \end{cases} \]

\[ G_1'(t, s) = \begin{cases} -t & \text{or} & (1-s), & t \leq s, \\ (1-t) & \text{or} & (-s), & t \geq s, \end{cases} \]

Proof: Suppose that \( y \) is a solution of (3.2). Then, for some constants \( b_0, b_1 \in R \), we have

\[ y(t) = I_0^\alpha \sigma(t) - b_0 - b_1 t, \]

\[ y'(t) = \int_{0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - b_1. \]

If \( t \in (t_1, t_2) \), then for some constants \( c_0, c_1 \in R \) we can write

\[ y(t) = \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_0 - c_1(t - t_1), \]

\[ y'(t) = \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_1. \]

Using impulse conditions \( \Delta y|_{t=t_1} = I_1(y(t_1)), \quad \Delta y|_{t=t_1} = J_1(y(t_1)), \)

\[ -c_0 = \int_{0}^{t_1} \frac{(t_1 - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - b_0 - b_1 t_1 + I_1(y(t_1)), \]

\[ -c_1 = \int_{0}^{t_1} \frac{(t_1 - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds - b_1 + J_1(y(t_1)). \]

Thus

\[ y(t) = \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \int_{0}^{t_1} \frac{(t_1 - s)^{q-2}(t - t_1) + (t_1 - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - b_0 - b_1 t + \]

\[ +J_1(y(t_1))(t - t_1) + I_1(y(t_1)) \]

\[ y'(t) = \int_{t_1}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \int_{0}^{t_1} \frac{(t_1 - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds - b_1 + J_1(y(t_1)). \]

If \( t \in (t_k, t_{k+1}) \), repeating the above procedure, we obtain

\[ y(t) = \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}(t - t_i) + (t_i - s)^{q-1}}{\Gamma(q-1)} \sigma(s)ds - b_0 \]

\[ -b_1 t + \sum_{i=1}^{k} J_i(y(t_i))(t - t_i) + \sum_{i=1}^{k} I_i(y(t_i)) \]

(3.10)
By the boundary conditions, we have

\[ b_0 = - \int_0^1 g(t)y(t)dt \]  

(3.11)

\[ b_1 = \int_{t_k}^{1} \frac{(1 - s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} + \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) \sigma(s)ds 
+ \sum_{i=1}^{k} J_i(y(t_i))(t - t_i) + \sum_{i=1}^{k} l_i(y(t_i)) \]  

(3.12)

Substituting (3.11) and (3.12) into (3.10), we have

\[ y(t) = \sum_{i=1}^{n+1} G_i(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \sum_{i=1}^{n} G_i(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} G_i(t, t_i) l_i(y(t_i)) 
- \sum_{i=1}^{n} G_i(t, t_i) J_i(y(t_i)) + \int_0^1 g(t)y(t)dt \]

\[ y(t) = \sum_{i=1}^{n+1} G_i(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \sum_{i=1}^{n} G_i(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} G_i(t, t_i) l_i(y(t_i)) 
- \sum_{i=1}^{n} G_i(t, t_i) J_i(y(t_i)) + \int_0^1 g(t)y(t)dt \]

Therefore, we have

\[ \int_0^1 g(s) y(s)ds = \frac{1}{1 - \int_0^1 g(s)ds} \left[ \int_0^1 g(u) \sum_{i=1}^{n+1} G_i(s, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)du ds 
- \int_0^1 g(u) \sum_{i=1}^{n} G_i(s, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)du ds 
- \int_0^1 g(s) \sum_{i=1}^{n} G_i(s, t_i) J_i(y(t_i))ds 
+ \int_0^1 g(s) \sum_{i=1}^{n} G_i(s, t_i) l_i(y(t_i))ds \right] \]

and

\[ y(t) = \sum_{i=1}^{n+1} G_i(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)ds - \sum_{i=1}^{n} G_i(t, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} G_i(t, t_i) l_i(y(t_i)) 
- \sum_{i=1}^{n} G_i(t, t_i) J_i(y(t_i)) \frac{1}{1 - \mu} \int_0^1 g(u) \sum_{i=1}^{n+1} G_i(s, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \sigma(s)du ds 
- \int_0^1 g(u) \sum_{i=1}^{n} G_i(s, t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \sigma(s)du ds 
- \int_0^1 g(s) \sum_{i=1}^{n} G_i(s, t_i) J_i(y(t_i))ds 
+ \int_0^1 g(s) \sum_{i=1}^{n} G_i(s, t_i) l_i(y(t_i))ds \]

