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EXISTENCE THEOREM AND EXTREMAL SOLUTIONS FOR PERTURBED MEASURE DIFFERENTIAL EQUATIONS WITH MAXIMA

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ABSTRACT

In this paper, an existence theorem and extremal solution for perturbed abstract measure differential equations with maxima is proved via hybrid fixed point theorem of Dhage [B.C.Dhage, On, on some nonlinear alternatives of Leray-Schauder type and functional integral equations, Arch. Math. (Brno) 42(2006) 11-23] under the mixed generalized Lipschitz and Caratheodory condition.

Keywords: *Abstract measure differential equation, Abstract measure- integro differential equation, Existence Theorem, Extremal Solution.*

1. INTRODUCTION

The study of abstract measure differential equations is initiated by Sharma [22, 23] and subsequently developed by Joshi [17], Shendge and Joshi [24]. Similarly, the study of abstract measure Integro-differential equation is studied by Dhage [10], Dhage and Bellale [13] for various aspects of solutions. In such models of differential equations, the ordinary derivative is replaced by the derivative of the set functions which there by gives the generalizations of the ordinary and measure differential equations. The various aspects of the solution of abstract measure differential equations have been studied in the literature using the fixed point techniques under continuous and discontinuous nonlinearies.

The perturbed ordinary differential equations have been treated in Krasnoselskli [18] and it is mentioned that the inverse of such equations yields the sum of two operators in appropriate function spaces. The Krasnoselskli [18] fixed point theorem is useful for proving the existence results for such perturbed differential equations under mixed geometrical and topological conditions on the nonlinearities involved in them.

The authors in [14] proved the existence and uniqueness results for abstract measure differential equations by using the Leray-schauder alternative [15] under Caratheodory conditions. In this paper by using the similar Leray-schauder alternative involving the sum of two operators, we extend the results of [14] to perturbed abstract measure differential equations for existence as well as for existence of the extremal solutions via hybrid techniques. Here our approach is different from that of Sharma, Joshi and Dhage.

In the present chapter we shall prove the existence result for perturbed abstract measure differential equations with maxima under Caratheodory condition. The existence of extremal solutions of the abstract measure integro-differential equations with maxima in question is also proved under certain monotonicity conditions of the nonlinearity involved in the equation. In the following section we give some preliminaries needed in the sequel

2. PRELIMINARIES

A mapping $T : X \to Y$ is called $D-$Lipschitz if there exists a continuous and nondecreasing function $g : IR^+ \to IR^+$ such that

$$\left\| T_x - T_y \right\| \le g\left(\left\| x - y \right\|\right) \text{ for all } x, y \in X \text{ where } g(0)=0$$

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In particular if $g(r)=\alpha r, \alpha > 0$, T is called a Lipschitz with a Lipschitz constant α. Further if $\alpha < 1$ then T is called a contraction on X with the contraction constant α.

Let X be a Banach space and $T: X \to X$. T is called compact if $\overline{T(X)}$ is a compact subset of X. $T(S)$ is a totally bounded subset of X. T is called completely continuous if T is continuous and totally bounded on X. Every compact operator is totally bounded but totally bounded operator may or may not be compact. The details of different types of nonlinear contraction, compact and completely continuous operators appears in Granas and Dugundji [15]

Let $(E, \leq, \| \|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be comparable if either the relation $x \leq y$ or $y \geq x$ holds. A non empty subset C of E is called a chain or totally ordered if all the elements of C are comparable. It is known that E is regular if $\{x_n\}$ is a nondecreasing (respectively nonincreasing) sequence E in such that as then for all the conditions guaranteeing the regularity of E may be found in Heikkila and Lakshmikantham [16].

We need the following definitions (see Dhage [6]) and the references there in what follows.

Definition 2.1: A mapping $T: E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\leq$, that is, if $x \leq y$ implies $T_x \leq T_y$ for all $x, y \in E$. Similarly, T is called monotone nonincreasing if $x \leq y$ implies $T_x \geq T_y$ for all $x, y \in E$. Finally T is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.

Definition 2.2: A mapping $T: E \to E$ is called partially continuous at a point $a \in E$ if $\in > 0$ there exists a $\delta > 0$ such that $\| T_x - T_a \| < \varepsilon$ whenever x is comparable to a and $\| x - a \| < \delta$. T is called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3: A non empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator T on a partially normed linear space E into itself is called partially bounded if $T(E)$ is a partially bounded subset of E. T is called uniformly partially bounded if all chains C in $T(E)$ are bounded by a unique constant.

