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COMMON FIXED POINTS IN COMPLEX-VALUED b-METRIC SPACES SATISFYING A SET OF RATIONAL NEQUALITIES

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ABSTRACT

*In 1989, Bakhtin introduced the notion of b-metric space (I. A. Bakhtin, The contraction principal in quasi-metric spaces, Functional Analysis **30**(1989), 26-37) as a generalization of metric space in which the triangle inequality was relaxed. Further, in 1993, Czerwik first proved a contraction mapping theorem for this space (S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis **1**(1993), 5-11) which generalized the well known Banach contraction mapping principal. In 2011, Azam et al. introduced the notion of complex-valued metric space (A. Azam, B. Fisher and M. S. Khan, Common fixed point theorems in complex-valued metric spaces, Numerical Functional Analysis & Optimization **32**(3)(2011), 243-253) to obtained a common fixed point result for a pair of self-mappings satisfying a rational inequality. Meanwhile, Jungck relaxed the concept of commutativity of a pair of mappings by compatibility [9], and further by weakly compatibility [10]. In this paper, we will prove some common fixed point theorems in complex-valued b-metric spaces for two pairs of self-mappings satisfying a set of rational inequalities using the weakly compatible mappings. Our result generalizes many results in the existing literature.*

AMS Subject Classification: *47H10, 54H25*

Keywords and Phrases: *Banach contraction mapping principal, common fixed point, complete metric space, complex-valued metric space, complex-valued b-metric space, weakly compatible mappings.*

Short Title: *Common fixed points in complex-valued b-metric spaces*

The first page will contain the title, the authors, an abstract, subject Classification same as the AMS, a short title, a list of key words and phrases and the complete address (es) of the author(s) with e-mail.

The novelty embodied in my work, or in the approach taken in my research: *In this paper, I have unified the notions of complex-valued metric space as well as of b-metric space. Since, every b-metric space is a metric spaces, my result will automatically apply for ordinary metric space (X, d). Also, since, a complex-valued metric **d** is a function from a set X×X into **C**, our theorem generalizes many quasi-contraction mapping on real-valued metric. This theorem also relaxes the commutativity and compatibility of mapping-pair, as well. Further, our theorem generalizes well known current results of complex-valued metric space like [1, 3, 4, 7, 11, 12, 16, 17, 19, 20, 21, 22, 25], results of b-metric spaces of [8, 15], and theorems on complex-valued b-metric spaces in the existing literature, like [14, 18].*

1. INTRODUCTION

Banach contraction principal [5] is a basic result in fixed point theory. This theorem has been generalized in many ways. Bakhtin [6] introduced the notion of b-metric space as a generalization of metric space in which the triangle inequality is relaxed. Czerwik [8] proved a contraction theorem in b-metric space which generalized the Banach contraction principal. Malhotra and Bansal [15] proved some common coupled fixed point theorems for generalized contraction in b-metric spaces. Azam *et al.* [2] introduced the notion of complex valued metric space as a generalization of metric space. They established sufficient conditions for the existence of common fixed points of a pair of mappings in this space satisfying a rational inequality. Jungck [9] introduced the notion of compatible mappings for a pair of mappings. This notion was further generalized to weakly compatible mappings [10]. In this line, Shukla-Pagey [20] and Verma-Pathak [24] proved some common fixed point theorems in complex valued metric spaces. More results on complex-valued metric spaces can be found in [1], [2], [3], [4], [7], [11], [12], [13], [16], [17], [19], [20], [21], [22], [23], [24], [25] etc. In 2014, Mukheimer [14] and Rao *et al.* [18] proved common fixed point theorems using complex valued b-metric space. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings in a complete complex valued b-metric space satisfying a set of contraction condition. Our theorem generalizes many results in the literature.

2. PRELIMINARIES

Let C be the set of complex numbers $z=a+ib$, where a, b are real numbers, a is called $Re(z)$ and b is called $Im(z)$. A complex valued metric d is a function from a set $X\times X$ into C. Let $z_1, z_2 \in C$. Define a partial order « on C as follows: $z_1 « z_2$ if and only if $Re(z_1) \le Re(z_2)$, $Im(z_1) \le Im(z_2)$. It follows that $z_1 « z_2$ if one of the following conditions satisfies:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In (i), (ii) and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So, $|z_1| \le |z_2|$ whenever $z_1 « z_2$. We will write $z_1 ‹ z_2$ if only (iii) satisfy. Hence $z_1 ‹ z_2$ implies $|z_1| < |z_2|$.

Remark ([11]): We note that the following statements hold. Let $z_1, z_2, z_3 \in C$.

(i) $a, b \in R$ and $a \le b$ implies $az « bz$,
(ii) $0 « z_1$ not« z_2 implies $|z_1| < |z_2|$,
(iii) $z_1 « z_2$ and $z_2 ‹ z_3$ implies $z_1 ‹ z_3$.

Azam *et al*. [2] defined complex-valued metric space (X, d) in the following way:

Definition 2.1 ([2]): Let X is a nonempty set. Suppose that the mapping $d: X\times X\to C$ satisfies the following conditions:
(A1) $0 « d(x, y)$, and $d(x, y) = 0$ if and only if $x=y$, for all $x, y \in X$;
(A2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(A3) $d(x, y) « d(x, z) + d(z, y)$, for all $x, y, z \in X$;
then d is called complex-valued metric, and (X, d) is called a complex-valued metric space.

Definition 2.2 ([15]): Let X is a nonempty set and let $s \ge 1$ be a real number. The mapping $d: X\times X\to R$ is called b-metric space if following three conditions satisfy:
(B1) $0 \le d(x, y)$, and $d(x, y) = 0$ if and only if $x=y$, for all $x, y \in X$,
(B2) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(B3) $d(x, y) \le s[d(x, z)+d(z, y)]$, for all $x, y, z \in X$. The number $s \ge 1$ is called the coefficient of b-metric space.

Example 2.3: Let $X = \{-1, 0, 1\}$. Define mapping $d: X\times X\to R^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 4$, $d(-1, 1) =1$, $d(0, 1) = 2$. Then (X, d) is a b-metric space, but not a metric space, since the triangle inequality is not satisfied. Indeed, $d(-1, 0)+d(0, 1)=4 + 2 = 6 > 1 = d(-1, 1)$ and $d(-1, 1)+d(-1, 0) = 1+4 =5>2= d(1, 0)$ are both true; but $d(-1, 1)+d(1, 0) = 1+2 = 3 < 4 = d(-1, 0)$ is not true. So, (X, d) is b-metric space with $s = 4/3$.

