ON $\mu\psi$-CLOSED SETS IN TOPOLOGICAL SPACES

K. ALLI1, M. DHANALAKSHMI2

1Assistant professor, Department of Mathematics, The M. D. T. Hindu College, Tirunelveli, Tamil Nadu, India-627010.
2Research scholar, Department of Mathematics, The M. D. T. Hindu College, Tirunelveli, Tamil Nadu, India-627010.

(Received On: 21-09-16; Revised & Accepted On: 25-10-16)

ABSTRACT

In this paper, we introduce a new class of sets namely, $\mu\psi$-closed sets and their properties. Applying these sets, we introduce and study some seven new spaces namely, $T\mu\psi$, $\alpha T\mu\psi$, $s T\mu\psi$, $p T\mu\psi$, $spT\mu\psi$, $\mu T\mu\psi$ and $\psi T\mu\psi$-spaces and some interrelationships between these spaces.

Keywords: $\mu\psi$-closed set, $\mu\psi$-open set $T\mu\psi$, $\alpha T\mu\psi$, $s T\mu\psi$, $p T\mu\psi$, $spT\mu\psi$, $\mu T\mu\psi$ and $\psi T\mu\psi$-spaces.

2010 Mathematics Subject Classification: 54A05, 54A10, 54D10.

1. INTRODUCTION

N. Levine [4] introduced the class of $g$–closed sets in 1970. Andrijevic [1], N. Levine [4], Mashoor et.al [7], have respectively introduced semipre-closed sets, semi-closed sets, pre-closed sets which are some weak forms of closed sets.

M. K. R. S. Veerakumar has introduced several generalized closed sets namely, $g^*$-closed sets, $g^*$-closed sets, $\alpha g$–closed sets, semi-open sets, semi-closed sets, $\mu$–closed sets, $\mu$–closed sets and $\mu$–closed sets. In this paper we introduce $\mu\psi$-closed sets and applying these sets seven new spaces namely $T\mu\psi$, $\alpha T\mu\psi$, $s T\mu\psi$, $p T\mu\psi$, $spT\mu\psi$, $\mu T\mu\psi$, $\psi T\mu\psi$ are introduced.

2. PRELIMINARIES

Throughout this paper, we consider spaces on which no separation axioms are assumed unless explicitly stated. For $A \subset X$, the closure and interior of $A$ is denoted by $cl(A)$ and $int(A)$ respectively. The complement of $A$ is denoted by $A^c$, the power set of $X$ is denoted by $P(X)$.

Definition 2.1: A subset $A$ of a topological space $(X, \tau)$ is called

1. a pre-open set [6] if $A \subseteq int(cl(A))$ and pre-closed if $cl(int(A)) \subseteq A$.
2. a semi-open set [3] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.
3. an $\alpha$-open set [7] if $A \subseteq int(cl(int(A)))$ and $\alpha$-closed set if $cl(int(cl(A))) \subseteq A$.
4. a semipre-open set [1] if $A \subseteq cl(int(cl(A)))$ and a semipre-closed set if $cl(int(cl(A))) \subseteq A$.
5. a regular open set [9] if $A = int(cl(A))$ and a regular closed set [19] if $cl(int(cl(A))) = A$.

The intersection of all semiclosed (resp. preclosed, semipreclosed, $\alpha$-closed) sets containing a subset $A$ of $X$ is called semiclosure (resp. preclosure, semipreclosure, $\alpha$-closure) of $A$ is denoted by $scl(A)$ (resp. pcl(A), spcl(A), acl(A)).

The union of all semiopen sets contained in $A$ is called semiinterior of $A$ and is denoted by $sint(A)$.