Definition 2.4: A non empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is a relatively compact subset of E. A mapping $T: E \to E$ is called partially compact if $T(E)$ is a partially relatively compact subset of E. T is called partially compact if T is a uniformly partially bounded if for any bounded subset of E, $T(S)$ is a partially relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Remark 2.1: Suppose that T is a nondecrasing operator on E into itself, then T is a partially bounded or partially compact if $T(c)$ is a bounded or relatively compact subset of E for each chain c in E.

Theorem 2.1: Let U and $\overline{U}$ denote respectively the open and closed bounded subset of a Banach algebra X such that $0 \in U$. Let $A: X \to X$ and $B: \overline{U} \to X$ be two operators such that $B: \overline{U} \to X$ and $A: X \to X$

(1) B is completely continuous, and
(2) A is contraction
Then either
(I) The operator equation $Ax + Bx = x$ has a solution in $\overline{U}$, or
(II) There is a point $u \in \partial U$ such that satisfying $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u$ for some $0 < \lambda < 1$, where ∂U is a boundary of U in X.

Corollary 2.1: Let $B_r(0)$ and $\overline{B}_r(0)$ denote respectively the open and closed balls in a Banach algebra X centered at origin '0' of radius 'r' for same real number $r > o$. Let $B: \overline{B}_r(0) \to X$ and $A: X \to X$ be two operators such that

(1) B is completely continuous and

(2) A is contraction
Then either

(I) The operator equation $Ax + Bx = x$ has a solution x in X with $\| x \| \le r$, or

(II) There is an $u \in X$ such that $\| u \| = r$ satisfying $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u$ for some $0 < \lambda < 1$

3. STATEMENT OF THE PROBLEM

Let X be a real Banach algebra with a convenient norm $\| . \|$. Let $x, y \in X$. then

$$\overline{xy} = \{ z \in x \mid z = x + r(y-x), 0 \le r \le 1 \} \tag{1}$$

is called as line segment $\overline{xy}$ in X.

For any $x \in \overline{x_0 z}$ where $x_0 \in X$ be a fixed point and $z \in X$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{ r x \mid -\infty < r < 1 \} \tag{2}$$

And

$$\overline{S_x} = \{ r x \mid -\infty < r \le 1 \} \tag{3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x1} \subset S_{x2}$.

Let M denote the σ - algebra of all subsets of X such that (X, M) is a measurable space.

Let $ca(X, M)$ be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on $ca(X, M)$ by

$$\| p \| = | p |(X), \tag{4}$$

And

$$| p |(X) = \sup \sum_{i=1}^{\infty} | p(E_i) | \quad , \quad Ei \subset X \tag{5}$$

Where $| p |$ is a total variation measure of p and supremum is taken over all possible partitions $\{ E_i : i \in N \}$ of X. it is known that $ca(X, M)$ is a Banach space with respect to the norm $\| . \|$ given by (4).

Let μ be a $\sigma -$ finite positive measure on X. and $P << \mu$ denotes the p is absolutely continuous with respect to the measure μ if $\mu(E) = 0 \Rightarrow p(E) = 0$ for some $E \in M$, where $p \in ca(X, M)$.

Let $x_0 \in X$ be fixed and Let M_0 denote the $\sigma -$ algebra on S_{x0}. Let $z \in X$ be such that $Z. > x_0$ and let M_z denote the $\sigma -$ algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.

Given that $p \in ca(X, M)$ with p is absolutely continuous with respect to the measure μ if $\mu(E) = 0 \Rightarrow p(E) = 0$ for some $E \in M$.

$$\frac{dp}{d\mu} = f\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi) \right) + g\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi) \right) \quad \text{a.e. } [\mu] \text{ on } \overline{X_0 Z} \tag{6}$$

And

$$P(E) = q(E), \quad E \in M_0 \tag{7}$$

Where $\frac{dp}{d\mu}$ is Radon-Nikodym derivative of p with respect to μ, $f, g : S_z \times IR \to IR$,

$f\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi) \right)$ and $g\left(x, \max_{0 \le \xi \le \overline{S}_x} p(\xi) \right)$ is μ - integrable function for each $p \in ca(S_z, M_z)$, q is given known vector measure.