Let

\[ H_1(t, s) = G_1(t, s) + \frac{1}{1 - \mu} \int_0^1 g(u) G_1(s, u)du \]

\[ H_2(t, s) = G_1'(t, s) + \frac{1}{1 - \mu} \int_0^1 g(u) G_1'(s, u)du \]
Then,
\[
y(t) = \sum_{i=1}^{n+1} H_2(t, t_i) \int_{t_{i-1}}^{t_i} \left( t - s \right)^{q-1} \frac{\sigma(s)ds}{\Gamma(q)} - \sum_{i=1}^{n} H_1(t, t_i) \int_{t_{i-1}}^{t_i} \left( t - s \right)^{q-2} \frac{\sigma(s)ds}{\Gamma(q-1)} + \sum_{i=1}^{n} H_2(t, t_i) J_i(y(t_i))
\]

**Proposition 3.2:** If \( (H_1) \) holds, then we have
\[
\begin{align*}
H_1(t, s) &> 0, \quad G_1(t, s) > 0, \quad \text{for all } t, s \in (0,1) \\
H_1(t, s) &\geq 0, \quad G_1(t, s) \geq 0, \quad \text{for all } t, s \in (0,1)
\end{align*}
\]

**Proof:** From the definition of \( H_1(t, s) \) and \( G_1(t, s) \), it is easy to obtain the result of (3.13) and (3.14).

**Proposition 3.3:** For \( t, s \in [0,1] \), we have
\[
e(t)e(s) \leq G_1(t, s) \leq G_1(s, s) = s(1-s) = e(s) \leq \bar{e} = \max_{t \in [0,1]} e(s) = \frac{1}{4}
\]

**Proof:** In fact, for \( t \in J, \ s \in (0,1) \), we have

**Case-1:** If \( 0 < t \leq s < 1 \), then
\[
\begin{align*}
\frac{G_1(t, s)}{G_1(s, s)} & = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \leq 1.
\end{align*}
\]

**Case-2:** If \( 0 < s \leq t < 1 \), then
\[
\begin{align*}
\frac{G_1(t, s)}{G_1(s, s)} & = \frac{s(1-t)}{s(1-s)} = \frac{1-t}{1-s} \leq \frac{1-s}{1-s} \leq 1.
\end{align*}
\]

In addition, by the definition of \( G_1(t, s) \), it is easy to obtain that
\[
G_1(t, s) \leq G_1(s, s), \quad \forall t \in J, \ s \in (0,1).
\]

Therefore
\[
G_1(t, s) \leq G_1(t, s) = e(s), \forall t, s \in J.
\]

Similarly, we can prove that
\[
G_1(t, s) \geq e(t)e(s).
\]

**Case-1:** If \( t \leq s \), then
\[
\begin{align*}
\frac{G_1(t, s)}{G_1(s, s)} & = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \geq t(1-t).
\end{align*}
\]

**Case-2:** If \( t \leq s \), then
\[
\begin{align*}
\frac{G_1(t, s)}{G_1(s, s)} & = \frac{s(1-t)}{s(1-s)} = \frac{1-t}{1-s} \geq (1-t) = t(1-t).
\end{align*}
\]

So we have
\[
G_1(t, s) \geq e(t)e(s), \quad \forall t, s \in J.
\]

**Proposition 3.4:** If \( (H_1) \) holds, then for \( t, s \in [0,1] \), we have
\[
\rho e(s) \leq H_1(t, s) \leq \gamma s(1-s) = \gamma e(s) \leq \frac{1}{4}
\]

where
\[
\begin{align*}
\gamma & = \frac{1}{1-\mu}, \\
\rho & = \frac{\int_0^1 e(\tau)g(\tau)d\tau}{1-\mu}
\end{align*}
\]

**Proof:** By (3.4) and (3.14), we have
\[
H_1(t, s) = G_1(t, s) + \frac{1}{1-\mu} \int_0^1 G_1(s, \tau)g(\tau)d\tau
\]
\[ \begin{align*} 
\frac{1}{1-\mu} \int_0^1 G_1(s, \tau) g(\tau) \, d\tau 
\geq \frac{1}{1-\mu} \int_0^1 e(\tau) g(\tau) \, d\tau 
= \rho \, e(s), \quad t \in [0,1]. 
\end{align*} \]