Corresponding Author: M. Dhanalakshmi2

2Research scholar, Department of Mathematics, The M. D. T. Hindu College, Tirunelveli, Tamil Nadu, India-627010.
Definition 2.2: A subset $A$ of a topological space $(X, \tau)$ is called
1. a generalized closed set (briefly $g$-closed) [4] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
2. an $\alpha$-generalized closed set (briefly $\alpha g$-closed) [6] if $\text{acl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
3. a semi generalized closed set (briefly $sg$-closed) [2] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
4. a $g^*$-closed set [11] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
5. an $g^*$-closed set [12] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g^*$-open in $(X, \tau)$.
6. a $g^*$-closed set [12] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g^*$-open in $(X, \tau)$.
7. a $g^*$-preclosed set (briefly $g^*$-closed) [13] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g^*$-open in $(X, \tau)$.
8. an $g^*$-semiclosed set [17] (briefly $gs^*$-closed) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g^*$-open in $(X, \tau)$.
9. a $\alpha g^*$-closed set [17] if $\text{cl}(A) \subseteq U$ whenever $\alpha cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.
10. a $\alpha g^*$-closed set [17] if $\text{cl}(A) \subseteq U$ whenever $\alpha cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.
11. a $\psi^*$-closed set [15] if $\text{cl}(A) \subseteq U$ whenever $\psi cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.
12. a $\gamma^*$-closed set [15] if $\text{cl}(A) \subseteq U$ whenever $\gamma cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.
13. a $\gamma^*$-closed set [15] if $\text{cl}(A) \subseteq U$ whenever $\gamma cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.
14. a $\alpha^* g^*$-closed set (briefly $\alpha^* g^*$-closed) [17] if $\text{pcl}(A) \subseteq U$ whenever $\alpha cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.
15. a $\alpha^* g^*$-closed set (briefly $\alpha^* g^*$-closed) [17] if $\text{pcl}(A) \subseteq U$ whenever $\alpha cl(A)$, and $U$ is $g^*$-open in $(X, \tau)$.

Notations 2.3
1. $\alpha C(X, \tau)$ is the class of $\alpha$-closed subsets of $(X, \tau)$.
2. $\psi C(X, \tau)$ is the class of $\psi$-closed subsets of $(X, \tau)$.
3. $\psi C(X, \tau)$ is the class of pre-closed subsets of $(X, \tau)$.
4. $\psi C(X, \tau)$ is the class of semi-pre-closed subsets of $(X, \tau)$.
5. $\psi C(X, \tau)$ is the class of $\mu$-closed subsets of $(X, \tau)$.
6. $\psi C(X, \tau)$ is the class of $\mu$-closed subsets of $(X, \tau)$.

3. PROPERTIES OF $\mu_\psi$-CLOSED SETS

We introduce the following definition.

Definition 3.1: A subset $A$ of $(X, \tau)$ is called $\mu_\psi$-closed set if $\mu cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\psi$-open in $(X, \tau)$. The class of $\mu_\psi$-closed subsets of $X$ is denoted by $\mu_\psi C(X, \tau)$.

Proposition 3.2: Every closed set is $\mu_\psi$-closed. But the converse is not true which can be seen from the following examples.

Example 3.3: Let $X = \{a, b, c\}$ and $\tau = \{x, \varphi, \{c\}, \{a, b\}\}$. Here, the set $\{b\}$ is $\mu_\psi$-closed but it is not closed.

Proposition 3.4: $\mu_\psi$-closedness is independent of $\alpha$-closedness and semi-closedness.

Proof: It follows from the following examples

Example 3.5: Let $X = \{a, b, c\}$, $\tau = \{x, \varphi, \{a\}\}$. Here the set $\{b\}$ is $\alpha$-closed and semi-closed but it is not $\mu_\psi$-closed.

Example 3.6: Let $X = \{a, b, c\}$, $\tau = \{x, \varphi, \{a, b\}\}$. Here the set $\{b, c\}$ is $\mu_\psi$-closed but it is neither $\alpha$-closed nor semi-closed.

Proposition 3.7: Every $\mu_\psi$-closed set is $g$-closed (resp. $\alpha g$-closed, $\alpha^* g$-closed). But the converses are not true as can be seen from the following examples.

Example 3.8: Let $X = \{a, b, c\}$, $\tau = \{\varphi, x, \{b\}\}$. Here the set $\{a\}$ is $g$-closed (resp. $\alpha g$-closed, $\alpha^* g$-closed) but it is not $\mu_\psi$-closed.

Proposition 3.9: $\mu_\psi$-closedness is independent of $*g$-closedness, $\alpha^* g$-closedness, $\psi$-closedness, $g^*$-$\psi$-closedness and $\mu$-closedness.

Proof: It follows from the following examples.

Example 3.10: Let $X = \{a, b, c\}$, $\tau = \{\varphi, x, \{a\}\}$. Here the set $\{b\}$ is both $*g$-closed and $\alpha^* g$-closed but it is not $\mu_\psi$-closed.