Definition 3.1: Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)$ $(z > x_0)$ is said to be a solution of abstract measure differential equation.

$$\frac{dp}{d\mu} = f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) + g\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) \text{ a.e.} [\mu] \text{ on } \overline{x_0 z}$$

and

$$p(E) = q(E), \; E \in M_0$$

P is absolutely continuous with respect to measure μ if $\mu(E) = 0 \Rightarrow p(E) = 0$ for some $E \in M$, p satisfies

$$\frac{dp}{d\mu} = f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) + g\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) \text{ a.e.} [\mu] \text{ on } \overline{x_0 z}.$$

Remark 3.1: The abstract measure differential equation (6) - (7) is equivalent to the abstract measure integral equation (In short AMIE)

$$P(E) = \int_E f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu + \int_E g\left(x, \max_{o \le \xi \le \bar{S}_x} p(\xi)\right) d\mu \tag{8}$$

If $E \in M_z, E \subset \overline{x_0 z}$ and $P(E) = q(E)$ if $E \in M_0$ (9)

A solution p of the abstract measure differential equation (6) - (7) on $\overline{x_0 z}$ will be denoted by $p\left(\bar{S}_{x0,} \; q\right)$.

4. EXISTENCE THEOREM

We need the following definition in what follows.

Definition 4.1: A function $\beta : S_z \times IR \to IR$ is called Caratheodory if

(1) $y \to \beta(x, y)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.

(2) $x \to \beta(x, y)$ is $\mu-$measurable for each $y \in IR$, and further a Caratheodory function $\beta(x, y)$ is called L^1_μ Caratheodory if

(3) For each real number $r > 0$ there exists a function $h_r \in L^1_\mu(S_z, IR)$ such that

$|\beta(x, y)| \le h_r(x) \;\; a.e.[\mu], \;\; x \in \overline{x_0 z}$ for all $y \in IR$ with $|y| \le r$.

We consider the following set of assumptions.

(H_1) There exists a $\mu-$integrable function $\alpha : S_z \to IR^+$ such that

$$\left| f(x, y_1) - f(x, y_2) \right| \le \alpha(x) \left| y_1 - y_2 \right| \; a.e. [\mu], \; x \in \overline{x_0 z} \text{ for all } y_{1,} \; y_2 \in IR$$

(H_2) For any $z > x_0$, the $\sigma-$ algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined on M_z by

$$d(E_{1,} E_2) \ge |\mu|(E_1 \Delta E_2) \text{ for all } E_1, E_2 \in M_z.$$

(H_3) $\mu(\{x_0\}) = 0.$

(H_4) q is continuous on M_z with respect to the pseudo-metric defined in assumptions (H_1)

(H_5) The function $g(x, y)$ is $L'^1_\mu-$ Caratheodory.

(H_6) There exists a function $\phi \in L'^1_\mu(S_z, M_z)$ such that $\phi(x) > 0$ *a.e.* $[\mu]$ on $\overline{x_o z}$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that $|g(x, y)| \le \phi(x)\psi(|y|)$ $a.e.[\mu]$ on $\overline{x_0 z}$ for all $y \in IR$.

Theorem 4.1: suppose that the assumption $(H_1) - (H_6)$ hold. Suppose $\exists \; r \in IR^+$ such that $\|\alpha\|_{L^1_\mu} < 1$ and

$$r > \frac{F_0 + \|q\| + \|\phi\|_{L^1_\mu} \psi(r)}{1 - \|\alpha\|_{L^1_\mu}} \tag{10}$$

Where $F_0 = \int_{x_0 z} |f(x, 0)| d\mu$, then abstract measure differential equations (6) - (7) has a solution on $\overline{x_0 z}$.

Proof: Consider the open ball $B_r(0)$ in $ca(S_z, M_z)$ centered at origin $'o'$ of radius $'r'$, where r is a positive real number satisfying the inequality (10).

We will define two operators such that

$$A : ca(S_z, M_z) \to ca(S_z, M_z) \quad B : B_r(0) \to ca(S_z, M_z) \text{ by } Ap(E) = \begin{cases} \int_E f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0 z} \\ 0, & E \in M_0 \end{cases} \tag{11}$$

And

$$Bp(E) = \begin{cases} \int_E g\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu, & \text{if } E \in M_Z, E \subset \overline{x_0 z} \\ q(E), & \text{if } E \in M_0 \end{cases} \tag{12}$$

We shall show that the operators A and B satisfy all the condition of corollary (2.1) on $\overline{B}_r(0)$.