On the other hand, noticing \( G_1(t, s) = s(1-s) \), we obtain
\[
H_1(t, s) = G_1(t, s) + \frac{1}{1-\mu} \int_0^1 G_1(s, \tau) g(\tau) \, d\tau 
= s(1-s) + \frac{1}{1-\mu} \int_0^1 s(1-s) g(\tau) \, d\tau 
= s(1-s) \left[ 1 + \frac{1}{1-\mu} \int_0^1 g(\tau) \, d\tau \right] 
= s(1-s) \left[ 1 + \frac{1}{1-\mu} \right] 
= ye(s), \quad t \in [0,1].
\]

The proof of Proposition 3.4 is complete.

**Proposition 3.5:** If \((H_4)\) holds, then for \(t, s \in [0,1]\), we have
\[
H_2(t, s) \leq \gamma (1-t) \leq \gamma
\]
(3.18)

**Proof:**
\[
H_2(t, s) = G_1'(t, s) + \frac{1}{1-\mu} \int_0^1 G_1'(s, \tau) g(\tau) \, d\tau 
= 1 + \frac{1}{1-\mu} \int_0^1 g(\tau) \, d\tau 
= 1 \leq \gamma
\]

It follows from Proposition (3.1) that:

**Lemma 3.1:** If \((H_1) - (H_2)\) hold, then a function \(y \in PC^1[J, R] \cap C^2[J', R]\) is a solution of problem (1.1) iff \(y \in PC^1[J, R]\) is a solution of the impulsive fractional integral equation
\[
y(t) = \sum_{i=1}^{n+1} H_2(t, t_i) \int_{t_i-1}^{t_i} \left[ \frac{(t_i-s)^{\gamma-1}}{\Gamma(\gamma)} \right] \omega(s) f(s, y(s), y'(s)) \, ds 
- \sum_{i=1}^{n} H_1(t, t_i) \int_{t_i-1}^{t_i} \left[ \frac{(t_i-s)^{\gamma-2}}{\Gamma(\gamma-1)} \right] \omega(s) f(s, y(s), y'(s)) \, ds 
+ \sum_{i=1}^{n} H_2(t, t_i) I_i(y(t_i)) 
- \sum_{i=1}^{n} H_1(t, t_i) I_i'(y(t_i)) 
k = 0, 1, 2, \ldots, n, \quad t_0 = 0, t_{n+1} = 1
\]
(3.19)

**Proof:** Let \(y \in PC^1[J, R] \cap C^2[J', R]\) be a solution of the boundary value problem (1.1), then by the same method as used in Proposition 3.1, we can prove that \(y\) is a solution of the fractional integral equation (3.19).

Conversely, assume that \(y\) satisfies the impulsive fractional integral equation (3.19).

If \(t \in [0, t_1]\) then \(y(0) = \int_0^{t_1} g(s) y(s) \, ds\), \(y(1) = \int_0^{t_1} g(s) y(s) \, ds\) and using the fact that \(^{CD}D^n\) is the left inverse of \(I^n\) we get
\[
^{CD}D^n y(t) = h(t) \text{ for each } t \in [0, t_1].
\]
If $t \in [t_k, t_{k+1})$, $k = 1, 2, ..., n$ and using the fact that $\frac{\mathrm{d}}{\mathrm{d}t} C = 0$, where $C$ is a constant, we get $\frac{\mathrm{d}}{\mathrm{d}t} y(t) = h(t)$ for each $t \in [t_k, t_{k+1})$.

Also we can easily show that $\Delta y|_{t=t_k} = I_k(y(t_k))$, $\Delta y|_{t=t_k} = I_k(y(t_k))$, $k = 1, 2, ..., n$.

Hence the proof.