Example 3.11: Let $X = \{a, b, c\}$, $\tau = \{\varphi, x, \{a, b, c\}\}$. Here the set $\{b\}$ is $\mu_\psi$-closed but it is not $*g$-closed and not $\alpha^* g$-closed.
Example 3.12: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{a\}\}$. Here the set $\{b\}$ is both $\psi$-closed and $g^*\psi$-closed but it is not $\mu_\psi$-closed.

Example 3.13: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{a\}, \{b, c\}\}$. Here the set $\{b\}$ is $\mu_\psi$-closed but it is not $\psi$-closed and not $g^*\psi$-closed.

Example 3.14: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{a\}, \{a, b\}\}$. Here the set $\{b\}$ is $\mu_\psi$-closed but it is not $g^*\psi$-closed.

Example 3.15: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{a\}, \{b, c\}\}$. Here the set $\{b\}$ is $\mu_\psi$-closed but it is not $\mu_\psi$-closed.

Proposition 3.16: Every $\mu_\psi$-closed set is $g^*p$-closed. But the converse is not true as can be seen from the following example.

Example 3.17: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{b, c\}\}$. Here the set $\{b\}$ is $g^*p$-closed but it is not $\mu_\psi$-closed.

Proposition 3.18: Every $\mu$-closed set is $\mu_\psi$-closed. But the converse is not true as can be seen from the following example.

Example 3.19: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{c\}, \{a, b\}\}$. Here the set $\{b\}$ is $\mu_\psi$-closed but it is not $\mu$-closed.

Proposition 3.20: Every $\mu_\psi$-closed set is $\mu_\psi$-closed. But the converse is not true as can be seen from the following example.

Example 3.21: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, x, \{a\}, \{a, b\}\}$. Here the set $\{b\}$ is $\mu_\psi$-closed but it is not $\mu_\psi$-closed.

Theorem 3.22: The union (intersection) of any two $\mu_\psi$-closed sets is also a $\mu_\psi$-closed set.

Proposition 3.23: Let $A$ and $B$ be any two subsets of the topology $(X, \tau)$. Then
1. $A$ is $\mu_\psi$-closed, then $\mu cl(A)$ does not contain any non empty $\psi$-closed set.
2. $A$ is $\mu_\psi$-closed and $A \subset B \subset \mu cl(A)$, then $B$ is $\mu_\psi$-closed.

Proof: Let $A$ be $\mu_\psi$-closed and suppose $\mu cl(A)$ is a non empty $\psi$-closed set $F$. Therefore, $F \subset \mu cl(A) \subset F'$, which is $\psi$-open. Since $A$ is $\mu_\psi$-closed, $\mu cl(A) \subset F'$ implies $F \subset (\mu cl(A))^\circ$, also $F \subset \mu cl(A)$ therefore $F \subset (\mu cl(A))^\circ \cap (\mu cl(A)) = \emptyset$.

Let $U$ be a $\psi$-open set such that $B \subset U$. Since $A \subset B \subset U$ and $U$ is $\psi$-open $\mu cl(A) \subset U$. Since $B \subset \mu cl(A)$, $cl(B) \subset \mu cl(\mu cl(A))$ implies $\mu cl(B) \subset \mu cl(A) \subset U$ therefore $B$ is $\mu_\psi$-closed.

Theorem 3.24: Let $A$ be a $\mu_\psi$-closed set of a topological space $(X, \tau)$. Then
1. $\text{sint}(A)$ is $\mu_\psi$-closed.
2. $\text{pcl}(A)$ is $\mu_\psi$-closed.
3. If $A$ is regular open, then $\text{pint}(A)$ and $\text{scl}(A)$ are also $\mu_\psi$-closed sets.

Proof: First we note that for a subset $A$ of $(X, \tau)$, $\text{scl}(A) = A \cup \text{int}(\text{cl}(A))$ and $\text{pcl}(A) = A \cup \text{cl}(\text{int}(A))$. Moreover $\text{sint}(A) = A \cap \text{cl}(\text{int}(A))$ and $\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$.

Since $\text{cl}(\text{int}(A))$ is a closed set, then $A$ and $\text{cl}(\text{int}(A))$ are $\mu_\psi$-closed sets. By the theorem 3.22, $A \cap \text{cl}(\text{int}(A))$ is also a $\mu_\psi$-closed set.