Step-I: *A is contraction on* $ca(S_z, M_z)$
Let $p_1, p_2 \in ca(S_z, M_z)$ be arbitrary then by assumption (H_1).

$$\begin{aligned} \left| Ap_1(E) - Ap_2(E) \right| &= \left| \int_E f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu - \int_E g\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu \right| \\ &\le \int_E \alpha(x) \left| p_1(E) - p_2(E) \right| d\mu \\ &\le \| \alpha \|_{L^1_\mu} \left| p_1 - p_2 \right| (E) \end{aligned}$$

For all $E \in M_z$, Hence by definition of the norm in $ca(S_z, M_z)$ one has

$$\| Ap_1 - Ap_2 \| \le \| \alpha \|_{L^1_\mu} \| p_1 - p_2 \|$$

For all $p_1, p_2 \in ca(S_z, M_z)$. Hence A is a contraction on $ca(S_z, M_z)$ with the contraction constant $\| \alpha \|_{L^1_\mu}$.

Step-II: *B is continuous on* $\overline{B}_r(0)$.
Let $\{P_n\}$ be a sequence of vector measures in $\overline{B}_r(0)$ converging to a vector measure P, then by Dominated convergence theorem,

$$\begin{aligned} \lim_{n \to \infty} BP_n(E) &= \lim_{n \to \infty} \int_E f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu \\ &= \int_E f\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi)\right) d\mu \end{aligned}$$

For all $E \in M_Z$, $E \subset \overline{x_0 z}$, Similarly, if $E \in M_Z$, then

$$\lim_{n \to \infty} \overline{B} p_n(E) = q(E) = Bp(E)$$

And so B is a continuous operator on $\overline{B}_r(0)$.

Step-III: $\{BP_n : n \in N\}$ *is uniformly bounded in* $ca(S_z, M_z)$.

Let $\{p_n\}$ be a sequence in $\overline{B}_r(0)$, then we have $\| p_n \| \le r$ for all $n \in N$.

Suppose $E \in M_z$ then there exists two subsets $F \in M_0$ and $G \in M_Z$, $G \subset \overline{x_0 z}$ such that $E = F \cup G$ and $F \cap G = \phi$.

Hence by definition of B.

$$\left| Bp_n(E) \right| \le \left| q(F) \right| + \int_G \left| g\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi) \right) \right| d\mu \text{ From equation (5)}$$

$$\le \| q \| + \int_G h_r(x) d\mu .$$

$$\le \| q \| + \int_E h_r(x) d\mu$$

$$= \| q \| + \| h_r \|_{L^1_\mu}$$

From equation (5)

$$\| BP_n \| = | Bp_n | \left(\max_{0 \le \xi \le S_x} p(\xi) \right)$$

$$= \sup_\sigma \sum_{i=1}^{\infty} \left| Bp_n(E_i) \right|$$

$$= \| q \| + \| h_r \|_{L^1_\mu}$$

For all $n \in N$. Hence the sequence $\{Bp_n\}$ is uniformly bounded in $B\left(\bar{B}_r(0)\right)$.

Step-IV: $\{BP_n : n \in N\}$ *is an equicontinuous sequence in* $ca(S_z, M_z)$..

Let $E_1, E_2 \in M_z$, then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_0$, $G_1 \subset \overline{x_0 z}$ and $G_2 \subset \overline{x_0 z}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \phi$$

And

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \phi ,$$

We know that,

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{13}$$

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2) \tag{14}$$

Hence,

$$Bp_n(E_1) - Bp_n(E_2) \le q(F_1) - q(F_2) + \int_{G_1 - G_2} g\left(x, \max_{0 \le \xi \le \bar{S}_x} P_n(\xi) \right) d\mu + \int_{G_2 - G_1} g\left(x, \max_{0 \le \xi \le \bar{S}_x} p_n(\xi) \right) d\mu$$