Example 2.4: Let $X = [0, 1]$ and $d: X\times X\to[0, \infty)$ be defined by $d(x, y) = (x - y)^2$, for all $x, y \in X$. Clearly, (X, d) is a b-metric space.

Example 2.5 ([15]): Let (X, d) be a metric space, $d^*(x, y) = (d(x, y))^p$ with $p>1$, then (X, d^*) is a b-metric space with $s=2^{p-1}$.

Remark 2.6 ([13]): In the b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.
(ii) Each convergent sequence is Cauchy sequence.
(iii) In general, a b-metric is not continuous (See Example in [13]).

On generalizing (A3), *i.e.*, triangle inequality, the complex valued b-metric space is defined in the following way:

Definition 2.7 ([14], [15], [18]): Let X be a nonempty set and mapping $d: X\times X\to C$ satisfy the following conditions:
(C1) $0 « d(x, y)$, and $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$,
(C2) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(C3) $d(x, y) « s[d(x, z)+d(z, y)]$, for all $x, y, z \in X$, where $s \ge 1$ is a real number, then d is called complex valued b-metric in X, and (X, d) is called complex valued b-metric space.

Example 2.8 ([18]): Let $X = [0, 1]$. Define a complex-valued metric $d: X\times X\to C$ by: $d(x, y) = |x-y|^2 + i|x-y|^2$, for all $x, y \in X$, then (X, d) is a complex valued b-metric space with $s=2$.

Remark 2.9: If $s=1$**,** then the complex-valued b-metric space always reduces to a complex-valued metric space. Thus every *complex-valued metric space is a complex valued b-metric space, but not conversely.* This generalizes the notion of a complex valued b-metric space over complex-valued metric space.

Definitions of interior point, limit point, open and closed sets, Hausdorff topology on X, the convergent sequence, Cauchy sequence and complete complex-valued metric space, can be seen from Rao *et.al.* [18]. Following lemmas are useful in our paper. Let us define weakly compatibility in the complex-valued b-metric space, below:

Lemma 2.10 ([18]): Let $\{x_n\}_{n\geq 1}$ be a sequence of a complex-valued b-metric space (X, d), then $\{x_n\}_{n\geq 1}$ converges to x if and only if $lim_{n\to\infty}|d(x_n , x)|=0$.

Lemma 2.11 ([18]): Let $\{x_n\}_{n\geq 1}$ be a sequence of a complex-valued b-metric space (X, d), then $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if and only if $lim_{n\to\infty}|d(x_n , x_{n+m})|=0$, where $m \in N$.

Definition 2.12 ([10]): Let (X, d) be a complex-valued metric space. A pair of self-mappings $A, S: X\to X$ is called weakly compatible if they commute at their coincidence points. That is, $ASu=SAu$, whenever $Au=Su$, for some $u \in X$.

Example 2.13: Let $X = R$. Define complex valued b-metric $d: X\times X\to C$ by $d(x, y) = |x-y|^2 + i|x-y|^2$, for all $x, y \in X$. Define mappings $f, g: X\to X$, by $f(x) = x/4$, $g(x) = x/5$, for all $x \in X$. Then f and g have coincidence point at $x = 0$. Now at this point, $fg(0) = gf(0)$. Thus (f, g) is weakly compatible at $x=0$.

In 2011, Azam *et al.* ([2]) proved the following result in complex valued metric space:

Theorem 2.14 ([2]): *Let (X, d) be a complete complex-valued metric space and let λ, μ be non-negative real numbers such that $\lambda+\mu< 1$. Suppose that $S, T: X\to X$ are mappings satisfying*

$$d(Sx, Ty) \ll \lambda.d(x, y)+ \mu.d(x, Sx)d(y, Ty)/\{1+d(x, y)\} \tag{2.1}$$

for all $x, y \in X$. Then S and T have a unique common fixed point in X.

This result was generalized by Mukheimer ([14]) for a complex valued b-metric space in the following way:

Theorem 2.15 ([14]): *Let (X, d) be a complete complex-valued b-metric space with parameter $s\geq 1$. Suppose that $S, T: X\to X$ are mappings satisfying:*

$$d(Sx, Ty) \ll \lambda.d(x, y)+\{\mu.d(x, Sx)d(y, Ty)/[1+d(x, y)]\},\ \text{for all } x, y \in X \tag{2.2}$$

where λ, μ are non-negative real numbers with $s\lambda+\mu< 1$. Then S and T have a unique common fixed point in X.

Mukheimer [14] also proved the following result, to generalize the main theorem of Bhatt *et al.*[3]:

Theorem 2.16 ([14]): *Let (X, d) be a complete complex valued b-metric space with coefficient $s \geq 1$ and let $S, T: X\to X$ be mappings satisfying:*

$$d(Sx, Ty) \ll [a\{d(x, Sx)d(x, Ty)+d(y, Ty)d(y, Sx)\}]/[d(x,Ty)+ d(y, Sx)]\} \tag{2.3}$$

for all $x, y \in X$, where $sa \in [0, 1)$. Then S and T have a unique common fixed point in X.

Other results in this line can be obtained in Ahmad *et.al.* [1] and Bhatt *et.al.* [3] for complex-valued metric space, and Mukheimer [14] for complex-valued b-metric space. In this paper, we prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying a set of rational inequalities in complex valued b-metric space.

3. MAIN RESULTS

Theorem 3.1: *Let $A, B, S, T: X\to X$ be self-mappings of a complete complex valued b-metric space (X, d), with metric parameter $s \geq 1$. Suppose that the set $B(X)$ is closed. If the following conditions satisfy:*
(i) $A(X)$ subseteq $T(X)$, $B(X)$ subseteq $S(X)$,
(ii) Let the set of inequalities is:

$$d(Ax,By) \ll q\ddot{U}_{x,y} (A, B, S, T),\ \text{for all } x, y \in X \tag{3.1}$$

where q is a non-negative real number such that $0\leq q<1/(s^2+s)$, and

*$\ddot{U}_{x, y} (A,B, S, T) \in \{d(Sx,Ty), d(Ax, Sx)d(By,Ty)/[d(Sx,Ty)+d(Ax,By)] , d(By,Sx)d(Ax,Ty)/[d(Sx,Ty)+d(Ax,By)] ,$
$d(Ax,Sx)d(By,Sx)/[d(Ax, By)+d(Sx,Ty)], d(Ax,Ty)d(By,Ty)/[1+d(Sx,Ty)], d(Ax,Sx)d(By,Ty)/[d(Sx,Ty)+d(Ax,By)],$
$d(Sx,Ty)d(Ax,By)/[d(Sx,Ty)+d(Ax, By)], d(By,Sx)d(Sx,Ty)/[d(Ax, By)+d(Sx,Ty)],$
$d(Ax,Sx)d(Sx,Ty)/[d(Ax,Sx)+d(Sx,Ty)], ½s.d(Sx,Ty)d(By,Ty)/[d(Ax,By)+d(Sx,Ty)]\}$,*
where $\ddot{U}_{x,y}(A,B, S, T)$ denotes the "selection of one co-ordinate amongst these options"
(iii) the pairs (A, S) and (B, T) are weakly compatible,
then A,B, S and T have unique common fixed point in X.