1. $\text{pcl}(A)$ is the union of two $\mu_\psi$-closed sets $A$ and $\text{cl}(\text{int}(A))$. Again by the theorem 3.22, $\text{pcl}(A)$ is $\mu_\psi$-closed.
2. Since $A$ is regular open, then $A = \text{int}(\text{cl}(A))$. Then $\text{cl}(A) = A \cup \text{int}(\text{cl}(A)) = A$. Thus, $\text{cl}(A)$ is $\mu_\psi$-closed. Similarly $\text{pint}(A)$ is also a $\mu_\psi$-closed set.

The converses of the statements in the above theorem are not true as we see from the following examples.

Example 3.25: Let $(X, \tau)$ be the space as in the example 3.14. $B = \{b\}$ is not $\mu_\psi$-closed set. However $\text{sint}(B) = \emptyset$ is a $\mu_\psi$-closed set.

Example 3.26: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, x, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Consider $A = \{c\}$. Clearly $A$ is not regular open. However $A$ is $\mu_\psi$-closed and $\text{scl}(A) = \text{pint}(A) = \emptyset$ is $\mu_\psi$-closed.

Remark 3.27: The following diagram shows the relationship established between $\mu_\psi$-closed set and some other sets $A \rightarrow B$ (resp. $A \leftrightarrow B$) represents $A$ implies $B$ but not conversely (resp. $A$ and $B$ are independent of each other).
From the above Propositions and Examples, we have the following diagram.

**Definition 3.28:** A subset A of a space X is said to be $\mu_\psi$–open if $A^c$ is $\mu_\psi$-closed. The class of all $\mu_\psi$-open subsets of X is denoted by $\mu_\psi O(X, \tau)$.

**Proposition 3.29:** A subset A of a topological space X is said to be $\mu_\psi$–open if and only if $F \subset \mu int(A)$ whenever $A \supset F$ and F is $\psi$-closed in X.

**Proof:** Suppose that A is $\mu_\psi$-open in X and $A \supset F$, where F is $\psi$-closed in X. Then $A^c \subset F^c$, where $F^c$ is $\psi$-open in X. Hence we get $\mu cl(A^c) \subset F^c$ implies $\mu int(A) \supset F$.

Conversely, suppose that $A^c \subset U$ and U is $\psi$-open in X then $A \supset U^c$ and $U^c$ is $\psi$-closed then by hypothesis $\mu int(A) \supset U^c$ implies $(\mu int(A))^c \subset U$. Hence $\mu cl(A^c) \subset U$ gives $A^c$ is $\mu_\psi$-closed.

**Proposition 3.30:** In a topological space X, for each $x \in X$, either $\{x\}$ is $\psi$–closed or $\mu_\psi$–open in X.

**Proof:** Suppose that $\{x\}$ is not $\psi$-closed in X. then $X - \{x\}$ is not $\psi$-open and the only $\psi$-open set containing $X - \{x\}$ is the space X itself. Therefore, $\mu cl(X - \{x\}) \subset X$ and so $X - \{x\}$ is $\mu_\psi$-closed gives $\{x\}$ is $\mu_\psi$–open.

4. **APPLICATION OF $\mu_\psi$-CLOSED SETS**

As an applications of $\mu_\psi$-closed sets, new spaces namely, $T_{\mu_\psi}$, $\alpha T_{\mu_\psi}$, $s T_{\mu_\psi}$, $p T_{\mu_\psi}$, $sp T_{\mu_\psi}$, $\mu T_{\mu_\psi}$, $\psi T_{\mu_\psi}$ spaces are introduced. First we introduce the following definitions.

**Definition 4.1:** A topological space $(X, \tau)$ is called a

1. $T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is closed.
2. $\alpha T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is $\alpha$-closed.
3. $s T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is semi-closed.
4. $p T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is pre-closed.
5. $sp T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is semipre-closed.
6. $\mu T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is $\mu$-closed.
7. $\psi T_{\mu_\psi}$-space if every $\mu_\psi$–closed set is $\psi$-closed.

**Example 4.2:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, x, \{a\}\}$. Here $\mu_\psi C(X, \tau) = \{x, \emptyset, \{b, c\}\}$. Then $(X, \tau)$ is $T_{\mu_\psi}$-space. The space in the following example is not a $T_{\mu_\psi}$-space. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, x, \{a, b\}\}$. Here $\mu_\psi C(X, \tau) = \{x, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}$.