Since $g(x, y)$ is L^1_μ- Caratheodory, we have

$$\left| Bp_n(E_1) - Bp_n(E_2) \right| \le \left| q(F_1) - q(F_2) \right| + \int_{G_1 \Delta G_2} \left| g\left(x, \max_{0 \le \xi \le \bar{S}_x} p(\xi) \right) \right| d\mu.$$

$$\le \left| q(F_1) - q(F_2) \right| + \int_{G_1 \Delta G_2} h_r(x) d\mu$$

Assume that,

$$d(E_1, E_2) = | \mu | (E_1 \Delta E_2) \to 0$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $| \mu | (G_1 \Delta G_2) \to 0$. As q is continuous on compact M_z. It is uniformly continuous and so

$$\left| Bp_n(E_1) - Bp_n(E_2) \right| \le \left| q(F_1) - q(F_2) \right| + \int_{G_1 \Delta G_2} h_r(x) d\mu \to 0 \text{ as } E_1 \to E_2$$

This shows that $\{Bp_n : n \in N\}$ is an equicontinuous set in $ca(S_z, M_z)$.

By application of the Arzela-Ascoli theorem yields that B is a totally bounded operator on $\overline{B}_r(0)$. Hence, B is continuous and totally bounded operator on $\overline{B}_r(o)$. So that the operator B is completely continuous operator on $\overline{B}_r(0)$.

By application of Corollary (2.1) yields that either the operator $Ax + Bx = x$ has a solution, or there is a $u \in ca\,(S_z, M_z)$ with $\|u\| = r$ such that $\lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u$ for some $0 < \lambda < 1$. we show that this assertion does not hold. Assume the contrary, then we have,

$$u(E) = \begin{cases} \lambda \int_E f\left(x, \max\limits_{0 \le \xi \le \overline{S}_x} u(\xi)\right) d\mu + \lambda \int_E g\left(x, \max\limits_{0 \le \xi \le \overline{S}_x} u(\xi)\right) d\mu, & if\ E \in M_z,\ E \subset \overline{x_0 z} \\ \lambda q(E), & if\ E \in M_o \end{cases}$$

For some $0 < \lambda < 1$.

If $E \in M_z$, then there exist sets $F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0\, z}$ such that $E = F \cup G$ and $F \cap G = \phi$. Hence,

$$u(E) = \lambda A\left(\frac{u(E)}{\lambda}\right) + \lambda Bu(E)$$

$$= \lambda q(F) + \lambda \int_G f\left(x, \frac{\max\limits_{0 \le \xi \le \overline{S}_x} u(\xi)}{\lambda}\right) d\mu + \lambda \int_G g\left(x, \max\limits_{0 \le \xi \le \overline{S}_x} u(\xi)\right) d\mu$$

Hence,

$$|u(E)| \le \lambda |q(F)| + \int_G \left[\, \left| f\left(x, \frac{\max\limits_{0 \le \xi \le \overline{S}_x} u(\xi)}{\lambda}\right) - f(x, 0) \right| + |f(x,0)| \right] d\mu + \lambda \int_G \left| g\left(x, \max\limits_{0 \le \xi \le \overline{S}_x} u(\xi)\right) \right| d\mu$$

$$\le \lambda \|q\| + \int_G \alpha(x) \left| \max\limits_{0 \le \xi \le \overline{S}_x} u(\xi) \right| d\mu + F_0 + \lambda \int_G \phi(x) \psi\left(\left| \max\limits_{0 \le \xi \le \overline{S}_x} u(\xi) \right| \right) d\mu$$

$$\le \|q\| + \|\alpha\|_{L^1_\mu} |u|(E) + F_0 + \int_G \phi(x)\, \psi\left(\max\limits_{0 \le \xi \le \overline{S}_x} |u|(\xi) \right) d\mu$$

$$\le \|q\| + \|\alpha\|_{L^1_\mu} |u|(E) + F_0 + \|\phi\|_{L^1_\mu} \psi\left(\max \|u\|\right)$$

Hence,

$$\|u\| \le \|q\| + \|\alpha\|_{L_\mu} \|u\| + F_0 + \|\phi\|_{L^1_\mu} \psi\left(\max\|u\|\right)$$

$$\|u\| \le \frac{\|q\| + F_0 + \|\phi\|_{L^1_\mu} \psi\left(\max\|u\|\right)}{1 - \|\alpha\|_{L^1_\mu}}$$

Substituting $\|u\| = r = \max\|u\|$ in the above inequality holds.

$$r \le \frac{\|q\| + F_0 + \|\phi\|_{L^1_\mu} \psi(r)}{1 - \|\alpha\|_{L^1_\mu}}.$$

Which is contraction to the equation no (10), the operator equation $p(E) = Ap(E) + Bp(E)$ has a solution $u\left(\overline{S}_{x_0}, q\right)$ in $ca(S_z, M_z)$ with $\|u\| \le r.$ so that abstract measure differential equation no (6) and (7) has a solution on $\overline{x_0\, z}$. Hence the proof.