Proof: Since $s\geq 1$ so that $1/(s^2+s) \leq ½$, and $0\leq q < 1/(s^2+s)$, so, $0\leq qs<1/(s+1) \leq ½<1$, and $1-qs>1-1/(s+1)$. Also, we have $0\leq q\leq ½$. Let x_0 be an arbitrary point of X. Define a sequence $\{x_n\}$ in X, such that

$$Ax_{2n}=Tx_{2n+1}=y_{2n} \quad \text{and} \quad Bx_{2n+1}=Sx_{2n+2}=y_{2n+1},\ \text{for all } n = 0, 1, 2, 3 \tag{3.2}$$

We will show that $\{y_n\}$ is a Cauchy sequence. If $x_{2n}\neq x_{2n+1}$ then putting $x=x_{2n}$ and $y=x_{2n+1}$ in condition (ii), and denoting $d_n=d(y_{n+1}, y_n)$, we have

$$d_{2n} = d(y_{2n+1},y_{2n}) = d(Ax_{2n},Bx_{2n+1}) \precsim q.\ddot{U}x2n,x2n+1\ (A, B, S, T) \tag{3.3}$$

where $\ddot{U}x_{2n},x_{2n+1}\ (A, B, S, T) \in \{d_{2n-1},\ d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}],\ 0,\ d_{2n-1}d(y_{2n-1},y_{2n+1})/[d_{2n}+d_{2n-1}],\ 0,\ d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}],\ d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}],\ d(y_{2n-1},y_{2n+1})d_{2n-1}/[d_{2n}+d_{2n-1}],\ d_{2n-1}d_{2n-1}/[d_{2n-1}+d_{2n-1}],\ ½sd_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]\}$

i.e., $\ddot{U}x_{2n},\ x_{2n+1}(A,B,S,T) \in\{d_{2n-1},d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}],0,sd_{2n-1}[d_{2n-1}+d_{2n}]/[d_{2n}+d_{2n-1}],0,d_{2n-1}d_{2n}/[d_{2n-1}+d_{2n}],$
$d_{2n-1}d_{2n}/[\ d_{2n-1}+d_{2n}],\ s.[d_{2n-1}+d_{2n}].d_{2n-1}/[\ d_{2n}+d_{2n-1}],d_{2n-1}d_{2n-1}/2d_{2n-1},\ ½sd_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]\}$ (using **(C3)**)

We have following cases to consider:

Case-1: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = d_{2n-1}$, then from (3.1), we have $d_{2n} \precsim q.d_{2n-1}$ implies $|d_{2n}| \le q|d_{2n-1}|$.

Case-2: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n})$, then from (3.1), $d_{2n} \precsim q.d_{2n-1}d_{2n}/(d_{2n-1}+d_{2n})$, i.e., $d_{2n} \precsim -(1-q).d_{2n-1}$, whence $|d_{2n}| \le (1-q)|d_{2n-1}|$.

Case-3: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n} \precsim q.0$ implies $|d_{2n}| \le 0$.

Case-4: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = s.d_{2n-1}(d_{2n-1}+d_{2n})/(d_{2n}+d_{2n-1})$, then from (3.1), $d_{2n} \precsim qs.d_{2n-1}$ implies $|d_{2n}|\le qs|d_{2n-1}|$.

Case-5: If $\ddot{U}x_{2n}\ ,\ x_{2n+1}(A,B, S, T) = 0$, then from (3.1), $d_{2n} \precsim q.0$ implies $|d_{2n}| \le 0$.

Case-6: If $\ddot{U}x_{2n}\ ,\ x_{2n+1}(A,B, S, T) = d_{2n-1}d_{2n}/(\ d_{2n-1}+d_{2n})$, then from (3.1), $d_{2n} \precsim q.d_{2n-1}d_{2n}/(\ d_{2n-1}+d_{2n})$ implies $d_{2n}\precsim -(1-q)d_{2n-1}$ i.e., $|d_{2n}| \le (1-q)|d_{2n-1}|$.

Case-7: If $\ddot{U}x_{2n}\ ,\ x_{2n+1}(A, B, S, T)=d_{2n-1}.d_{2n}/[\ d_{2n-1}+d_{2n}]$, then $d_{2n}\precsim -(1-q)d_{2n-1}$, which implies $|d_{2n}| \le (1-q).|d_{2n-1}|$.

Case-8: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T)=s.[d_{2n-1}+d_{2n}].d_{2n-1}/[\ d_{2n}+d_{2n-1}]$, then $d_{2n}\precsim qs.d_{2n-1}$, i.e., $|d_{2n}| \le qs.|d_{2n-1}|$, where $qs<1$.

Case-9: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T)=q.d_{2n-1}d_{2n-1}/2d_{2n-1}$, then from (3.1), $d_{2n}\precsim ½q.d_{2n-1}$, i.e., $|d_{2n}|\le ½q.|d_{2n-1}|$

Case-10: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T)=½s.d_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]$, then from (3.1), $d_{2n}\precsim ½qs.d_{2n-1}d_{2n}/[d_{2n}+d_{2n-1}]$, or, $d_{2n}\precsim -(1-½qs).d_{2n-1}$, implies $|d_{2n}|\le (1-½qs)|d_{2n-1}|$.

Hence in all cases, we obtain $\boldsymbol{|d_{2n}|\le \lambda.|d_{2n-1}|}$ (3.4),
where $\lambda=max\{q,\ 1-q,\ 0,\ qs,\ ½q,\ 1-½qs\}<1$, as $qs\le½$ and $0\le q<1/(s^2+s) \le½$.