Define $T : PC^1[J, R] \rightarrow PC^1[J, R]$ by

$$Ty(t) = \sum_{i=1}^{n+1} H_2(t, t_i) \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) [\omega(s)f(s, y(s), y'(s))] \, ds$$

$$- \sum_{i=1}^{n} H_1(t, t_i) \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \right) [\omega(s)f(s, y(s), y'(s))] \, ds + \sum_{i=1}^{n} H_2(t, t_i) I_i(y(t_i))$$

$$- \sum_{i=1}^{n} H_1(t, t_i) J_i(y(t_i)), \quad k = 0, 1, 2, ..., n, t_0 = 0, t_{n+1} = 1 \quad (3.20)$$

Using Lemma (3.1), Problem (1.1) reduces to a fixed point problem $T(x) = x$, where $T$ is given by (3.20). Thus, The problem (1.1) has a solution iff operator $T$ has a fixed point.

**Lemma 3.2:** Assume that $(H_1) - (H_4)$ hold. Then $T : PC^1[J, R] \rightarrow PC^1[J, R]$ is completely continuous.

**Proof:** Note that the continuity of $f, g, \omega, I_k$ and $J_k$ together with $H_1(t, s)$ and $H_2(t, s)$ ensures the continuity of $T$.

Let $\Omega \subset PC^1[J, R]$ be bounded. Then there exists positive constants $\lambda_1, \lambda_2, \lambda_3$, such that $|f(t, y(t), y'(t))| \leq \lambda_1, |g(y)| \leq \lambda_2$ and $|I_k(y)| \leq \lambda_3 \forall y \in \Omega$. Thus $\forall y \in \Omega$, we have

$$|(Ty)(t)| \leq \sum_{i=1}^{n+1} |H_2(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) |\omega(s)f(s, y(s), y'(s))| \, ds$$

$$+ \sum_{i=1}^{n} |H_1(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \right) |\omega(s)f(s, y(s), y'(s))| \, ds + \sum_{i=1}^{n} |H_2(t, t_i)||I_i(y(t_i))|$$

$$+ \sum_{i=1}^{n} |H_1(t, t_i)||J_i(y(t_i))|$$

$$\leq \sum_{i=1}^{n+1} |H_2(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) [c\lambda_1] \, ds + \sum_{i=1}^{n} |H_1(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \right) [c\lambda_1] \, ds + \lambda_2 \sum_{i=1}^{n} |H_2(t, t_i)|$$

$$+ \lambda_3 \sum_{i=1}^{n} |H_1(t, t_i)|$$

$$\leq \frac{c(n+1)\lambda_1}{\Gamma(q + 1)} + \frac{ncy\lambda_1}{4\Gamma(q)} + n\lambda_2 + \frac{ny\lambda_3}{4} = \lambda \quad (3.21)$$

where $c = \max_{t \in J} \omega(t)$

Furthermore, for any $t \in (t_k, t_{k+1}]$

$$|(Ty)(t)| \leq \sum_{i=1}^{n+1} |G^{\alpha}_{2}(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) [\omega(s)f(s, y(s), y'(s))] \, ds$$

$$+ \sum_{i=1}^{n} |G_{1}(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \right) [\omega(s)f(s, y(s), y'(s))] \, ds + \sum_{i=1}^{n} |G^{\alpha}_{1}(t, t_i)||I_i(y(t_i))|$$

$$+ \sum_{i=1}^{n} |G_{1}(t, t_i)||J_i(y(t_i))| \quad (3.22)$$
On account of (4.1), we can choose \( \theta' > \theta, \theta'_1 > \theta_1 \) and \( \theta'_2 > \theta_2 \) such that

\[
\delta' = \frac{2c(n + 1)\theta'}{\Gamma(q + 1)} + \frac{2cn\theta'}{4\Gamma(q)} + \frac{ny\theta'_1}{4} + \frac{ny\theta'_2}{4} \leq \frac{2c(n + 1)\theta}{\Gamma(q + 1)} + \frac{2cn\theta}{4\Gamma(q)} + n\theta_1 + n\theta_2,
\]

and

\[
\delta'_2 = \frac{2c(n + 1)\theta'}{\Gamma(q + 1)} + \frac{2cn\theta'}{4\Gamma(q)} + n\theta_1 + n\theta_2.
\]

By the definition of \( \theta' \), there exists \( l > 0 \) such that

\[
|f(t, y, z)| < \theta'(|y| + |z|), \quad \forall \ t \in J, \ |y| + |z| > l,
\]

\[
|f(t, y, z)| < \theta'(|y| + |z|) + M_l, \quad \forall \ t \in J, \ y, z \in \mathbb{R},
\]

where

\[
M_l = \max_{t \in J, |y| + |z| \leq l} |f(t, y, z)| < +\infty.
\]