5. EXITENCE OF EXTREMAL SOLUTIONS

Here we will prove the existence of the extremal solutions for the abstract measure differential equation (6) - (7) on $\overline{x_0\, z}$ under certain monotonicity conditions. We will define an order relation $\le$ in $ca\,(S_z, M_z)$ with the help of the cone K in $ca(S_z, M_z)$

$$K = \left\{ p \in ca(S_z, M_z) \,\middle|\, p(E) \ge 0\ for\ all\ E \in M_z \right\} \tag{15}$$

Thus for any $P_1, P_2 \in ca\left(S_z, M_z\right)$, one has

$$P_1 \le P_2 \quad \Leftrightarrow P_2 - P_1 \in K \tag{16}$$

Or equivalenty

$$P_1 \le P_2 \Leftrightarrow P_1(E) \le P_2(E)$$

For all $E \in M_z$. A cone K in $ca\left(S_z, M_z\right)$ is called normal if the norm is semi-monotone on K.

The detail of different properties of cones in Banach space appears in Heikkila and Lakshmikantham [16]

The bellow lemma gives the definition of the cone in $ca\left(S_z, M_z\right)$.

Lemma 5.1: The cone K defined by (15) is normal in $ca\left(S_z, M_z\right)$.

Proof: The proof of the lemma appears in Dhage *et al.* [14] and hence we omit the details

Theorem 5.1: Let K be a cone in real Banach space X and Let $A, B: X \to X$ be two nondecreasing operators such that

(a) A is contraction
(b) B is completely continuous, and
(c) There exist elements $u, v \in X$ such that $u \le v$ satisfying $u \le Au + Bu$ and $Av + Bv \le v$.

Further if the cone K is normal then the operator equation $Ax + Bx = x$ has a minimal and a maximal solution in $[u, v]$

Definition 5.1: A vector measure $u \in ca\left(S_z, M_z\right)$ is called a lower solution of abstract measure differential equation (6) – (7) if

$$\frac{du}{d\mu} \le f\left(x, \max_{0 \le \xi \le \bar{S}_x} u(\xi)\right) + g\left(x, \max_{0 \le \xi \le \bar{S}_x} u(\xi)\right) \quad a.e. \ [\mu] \ on \ \overline{x_0 z}$$

And

$$u(E) \le q(E), \qquad E \le M_0.$$

Similarly, a vector measure $v \in ca\left(S_z, M_z\right)$ is called an upper solution to abstract measure differential equation (6) - (7) if

$$\frac{dv}{d\mu} \ge f\left(x, \max_{0 \le \xi \le \bar{S}_x} v(\xi)\right) + g\left(x, \max_{0 \le \xi \le \bar{S}_x} v(\xi)\right) a.e. [\mu] \text{ on } \overline{x_0 z}.$$

And

$$u(E) \ge q(E), \qquad E \ge M_0.$$

A vector measure $p \in ca\left(S_z, M_z\right)$ is a solution to abstract measure differential equation (6)-(7) if is upper as well as lower solution to abstract measure differential equation On $\overline{x_0 z}$.

Definition 5.2: A solution P_M is called a maximal solution for the abstract measure differential equation (6) – (7) if for any other solution $p\left(\bar{S}_{x_0}, q\right)$ we have that

$$P(E) \le p_M(E), \qquad \forall E \in M_z,$$

Similarly, a minimal solution $P_m\left(\bar{S}_{x_0}, q\right)$ for the abstract measure differential equation (6) - (7) is defined on $\overline{x_0 z}$.

We consider the following assumptions.

(H_7) The functions $f(x, y)$ and $g(x, y)$ are nondecreasing in y a.e. $[\mu]$ for $x \in \overline{x_0 z}$.
(H_8) The abstract measure differential equation (6) - (7) has a lower solution u and an upper solution v such that $u \le v$ on M_z.

Theorem 5.2: Suppose that assumptions $(H_1)-(H_6)$, $(H_7)-(H_8)$ hold. Further suppose that $\|\alpha\|_{L^1_\mu} < 1$, then abstract measure differential equation (6) - (7) has a minimal and a maximal solution on $\overline{x_0\, z}$.