Similarly, by putting $x=x_{2n+2}$ and $y=x_{2n+1}$ in condition (ii), we have:

$d(Ax_{2n+2},Bx_{2n+1}) = d_{2n+1} \precsim q\ddot{U}x_{2n+2},x_{2n+1}\ (A, B, S, T)$, where $0<q \in R$ such that $0 \le q < 1/(s^2+s)$, and

$\ddot{U}_{x,\ y}(A,B,\ S,\ T) \in \{d(y_{2n+1},\ y_{2n}\ ),\ d(y_{2n+2},\ y_{2n+1})d(y_{2n+1},\ y_{2n}\ )/[d(y_{2n+1},\ y_{2n}\ )+d(y_{2n+2},\ y_{2n+1})]\ ,\ d(y_{2n+1},\ y_{2n+1})d(y_{2n+2},\ y_{2n})/[d(y_{2n+1},\ y_{2n})+d(y_{2n+2},\ y_{2n+1})]\ ,\ d(y_{2n+2},\ y_{2n+1})d(y_{2n+1},\ y_{2n+1})/[d(y_{2n+2},\ y_{2n+1})+d(y_{2n+1},\ y_{2n}\ )],\ d(y_{2n+2},\ y_{2n}\ ).d(y_{2n+1},\ y_{2n}\ )/[1+d(y_{2n+1},\ y_{2n}\ )],\ d(y_{2n+2},\ y_{2n+1})d(y_{2n+1},\ y_{2n}\ )/[d(y_{2n+1},\ y_{2n}\ )+d(y_{2n+2},\ y_{2n+1})],\ d(y_{2n+1},\ y_{2n}\ )d(y_{2n+2},\ y_{2n+1})/[\ d(y_{2n+1},\ y_{2n}\ )+d(y_{2n+2},\ y_{2n+1})],\ d(y_{2n+1},\ y_{2n+1})d(y_{2n+1},\ y_{2n}\ )/[\ d(y_{2n+2},\ y_{2n+1})+d(y_{2n+1},\ y_{2n}\ )],\ d(y_{2n+2},\ y_{2n+1})d(y_{2n+1},\ y_{2n}\ )/[d(y_{2n+2},y_{2n+1})+d(y_{2n+1},\ y_{2n}\ )],\ d(y_{2n+1},\ y_{2n})d(y_{2n+1},\ y_{2n}\ )/[d(y_{2n+2},\ y_{2n+1})+d(y_{2n+1},\ y_{2n}\ )]\},$

$\precsim \{d_{2n},\ d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1}),\ 0,\ 0,\ (d_{2n+1}+d_{2n}\ ).d_{2n}/(1+d_{2n}\ ),\ d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1}),\ d_{2n}\ .d_{2n+1}/(d_{2n}+d_{2n+1}),\ 0,\ d_{2n+1}d_{2n}/(d_{2n+1}+d_{2n}),\ d_{2n}.d_{2n}/(d_{2n+1}+d_{2n})\},$

We have the following cases to consider:

Case-1: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = d_{2n}$, then from (3.1), we have $d_{2n+1} \precsim q.d_{2n}$ implies $|d_{2n+1}\ | \le q|d_{2n}|$.

Case-2: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1})$, then from (3.1), $d_{2n+1} \precsim q.d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1})$, i.e., $d_{2n+1} \precsim -(1-q).d_{2n}$, whence $|d_{2n+1}| \le (1-q)|d_{2n}|$.

Case-3: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n+1} \precsim q.0$ implies $|d_{2n+1}| \le 0$.

Case-4: If $\ddot{U}x_{2n},\ x_{2n+1}(A, B, S, T) = 0$, then from (3.1), $d_{2n+1} \precsim q.0$ implies $|d_{2n+1}| \le 0$.

Case-5: If $\ddot{U}x_{2n}\ ,\ x_{2n+1}(A,B, S, T) = (d_{2n+1}+d_{2n}\ ).d_{2n}/(1+d_{2n}\ )$, then from (3.1), $d_{2n+1} \precsim q.d_{2n}^{\ 2}$.

Case-6: If $Ü x_{2n}, x_{2n+1}(A,B,S,T) = d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1})$, then from (3.1), $d_{2n+1} \ll q.\, d_{2n+1}d_{2n}/(d_{2n}+d_{2n+1})$, implies $d_{2n+1} \ll -(1-q)d_{2n}$ i.e., $|d_{2n+1}| \le (1-q)|d_{2n}|$.

Case-7: If $Ü x_{2n}, x_{2n+1}(A, B, S, T)=d_{2n+1}d_{2n}/(d_{2n+1}+d_{2n})$, then $d_{2n+1} \ll -(1-q)d_{2n}$, whence $|d_{2n+1}| \le (1-q)|d_{2n}|$.

Case-8: If $Ü x_{2n}, x_{2n+1}(A, B, S, T)=0$, then $d_{2n+1} \ll q.0$, i.e., $|d_{2n+1}| \le 0$.

Case-9: If $Ü x_{2n}, x_{2n+1}(A, B, S, T)=d_{2n+1}d_{2n}/(d_{2n+1}+d_{2n})$, then from (3.1), $d_{2n+1} \ll -(1-q).d_{2n}$, i.e., $|d_{2n+1}| \le (1-q)|d_{2n}|$

Case-10: If $Ü x_{2n}, x_{2n+1}(A, B, S, T)= d_{2n}^2/(d_{2n+1}+d_{2n})$, then from (3.1), $d_{2n+1} \ll ½qs.d_{2n}d_{2n}/(d_{2n+1}+d_{2n})$, i.e., $d_{2n+1}^2 + d_{2n}.d_{2n+1} \ll ½qs\, d_{2n}^2$, which implies $(d_{2n+1} + ½d_{2n})^2 \ll (½qs+1/4)\, d_{2n}^2$,

So that $d_{2n+1} \ll [-½\pm\sqrt{(½qs+1/4)}]d_{2n}$, whence $|d_{2n+1}|<|-½\pm\sqrt{(½qs+1/4)}|\, |d_{2n}|$, where $|-½\pm\sqrt{(½qs+1/4)}|<1$, as $0\le qs\le ½$.

Hence in all cases, we obtain $|\boldsymbol{d_{2n+1}}| \le \boldsymbol{\mu}.|\boldsymbol{d_{2n}}|$ (3.5)
where $\boldsymbol{\mu}=max\{q, 1-q, 0, |-½\pm\sqrt{(½qs+1/4)}|\}<1$.