Similarly, we have

\[
\begin{align*}
|u_k(y)| &< \theta'_1 |x| + M_k, \quad \forall \ y \in \mathbb{R}, \quad k = 1, 2, \ldots, n \\
|u_k(y)| &< \theta'_2 |x| + \bar{M}_k, \quad \forall \ y \in \mathbb{R}, \quad k = 1, 2, \ldots, n
\end{align*}
\]

where \( M_1, M_2, M_k, \bar{M}_k \) are positive constants.

To prove our main result, we also need the following lemma.

**Lemma 3.3 (Schauder Fixed point Theorem):** Let \( D \) be non empty, closed, bounded, convex subset of a banach space \( X \), and suppose that \( T: D \to D \) is completely continuous operator. Then \( T \) has fixed point \( x \in D \).

**4. EXISTENCE OF SOLUTIONS**

In this section we apply Lemma 3.3 to establish the existence of solutions to problem (1.1). Let us define

\[
\begin{align*}
\theta & = \lim_{|y| + |z| \to \infty} \left( \max_{t \in J} \frac{|f(t, y, z)|}{|y| + |z|} \right) \\
\theta_1 & = \lim_{|y| \to \infty} \frac{|u_1(y)|}{|y|}, \quad \theta_2 = \lim_{|y| \to \infty} \frac{|u_k(y)|}{|y|}, \quad k = 1, 2, \ldots, n
\end{align*}
\]

**Theorem 4.1:** Assume that \( (H_1) - (H_8) \) hold. Suppose further that

\[
\delta = \max\{\delta_1, \delta_2\} < 1
\]

where

\[
\delta_1 = \frac{2c(n + 1)\theta}{\Gamma(q + 1)} + \frac{2cn\theta}{4\Gamma(q)} + ny\theta_1 + \frac{ny\theta'_1}{4}
\]

and

\[
\delta_2 = \frac{2c(n + 1)\theta}{\Gamma(q + 1)} + \frac{2cn\theta}{4\Gamma(q)} + n\theta_1 + n\theta_2.
\]

Proof: We shall use Schauder’s fixed point theorem to prove that \( T \) has a fixed point. First, recall that the operator \( T: PC^1[J, R] \to PC^1[J, R] \) is completely continuous (see the proof of lemma (3.2)).
It follows from (4.6) and (4.7) that

\[
|T(y)(t)| \leq \sum_{i=1}^{n+1} |H_2(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) \left[ \omega(s) f(s, y(s), z(s)) \right] ds
+ \sum_{i=1}^{n} |H_1(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q - 1)} \right) \left[ \omega(s) f(s, y(s), z(s)) \right] ds
+ \sum_{i=1}^{n} |H_2(t, t_i)| \left| l_i(y(t_i)) \right|
\leq \sum_{i=1}^{n+1} |H_2(t, t_i)| \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) \left[ \gamma \left( \frac{|\gamma(y)| + |z|}{M + 1} \right) \right] ds
+ \sum_{i=1}^{n} |H_1(t, t_i)| \left| l_i(y(t_i)) \right|
\leq \sum_{i=1}^{n+1} \frac{(n + 1)c}{\Gamma(q + 1)} \left( \frac{2\theta(\|y\|_{PC}^1) + M}{1 + \theta_1(\|y\|_{PC}^1)} \right)
+ \frac{nc}{\Gamma(q)} \left( \frac{2\theta(\|y\|_{PC}^1) + M}{1 + \theta_1(\|y\|_{PC}^1)} \right) + M_k
+ \frac{n}{4} \theta_2(\|y\|_{PC}^1) + \bar{M}_k
\]

(4.6)

Where \( \delta_1 \) is defined by (4.2) and \( M^{(1)} \) is defined by

\[
M^{(1)} = \frac{M(n + 1)c}{\Gamma(q + 1)} + \frac{Mnc}{\Gamma(q)} + nM_k + \frac{n}{4} \bar{M}_k
\]