Proof: Abstract measure differetial equation (6) - (7) is equivalent to the Abstract integral Equation

$$p(E)=\int_E f\left(x, \max_{0\le\xi\le \bar{S}_x} p(\xi)\right) d\mu + \int_E g\left(x, \max_{0\le\xi\le \bar{S}_x} p(\xi)\right) d\mu,\ E\in M_{Z,}\quad E\subset \overline{x_0\, z} \tag{17}$$

And

$$P(E)=q(E),\qquad E\in M_0 \tag{18}$$

Define the operators $A, B:[u,v]\to ca(S_z, M_z)$ by

$$Ap(E)=\begin{cases}\int_E f\left(x, \max_{0\le\xi\le \bar{S}_x} p(\xi)\right), & E\in M_z,\ \ E\subset \overline{x_0\, z}\\ 0 & , if\ \ E\in M_0\end{cases} \tag{19}$$

And

$$Bp(E)=\begin{cases}\int_E g\left(x, \max_{0\le\xi\le \bar{S}_x} p(\xi)\right) d\mu & , if\ \ E\in M_z\ , E\subset \overline{x_0 z}\\ q(E), & if\ E\in M_0\end{cases} \tag{20}$$

Then the abstract measure integral equation (6) – (7) is equivalent to the operator equation

$$p(E)=Ap(E)+Bp(E),\qquad E\in M_z \tag{21}$$

To show that A and B satisfy all the condition of theorem (5.1) on $ca(S_z, M_z)$, since μ is a positive measure from assumptions (H_7), The operator A and B are nondecreasing on $ca(S_z, M_z)$.

Let $p_1, p_2\in ca(S_z, M_z)$ be such that $p_1\le p_2$ on M_z. Hence from assumptions (H_8) it follow that

$$\begin{aligned}Ap_1(E)&=\int_E f\left(x, \max_{0\le\xi\le \bar{S}_x} p_1(\xi)\right) d\mu\\ &\le\int_E f\left(x, \max_{0\le\xi\le \bar{S}_x} p_2(\xi)\right) d\mu\\ &=Ap_2(E)\end{aligned}$$

For all $E\in M_z,\ E\subset\overline{x_0\, z}$, again if $E\in M_0$, then

$$Bp_1(E)=q(E)=Bp_2(E).$$

Hence the operator B is nondecreasing on $ca(S_{z,} M_z)$. And already we have proved that in theorem (4.1) the operator A is a contraction with the contraction constant $\|\alpha\|_{L^1_\mu}$ and the operator B is completely continuous on X. Hence u is a lower solution of abstract measure differential equation (6) - (7)

$$u(E)\le\int_E f\left(x, \max_{0\le\xi\le \bar{S}_x} u(\xi)\right) d\mu + \int_E g\left(x, \max_{0\le\xi\le \bar{S}x} u(\xi)\right) d\mu,\quad E\in M_z, E\subset\overline{x_0\, z}$$

And

$$u(E)\le q(E),\qquad E\in M_{0.}$$

From equation no. (19) and (20) it follows that,

$$u(E)\le Au(E)+Bu(E),\qquad if\ E\in M_z$$

So that,

$$u \le Au + Bu$$

Similarly, since $v \in ca(S_z, M_z)$ is an upper solution of abstract measure differential equation,

$$v(E) \ge \int_E f\left(x, \max_{0 \le \xi \le \overline{S_x}} v(\xi)\right) d\mu + \int_E g\left(x, \max_{0 \le \xi \le \overline{S_x}} v(\xi)\right) d\mu, \qquad E \in M_{z,}\ \ E \subset \overline{x_0 z}$$

and

$$v(E) \ge q(E), \qquad E \in M_0 .$$

From equation no (19) and (20) it follow that

$v(E) \ge Av(E) + Bv(E)$ for all $E \in M_z$

And consequently

$v(E) \ge Av(E) + Bv(E)$ on M_z.

Thus hypothesis (a) - (c) of theorem (5.1) are satisfied .

Thus the operators A and B satisfy all the condition of theorem (5.1) and so an application of it yields that the operators equation $Ap + Bp = p$ has a maximal and minimal solution in $[u, v]$ which implies that abstract measure differential equation (6) - (7) has a maximal and minimal solution on $\overline{x_0 z}$. Hence the proof.

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