If we write $k=max\{\lambda, \mu\}=max\{q, 1-q, 0, qs, ½q, 1-½qs, |-½\pm\sqrt{(½qs+1/4)}|\}<1$, (3.6)
then for each $k \in N$, we have

$$|\boldsymbol{d_{n+1}}| \le \boldsymbol{k}|\boldsymbol{d_n}|, \text{ where } k<1 \tag{3.7}$$

From (3.7) we have

$$|d_n| \le k|d_{n-1}| \le k^2|d_{n-2}| \le k^3|d_{n-3}| \le \dots \le k^{n-1}|d_1| \le k^n|d_0| \tag{3.8}$$

In order to prove the Cauchy condition, let $m, n \in N$, with $m>n$, we have on using (C3):

$$\begin{aligned}|d(y_m, y_n)| &\le s(|d(y_n, y_{n+1})+d(y_{n+1}, y_m)|=s|d_n|+s|d(y_{n+1}, y_m)|\\ &\le s|d_n|+s\{s|d(y_{n+1}, y_{n+2})+d(y_{n+2}, y_m)|\}= s|d_n|+s^2|d_{n+1}|+s^2|\, d(y_{n+2}, y_m)|\\ &\le s|d_n|+s^2|d_{n+1}|+s^2\{s|d(y_{n+2}, y_{n+3})+d(y_{n+3}, y_m)|\}= s|d_n|+s^2|d_{n+1}|+s^3|d_{n+2}|+s^3(y_{n+3}, y_m)|\}\\ &\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\le s|d_n|+s^2|d_{n+1}|+s^3|d_{n+2}|+\dots\dots+ s^{m-n-2}|d_{m-3}|+ s^{m-n-1}|d_{m-2}|+ s^{m-n}|d_{m-1}|\end{aligned}$$

Using (3.8), it yields

$$\begin{aligned}|d(y_m, y_n)| &\le sk^n|d_0|+s^2k^{n+1}|d_0|+s^3k^{n+2}|d_0|+\dots\dots+ s^{m-n-2}k^{m-3}|d_0|+ s^{m-n-1}k^{m-2}|d_0|+ s^{m-n}k^{m-1}|d_0|\\ &= \textstyle\sum_{i=1}^{i=m-n} s^i k^{i+n-1}|d_0|\\ &\le \textstyle\sum_{i=1}^{i=m-n} s^{i+n-1}k^{i+n-1}|d_0|\\ &= \textstyle\sum_{t=n}^{t=m-1} s^t k^t|d_0|\\ &\le \textstyle\sum_{t=n}^{t=\infty} s^t k^t|d_0|\\ &= [(sk)^t/(1-sk)].|d_0|,\end{aligned}$$

This tends to zero as $m, n\to \infty$. It shows that $\{y_n\}$ is a Cauchy sequence in X. Further, since X is complete, there exists a point z in X such that $lim_{n\to\infty}y_n=z$.

Now, since $B(X)$ is closed, let $z \in B(X)$. Since, $B(X)$ subseteq $S(X)$, there exist $u \in X$ such that $z=Su$. We claim that $Au=z$. If not, then putting $x=u$, $y=x_{2n+1}$ in condition (ii), and using (3.1), we have

$$d(Au, y_{2n+1})=d(Au, Bx_{2n+1}) \ll qÜ_{u,\, x2n+1}(A, B, S, T) \tag{3.9}$$

where
$Ü_{u, y}(A,B, S, T) \in \{d(Su,Tx_{2n+1}), d(Au, Su)d(B x_{2n+1},Tx_{2n+1})/[d(Su, Tx_{2n+1})+d(Au,Bx_{2n+1})], d(Bx_{2n+1}, Su)d(Au,Tx_{2n+1})/[d(Su,Tx_{2n+1})+d(Au, Bx_{2n+1})], d(Au, Su)d(Bx_{2n+1},Su)/[d(Au, Bx_{2n+1})+d(Su, Tx_{2n+1})], d(Au, Tx_{2n+1})d(Bx_{2n+1},Tx_{2n+1})/[1+d(Su,Tx_{2n+1})], d(Au, Su)d(Bx_{2n+1},Tx_{2n+1})/[d(Su, Tx_{2n+1})+d(Au,Bx_{2n+1})], d(Su, Tx_{2n+1})d(Au, Bx_{2n+1})/[d(Su, Tx_{2n+1})+d(Au, Bx_{2n+1})], d(Bx_{2n+1}, Su)d(Su,Tx_{2n+1})/[d(Au, Bx_{2n+1})+d(Su,Tx_{2n+1})], d(Au, Su)d(Su, Tx_{2n+1})/[d(Au, Su)+d(Su,Tx_{2n+1})], ½s.d(Su, Tx_{2n+1})d(Bx_{2n+1},T x_{2n+1})/[d(Au, Bx_{2n+1})+d(Su, Tx_{2n+1})]\}$,

$=\{d(z,y_{2n}), d(Au, Su)d(y_{2n+1},y_{2n})/[d(z, y_{2n})+d(Au,y_{2n+1})], d(y_{2n+1},z)d(Au,y_{2n})/[d(z,y_{2n})+d(Au, y_{2n+1})], d(Au, z)d(y_{2n+1}, z)/[d(Au, y_{2n+1})+d(z,y_{2n})], d(Au,y_{2n})d(y_{2n+1},y_{2n})/[1+d(z,y_{2n})], d(Au, z)d(y_{2n+1},y_{2n})/[d(z,y_{2n})+d(Au, y_{2n+1})], d(z,y_{2n})d(Au,y_{2n+1})/[d(z,y_{2n})+d(Au, y_{2n+1})], d(y_{2n+1}, z)d(z, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})], d(Au, z)d(z, y_{2n})/[d(Au, z)+d(z, y_{2n})], ½s.d(z, y_{2n})d(y_{2n+1},y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]\}$, (using (3.2))

We will take following cases, as follows:

Case-1: If $Ü_{u,\, x2n+1}(A, B, S, T) = d(z,y_{2n})$, then we have $d(Au, y_{2n+1}) \ll q.\, d(z, y_{2n})$, implies $|d(Au, y_{2n+1})| \le q|d(z, y_{2n})|$. Taking $n\to\infty$, it yields $|d(Au, z)| \le q|d(z, z)|=0$. Thus $Au=z$.

Case-2: If $\ddot{U}_{u,\,x2n+1}(A, B, S, T) = d(Au, Su)d(y_{2n+1},y_{2n})/[d(z, y_{2n})+d(Au,y_{2n+1})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(Au, Su)d(y_{2n+1},y_{2n})/[d(z, y_{2n})+d(Au,y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \le q|d(Au, Su)d(y_{2n+1},y_{2n})|/|\, d(z, y_{2n})+d(Au,y_{2n+1})|$
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.|d(Au, z)|.0/|d(Au, z)|=0$. Thus $Au=z$.