Similarly, from (3.20), (4.4) and (4.5), we get

\[
|T'(y)(t)| \leq \sum_{i=1}^{n+1} G''_{1st}(t, t_i) \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) \left[ \omega(s) f(s, y(s), z(s)) \right] ds
+ \sum_{i=1}^{n} G'_{1st}(t, t_i) \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-2}}{\Gamma(q - 1)} \right) \left[ \omega(s) f(s, y(s), z(s)) \right] ds
+ \sum_{i=1}^{n} G''_{1st}(t, t_i) \left| l_i(y(t_i)) \right|
\leq \sum_{i=1}^{n+1} G''_{1st}(t, t_i) \int_{t_{i-1}}^{t_i} \left( \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right) \left[ \gamma \left( \frac{|\gamma(y)| + |z|}{M + 1} \right) \right] ds
+ \sum_{i=1}^{n} G'_{1st}(t, t_i) \left| l_i(y(t_i)) \right|
\leq \sum_{i=1}^{n+1} \frac{(n + 1)c}{\Gamma(q + 1)} \left( \frac{2\theta(\|y\|_{PC}^1) + M}{1 + \theta_1(\|y\|_{PC}^1)} \right)
+ \frac{nc}{\Gamma(q)} \left( \frac{2\theta(\|y\|_{PC}^1) + M}{1 + \theta_1(\|y\|_{PC}^1)} \right) + M_k
+ \frac{n}{4} \theta_2(\|y\|_{PC}^1) + \bar{M}_k
\]

(4.7)

Where \( \delta_2 \) is defined by (4.3) and \( M^{(2)} \) is defined by

\[
M^{(2)} = \frac{M(n + 1)c}{\Gamma(q + 1)} + \frac{Mnc}{\Gamma(q)} + nM_k + \frac{n}{4} \bar{M}_k
\]

It follows from (4.6) and (4.7) that

\[
\|Ty\|_{PC}^1 \leq \delta^{*} \|Ty\|_{PC}^1 + M', \forall y \in PC^1[J, R]
\]

where

\[
\delta^{*} = \max(\delta_1, \delta_2) < 1, \quad M' = \max(M^{(1)}, M^{(2)})
\]

Hence, we can choose a sufficiently large \( r > 0 \) such that \( T(B_r) \subset B_r \), where \( B_r = \{ y \in PC^1 : \|y\|_{PC}^1 \leq r \} \)
We consider the following boundary value problem:

\[\text{This problem has at least one solution in} \quad \{0, \frac{1}{2} \} \cup \left( \frac{1}{2}, 1 \right).\]

**Remark 4.1:** Condition (4.1) is certainly satisfied if \( \frac{f(t,y,x)}{|y|+|x|} \to 0 \) uniformly in \( t \in J \) as \( |y|+|z| \to +\infty \), \( \frac{|y_i(y)|}{|y|} \to 0 \) as \( |y| \to +\infty \), \( |y_i(y)| \to 0 \) as \( |y| \to +\infty \), \( (k = 1, 2 \ldots n) \).

### 5. EXAMPLE

We consider the following boundary value problem:

\[
\begin{align*}
\text{We have} & \quad \omega(t) = b_0 t_0^\frac{1}{2}, \\
\text{It follows from (5.1) that} & \quad f(t,x,x) = b_0 \sqrt{b_2 t - x + x} - \frac{1}{20} x' - b_3 \int_0^x (1 + x^2) \\
\text{From the definition of} & \quad \omega, f, I_1 \text{and} J_1, \text{it is easy to see that} (H_1) - (H_4) \text{hold. So} \\
\text{So} & \quad \theta_1 \leq \frac{1}{20}, \theta_2 \leq \frac{1}{6}, \mu = \int_0^1 \frac{1}{25} dt. \text{We have} \\
G_1(t,s) & = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1, \end{cases} \\
G_1^2(t,s) & = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ (s-1), & 0 \leq t \leq s \leq 1, \end{cases} \\
G_1^3(t,s) & = \begin{cases} (t-1), & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1, \end{cases} \\
G_1^4(t,s) & = 1 \\
\mu & \leq \frac{1}{25}, \quad H_1(t,s) \leq \frac{25}{96}, \quad H_2(t,s) \leq \frac{25}{24} \\
\delta_1 & \leq 0.25, \quad \delta_2 \leq 0.4. \\
\text{Thus, our conclusion follows from theorem 4.1.} \\
\end{align*}
\]

### REFERENCES


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