**Example 4.3:** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, x, \{b\}\}$. Here $\mu_\psi C(X, \tau) = \{x, \emptyset, \{a, c\}\}$. Then $(X, \tau)$ is $\alpha T_{\mu_\psi}$-space. The space in the following example is not $\alpha T_{\mu_\psi}$-space. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, x, \{a, b\}\}$. Here $\mu_\psi C(X, \tau) = \{x, \emptyset, \{c\}, \{b, c\}, \{a, c\}\}$ and $\alpha C(X, \tau) = \{\emptyset, x, \{a, b\}\}$.

**Proposition 4.4:** If $(X, \tau)$ is a $\alpha T_{\mu_\psi}$-space then every singleton of X is either $\psi$–closed or $\mu$-open.

**Proof:** Let $x \in X$. Suppose $\{x\}$ is not $\psi$–closed, then $X - \{x\}$ is not $\psi$–open. This implies that X is the only $\psi$–open set containing X-{$x$}. So X-{$x$} is $\mu_\psi$-closed of $(X, \tau)$. Since $(X, \tau)$ is $\alpha T_{\mu_\psi}$-space, X-{$x$} is $\alpha$-closed and every $\alpha$-closed
is µ-closed implies X-{x} is µ–closed or equivalently {x} is µ–open. The converse of the above proposition is not true as it can be seen from the following example.

**Example 4.5:** Let X = {a, b, c} and τ = {∅, x, {a, c}}. Here every singleton of X is either ψ–closed or µ–open but is not αTµψ-space.

**Proposition 4.6:** Every αTµψ (resp. sTµψ)-space is pTµψ-space.

**Proof:** It follows from the fact that every α-closed (resp. semi-closed) is pre-closed. The converse of the above proposition is not true as it can be seen by the following example.

**Example 4.7:** Let X = {a, b, c} and τ = {∅, x, {a}, {b, c}}. Here (X, τ) is pTµψ–space but it is not a αTµψ (resp. not a sTµψ)-space.

**Proposition 4.8:** Every Tµψ-space is pTµψ-space, spTµψ-space, µTµψ–space and ψTµψ-space but not conversely.

**Example 4.9:** The space (X, τ) in Example 4.5 is pTµψ-space, spTµψ-space, µTµψ–space and ψTµψ-space but not Tµψ-space.

**Proposition 4.10:** Every Tµψ (resp. α T µψ) space is μTµψ-space, but not conversely.

**Proof:** Let A be µψ-closed set in a topological space X, which is T µψ-space. Hence A is closed implies A is µ-closed. Therefore T µψ-space is μ T µψ-space. Similarly A is µψ-closed set in topological space X which is α T µψ-space. Hence A is α-closed implies A is µ-closed. Therefore αTµψ-space is μ T µψ-space.

Converse is not true as it can be seen by the following example. The space (X, τ) in the example 4.9 is µT µψ–space but it is neither T µψ-space nor αT µψ-space.

**Theorem 4.11:** The following statements are true but the respective converses are not true in general.

1. If (X, τ) is a T µψ-space, then every singleton of X is either ψ-closed or pre-open.
2. If (X, τ) is a αT µψ-space, then every singleton of X is either ψ-closed or pre-open.
3. If (X, τ) is a sT µψ-space, then every singleton of X is either ψ-closed or pre-open.
4. If (X, τ) is a ψT µψ-space, then every singleton of X is either sg-closed or µψ-open.

**Proof:**

1. Let x ∈ X and suppose that {x} is not a ψ-closed of (X, τ). This implies that X-{x} is not ψ–open set. So X is the only ψ–open set such that X-{x} ⊆ X. Then X-{x} is a µψ –closed set of(X, τ). Since is a T µψ–space, then X-{x} is closed or equivalently {x} is open.
2. The proofs for the first assertions of 2 to 5 are similar to as that of the first assertions of (1). The space (X, τ) as in the example 4.7 shows that the converses of 1 to 5 need not be true.

**Remark 4.12:** The following diagram shows relationship among the spaces considered in this paper.
REFERENCES


Source of support: Nil, Conflict of interest: None Declared

[Copy right © 2016. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]