Case-3: If $\ddot{U}_{u,\,x2n+1}(A, B, S, T) = d(Bx_{2n+1},z)d(Au,Tx_{2n+1})/[d(z,Tx_{2n+1})+d(Au, Bx_{2n+1})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(y_{2n+1},z)d(Au,y_{2n})/[d(z,y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})|\le q|\, d(y_{2n+1},z)d(Au,y_{2n})\, |/|\, d(z,y_{2n})+d(Au, y_{2n+1})\, |$.
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.0.d(Au, z)/|d(Au, z)|=0$, or, $|d(Au, z)| \le 0$. Thus $Au=z$.

Case-4: If $\ddot{U}_{u,\,x2n+1}(A, B, S, T) = d(Au, z)d(y_{2n+1}, z)/[d(Au, y_{2n+1})+d(z,y_{2n})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(Au, z)d(y_{2n+1}, z)/[d(Au, y_{2n+1})+d(z,y_{2n})]$,
i.e., $|d(Au, y_{2n+1})|\le q|d(Au, z)d(y_{2n+1}, z)|/|d(Au, y_{2n+1})+d(z, y_{2n})\, |$.
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.d(Au, z).0/d(Au, z)]=0$, i.e., $|d(Au, z)| \le 0$. Thus $Au=z$.

Case-5: If $\ddot{U}_{u\,,\,x2n+1}(A,B, S, T) = d(Au,y_{2n})d(y_{2n+1},y_{2n})/[1+d(z,y_{2n})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(Au,y_{2n})d(y_{2n+1},y_{2n})/[1+d(z,y_{2n})]$
i.e., $|d(Au, y_{2n+1})| \le q.\, |d(Au,y_{2n})d(y_{2n+1},y_{2n})|/|1+d(z,y_{2n})|$.
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.|d(Au, z)|.0/(1+0)=0$, i.e., $|d(Au, z)| \le 0$. Thus $Au=z$.

Case-6: If $\ddot{U}_{u,\,x2n+1}(A, B, S, T) = d(Au, z)d(y_{2n+1},y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(Au, z)d(y_{2n+1},y_{2n})/[d(z, y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \le q|d(Au, z)d(y_{2n+1},y_{2n})|/|d(z, y_{2n})+d(Au, y_{2n+1})|$,
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.|d(Au, z)|.0/|d(Au, z)|=0$. Thus $Au=z$.

Case-7: If $\ddot{U}_{u,\,x2n+1}(A, B, S, T)= d(z,y_{2n})d(Au,y_{2n+1})/[d(z,y_{2n})+d(Au, y_{2n+1})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(z,y_{2n})d(Au,y_{2n+1})/[d(z,y_{2n})+d(Au, y_{2n+1})]$,
i.e., $|d(Au, y_{2n+1})| \le q|\, d(z,y_{2n})d(Au,y_{2n+1}\, |/|d(z,y_{2n})+d(Au, y_{2n+1})|$,
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.0.|d(Au, z)|/|d(Au, z)|=0$. Thus $Au=z$.

Case-8: If $\ddot{U}_{u},\, x_{2n+1}(A, B, S, T) = d(y_{2n+1}, z)d(z, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(y_{2n+1}, z)d(z, y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$,
i.e., $|d(Au, y_{2n+1})| \le q|d(y_{2n+1}, z)d(z, y_{2n})|/|d(Au, y_{2n+1})+d(z, y_{2n})|$,
Taking $n\to\infty$, it yields $|d(Au, z)|\le q.0.0/|d(Au, z)|=0$. Thus $Au=z$.

Case-9: If $\ddot{U}_{u},\, x_{2n+1}(A, B, S, T) = d(Au, z)d(z, y_{2n})/[d(Au, z)+d(z, y_{2n})]$, then we have
$d(Au, y_{2n+1}) \ll q.\, d(Au, z)d(z, y_{2n})/[d(Au, z)+d(z, y_{2n})]$,
i.e., $|d(Au, y_{2n+1})| \le q|d(Au, z)d(z, y_{2n})|/|d(Au, z)+d(z, y_{2n})|$,
Taking $n\to\infty$, it yields $|d(Au, z)| \le q.|d(Au, z)|.0/|d(Au, z)|=0$. Thus $Au=z$.

Case-10: If $\ddot{U}_{u},\, x_{2n+1}(A, B, S, T) = ½s.d(z, y_{2n})d(y_{2n+1},y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$, then we have
$d(Au, y_{2n+1}) \ll ½qs.d(z, y_{2n})d(y_{2n+1},y_{2n})/[d(Au, y_{2n+1})+d(z, y_{2n})]$
i.e., $|d(Au, y_{2n+1})|\le ½qs|d(z, y_{2n})d(y_{2n+1},y_{2n})|/|d(Au, y_{2n+1})+d(z, y_{2n})|$
Taking $n\to\infty$, it yields $|d(Au, z)| \le ½qs.0.0/|d(Au, z)|=0$, or, $|d(Au, z)| \le 0$. Thus $Au=z$.

Hence in all cases $Au=z=Su$, i.e., $u \in X$ is a coincidence point of (A, S). Further, since $z=Au$ and *AX subseteq TX*, there exists $v \in X$ such that $Au=Tv$. We claim that $Bv=z$. If not, then putting $x=u$, $y=v$ in condition (ii), and using $Au=Su =Tv =z$, we have

$$d(z, Bv)=d(Au, Bv) \ll q.\ddot{U}_{u,\,v}\,(A, B, S, T) \tag{3.10}$$

where $\ddot{U}_{u,\,v}\,(A, B, S, T) \in \{d(Su,Tv), d(Au,Su)d(Bv,Tv)/[d(Su,Tv)+d(Au,Bv)], d(Bv,Su)d(Au,Tv)/[d(Su,Tv)+d(Au,Bv)]\,, d(Au,Su)d(Bv,Su)/[d(Au, Bv)+d(Su,Tv)], d(Au,Tv)d(Bv,Tv)/[1+d(Su,Tv)], d(Au,Su)d(Bv,Tv)/[d(Su,Tv)+d(Au, Bv)],$

$d(Su,Tv)d(Au,Bv)/[d(Su,Tv)+d(Au, Bv)], d(Bv,Su)d(Su,Tv)/[d(Au, Bv)+d(Su,Tv)], d(Au,Su)d(Su,Tv)/[d(Au,Su)+d(Su,Tv)], ½sd(Su,Tv)d(Bv,Tv)/[d(Au,Bv)+d(Su,Tv)]\}$,

$= \{0,0,0,0,0,0, 0,0, d(z,z)d(z,z)/2d(z,z), ½s.d(z,z)d(Bv,z)/[d(z,Bv)+d(z,z)]= \{0, ½d(z,z)\}=0.$

Thus $Au = Su = z = Bv = Tv$, i.e., u and v are coincidence points of (A, S) and (B, T) respectively. These two pairs are weakly compatible therefore, from condition (iii), we have $ASu=SAu=Az=Sz$ and $BTv=TBv=Bz=Tz$. We now claim that $Az=Bz$. If not, then putting $x=z$, $y=z$ in condition (ii), we have

$$d(Az,Bz) \ll q\ddot{U}z,\, z\,(A, B, S, T) \tag{3.11}$$

where $\ddot{U}_{z,z}(A,B,S,T) \in \{d(Sz,Tz), d(Az,Sz)d(Bz,Tz)/[d(Sz,Tz)+d(Az,Bz)], d(Bz,Sz)d(Az,Tz)/[d(Sz,Tz)+d(Az,Bz)], d(Az,Sz)d(Bz,Sz)/[d(Az,Bz)+d(Sz,Tz)], d(Az,Tz)d(Bz,Tz)/[1+d(Sz,Tz)], d(Az,Sz)d(Bz,Tz)/[d(Sz,Tz)+d(Az,Bz)], d(Sz,Tz)d(Az,Bz)/[d(Sz,Tz)+d(Az,Bz)], d(Bz,Sz)d(Sz,Tz)/[d(Az,Bz)+d(Sz,Tz)], d(Az,Sz)d(Sz,Tz)/[d(Az,Sz)+d(Sz,Tz)], \tfrac{1}{2}s.d(Sz,Tz)d(Bz,Tz)/[d(Az,Bz)+d(Sz,Tz)]\},$

$$= \{0, \tfrac{1}{2}d(Az,Bz), 0, 0, 0, \tfrac{1}{2}d(Az,Bz), \tfrac{1}{2}d(Az,Bz), 0, 0\}.$$

If 0 is chosen then $d(Az,Bz) \ll q.0$ yields $Az=Bz$. Also, if $\tfrac{1}{2}d(Az,Bz)$ is chosen then (3.11) implies $d(Az,Bz) \ll \tfrac{1}{2}q\, d(Az,Bz)$, *i.e.*, $|d(Az,Bz)| \le \tfrac{1}{2}q.|d(Az,Bz)|$, a contradiction. Thus $Az=Bz$. Therefore z is a coincidence point of A, B, S and T.

We now claim that z is a common fixed point of these four mappings. For, putting $x=u$, $y=z$ in condition (ii), and using $Au=Su=z$, $Bz=Tz$, we have

$$d(z,Bz)=d(Au,Bz) \ll q\ddot{U}_{u,z}(A,B,S,T), \tag{3.12}$$

where $\ddot{U}_{u,z}(A,B,S,T) \in \{d(Su,Tz), d(Au,Su)d(Bz,Tz)/[d(Su,Tz)+d(Au,Bz)], d(Bz,Su)d(Au,Tz)/[d(Su,Tz)+d(Au,Bz)], d(Au,Su)d(Bz,Su)/[d(Au,Bz)+d(Su,Tz)], d(Au,Tz)d(Bz,Tz)/[1+d(Su,Tz)], d(Au,Su)d(Bz,Tz)/[d(Su,Tz)+d(Au,Bz)], d(Su,Tz)d(Au,Bz)/[d(Su,Tz)+d(Au,Bz)], d(Bz,Su)d(Su,Tz)/[d(Au,Bz)+d(Su,Tz)], d(Au,Su)d(Su,Tz)/[d(Au,Su)+d(Su,Tz)], \tfrac{1}{2}sd(Su,Tz)d(Bz,Tz)/[d(Au,Bz)+d(Su,Tz)]\},$

$$= \{d(z,Bz), 0, \tfrac{1}{2}d(Bz,z), 0, 0, 0, \tfrac{1}{2}d(z,Bz), \tfrac{1}{2}.d(Bz,z), 0, \tfrac{1}{2}s.d(z,Bz)\}=\{d(z,Bz), 0, \tfrac{1}{2}s.d(z,Bz)\}.$$

If $d(z,Bz)$ is chosen, then $d(z,Bz) \ll q.\,d(z,Bz)$ implies $|d(z,Bz)| \le q.|d(z,Bz)|$, a contradiction, thus $Bz=z$. If $\tfrac{1}{2}d(z,Bz)$ is chosen, then $d(z,Bz) \ll \tfrac{1}{2}qd(z,Bz)$ implies $|d(z,Bz)| \le \tfrac{1}{2}q|d(z,Bz)|$, a contradiction, thus $Bz=z$. If 0 is chosen then also, $Bz=z$. Therefore z is a common fixed point of A, B, S and T. This common fixed point is unique. If not, then assume $w \in X$ be another common fixed point such that $z \neq w$. Now, by putting $x=z$ and $y=w$ in condition (ii) and using $Az=Sz=z$, $Bw=Tw=w$, we get a contradiction. Thus z is the unique common fixed point of A, B, S and T. This completes the proof.

If $A=B=f$ and $S=T=g$ then Theorem 3.1 reduces to following corollary:

Corollary 3.2: *Let (X, d) be a complete complex valued b-metric space with metric parameter $s \ge 1$ and $f, g: X \to X$ be self-mappings on X. Suppose that the set $f(X)$ is closed, and $f(X)$ subseteq $g(X)$. If the following conditions satisfy:*

$$d(fx,fy) \ll q.\ddot{U}_{x,y}(f,g), \text{ for all } x,y \in X, \text{ where } q \text{ is a non-negative real number such that } 0 \le q < 1/(s^2+s) \tag{3.12}$$

where $\ddot{U}_{x,y}(f,g) \in \{d(gx,gy), d(fx,gx)d(fy,gy)/[d(gx,gy)+d(fx,fy)], d(fy,gx)d(fx,gy)/[d(gx,gy)+d(fx,fy)], d(fx,gx)d(fy,gx)/[d(fx,fy)+d(gx,gy)], d(fx,gy)d(fy,gy)/[1+d(gx,gy)], d(fx,gx)d(fy,gy)/[d(gx,gy)+d(fx,fy)], d(gx,gy)d(fx,fy)/[d(gx,gy)+d(fx,fy)], d(fy,gx)d(gx,gy)/[d(fx,fy)+d(gx,gy)], d(fx,gx)d(gx,gy)/[d(fx,gx)+d(gx,gy)], \tfrac{1}{2}sd(gx,gy)d(fy,gy)/[d(fx,fy)+d(gx,gy)]\}.$

If the pair (f, g) is weakly compatible, then A, B, S and T have unique common fixed point in X.

If we put $S=T=I$, the identity mapping, then condition (i) of Theorem 3.1 reduces to $A(X)$ *subseteq* X, $B(X)$ *subseteq* X, which is always true. Further, the weakly compatibility reduces to $AIx=IAx$, and $BIx=IBx$, where $Ix=x$; thus weakly compatibility condition always hold. Therefore, Theorem 3.1 reduces to following corollary:

Corollary 3.2: *Let (X, d) be a complete complex valued b-metric space and $A, B: X \to X$ be self-mappings of (X, d) with metric parameter $s \ge 1$. Suppose that the set $B(X)$ is closed. If the following inequality:*

$$d(Ax,By) \ll q\ddot{U}_{x,y}(A,B), \text{ for all } x,y \in X, \text{ where } q \text{ is a non-negative real number such that } 0 \le q < 1/(s^2+s) \tag{3.13}$$

where $\ddot{U}_{x,y}(A,B) \in \{d(x,y), d(Ax,x)d(By,y)/[d(x,y)+d(Ax,By)], d(By,x)d(Ax,y)/[d(x,y)+d(Ax,By)], d(Ax,x)d(By,x)/[d(Ax,By)+d(x,y)], d(Ax,y)d(By,y)/[1+d(x,y)], d(Ax,x)d(By,y)/[d(x,y)+d(Ax,By)], d(x,y)d(Ax,By)/[d(x,y)+d(Ax,By)], d(By,x)d(x,y)/[d(Ax,By)+d(x,y)], d(Ax,x)d(x,y)/[d(Ax,x)+d(x,y)], \tfrac{1}{2}s.d(x,y)d(By,y)/[d(Ax,By)+d(x,y)]\},$
then A,B, S and T have unique common fixed point in X.

Remark 3.3: Corollary 3.2 generalizes Banach contraction mapping principal [5] in complex-valued b-metric space, if only first co-ordinate is chosen. Also, since $q<\tfrac{1}{2}$, the sum of product of first co-ordinate with λ and product of fifth co-ordinate with μ, will produce the inequality of Azam *et al.* [2] with $\lambda+\mu<1$. Hence we have generalized [2] in complex-valued b-metric spaces.

REFERENCES

1. J. Ahmad, A. Azam and S. Saejung, Common fixed point results for contractive mappings in complex-valued metric spaces, *Fixed Point Theory& Appication* 67(2014),
2. A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex-valued metric spaces, *Numerical Functional Analysis & Optimization* 32(3)(2011), 243-253
3. S. Bhatt, S. Choukiyal and R. C. Dimri, Common fixed point of mappings satisfying rational inequality in complex-valued metric spaces, *International Journal of Pure & Applied Mathematics* 73(2)(2011), 159-164
4. S. Bhatt, S. Choukiyal and R. C. Dimri, A common fixed point theorem for weakly compatible maps in complex-valued metric spaces, *International Journal of Mathematical Analysis* 1(3), (2011), 1385-1389
5. S. Banach, S¨ur les op´erations dans les ensembles abstraits et leur application aux ˜equations int´egrales, *Fund. Math*. 3(1922), 133-181
6. I. A. Bakhtin, The contraction principal in quasi-metric spaces, *Functional Analysis* 30(1989), 26-37
7. S. Chandok and D. Kumar, Some common fixed point results for rational type contraction mappings in complex valued metric spaces, *J. Oper.* 2013, Article ID813707 (2013)
8. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis* 1(1993), 5-11
9. G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics & Mathematical Sciences* 9(4)(1986), 771-779
10. G. Jungck, Common fixed points for non-continuous non-self mappings on a non-numeric spaces, *Far East J. Math. Sci.* 4(2)(1996), 199-212
11. M. Kumar, P. Kumar and S. Kumar, Common fixed point theorems in complex valued metric spaces, *Journal of Analysis & Number Theory* 2(2)(2014), 103-109
12. M. Kumar, P. Kumar, S. Kumar and S. M. Kang, Common fixed points for weakly compatible mappings in complex valued metric spaces, *International Journal of Pure & Applied Mathematics* 92(3)(2014), 403-419
13. S. K. Mohanta, Coincidence points and common fixed points for expansive type mappings in b-metric spaces, Iranian Journal of Mathematical Sciences and Informatics Vol. 11, No. 1 (2016), pp 101-113, DOI: 10.7508/ijmsi.2016.01.009
14. S. K. Mohanta and R. Maitra, Common fixed points for ϕ-pairs in complex valued metric spaces, *Int. J. Math. Comp. Res*. 1(9)(2013), 251-256
15. A. Mukheimer, Some common fixed point theorems in complex valued b-metric spaces, *Sci. World. Jour.* 2014 Article ID 587825(2014)
16. N. Malhotra and B. Bansal, Some common coupled fixed point theorems for generalized contraction in b-metric spaces, *Jour. Nonlinear Sci. Appl.* 8(2015), 8-16
17. M.¨Ozt¨urk, Common fixed point theorems satisfying contractive type conditions in complex valued metric space, *Abst. Appl. Anal.* 2014, Article ID 598465(2014)
18. M.¨Ozt¨urk and N. Kaplan, Common fixed points of f-contraction mappings in complex-valued metric spaces, *Math. Sci.* 2014, 8:129(2014)
19. K. Rao, P. Swami and J. Prasad, A common fixed point theorem in complex valued b-metric spaces, *Bull. Math. Stat. Res.* 1(1)(2013), 1-8
20. F. Rouzkard and M. Imdad, Some common fixed point theorems on complex valued metric spaces, *Comp. Math. Appl* .64(6), (2012), 1866-1874
21. S. Shukla and S. S. Pagey, Some common fixed point theorems in complex valued metric spaces satisfying (E.A) property and (CLR) property, *Int.. Jour. Scientific Inno. Math. Res.* 2(4)(2014), 399-407
22. Y. R. Sharma, Common fixed point theorems in complex valued metric spaces, *Int. J. Inno. Res. Sci. Engg. Tech.* 2(12)(2013), 8282-8286
23. W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, *J. Inequal. Appl.* 84(2012), (2012)
24. W. Sintunavarat, Y. J. Cho and P. Kumam, Ursyhon integral equations approach by common fixed point theorems in complex valued metric spaces, *Advances in Differential Equations* 49(2013), (2013)
25. R. K. Verma and H. K, Pathak, Common fixed point theorems using property (E.A) in complex valued metric spaces, *Thai. J. Math.* 11(2)(2013), 347-355
26. K. Sitthikul and S. Saejung, Some fixed point theorems in complex valued metric spaces, *Fixed Point Theory & Appl.* 189(2012)(2012)

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