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STRONG ROMAN DOMINATION NUMBER OF CERTAIN CLASSES OF GRAPHS

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ABSTRACT

A Roman dominating function on a graph G = (V; E) is a function : V ⟶ {0, 1, 2} satisfying the condition that every vertex u for which (u) = 0 is adjacent to at least one vertex v for which f(v) = 2: The weight of a Roman dominating function is the value f(V) =$\sum_{u\in V}$ f(u): The minimum weight of a Roman dominating function on a graph G is called the Roman dominating number of G: In this paper we study the strong Roman domination number of certain classes of graphs.

Key Words: *Strong Roman domination, Strong Roman domination number.*

1. INTRODUCTION

Let G = (V; E) be a graph of order |V| = n: For any vertex v ∈ V; the open neighborhood of v is the set N(v) = {u ∈ V: uv ∈ E} and the closed neighborhood is the set N[v] = n(v)∪{v}: For a set S⊆V; the open neighborhood is N(S) = $\bigcup_{v\in S}$ N(v) and the closed neighborhood is N[S] = N(S) ∪ S:

Let v∈ S⊆V: Vertex u is called a private neighbor of v with respect to S (denoted by u is an S pn of v) if u ∈ N[v] \ N[S\{v}]: An S pn of v is external if it is a vertex of V S: The set pn(v, S) = N[v] \N[S\{v}] of all S pn's of v is called the private neighborhood set of v with respect to S: The set S is said to be irredundant if every v ∈ S; pn(v; S) ≠∅

A set S V is a dominating set if N[S] = V; or equivalently, every vertex in V\ S is adjacent to at least one vertex in S: The domination number γ(G) is the minimum cardinality of a dominating set in G; and a dominating set S of minimum cardinality is called a γ -set of G:

A set S of vertices is called a 2-packing if for every pair of vertices u; v ∈S; N[u] \ N[v] = ∅. The 2-packing number P_2(G) of a graph G is the maximum cardinality of a 2-packing in G: A set S of vertices is called a vertex cover if for every edge uv ∈ E; either u ∈ S or v ∈ S:

A Roman dominating function (RDF) on a graph G = (V, E) is defined in [8] as a function f: V⟶ {0 , 1, 2} satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 2: The weight of a RDF is the value f(V)= $\sum_{u\in V}$ f(u). The Roman domination number of a graph is denoted by γ_R(G)

Stated in other words, a Roman dominating function is a colouring of the vertices of a graph with the colours {0, 1, 2} such that every vertex coloured 0 is adjacent to at least one vertex coloured 2: The idea is that colours 1 and 2 represent either one or two Roman legions stationed at a given location (vertex v). A nearby location (an adjacent vertex u) is considered to be unsecured if no legions are stationed there (ie f(u) = 0). An unsecured location (u) can be secured by sending a legion to u from an adjacent location (v): But Emperor Constantine the Great, in the fourth century A:D:, decreed that a legion cannot be sent from a location v if doing so leaves that location unsecured (ie if f(v) = 1). Thus, two legions must be stationed at a location (f(v) = 2) before one of the legions can be sent to an adjacent location.

In 2004, Cockayne *et al.* [2] studied the graph theoretic properties of Roman dominating sets. In recent years many authors studied the concept of Roman dominating functions and Roman domination numbers [2]-[9].

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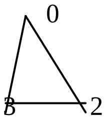
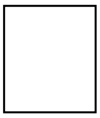
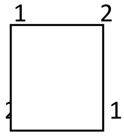
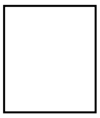
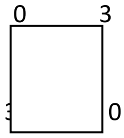
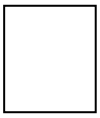
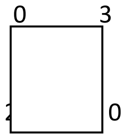
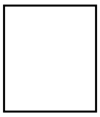
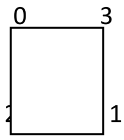
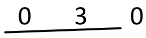
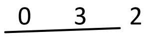
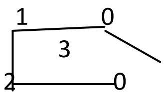
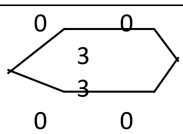
Recently [1] we introduced the concept of Strong Roman domination which is the generalization of Roman domination. A Strong Roman dominating function (SRDF) is a function f: V→ {0, 1, 2, 3} satisfying the condition that every vertex u for which f(u) = 0 is adjacent to at least one vertex v for which f(v) = 3 and every vertex u for which f(u) = 1 is adjacent to at least one vertex v for which f(v) = 2: The weight of a SRDF is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a SRDF on a graph G is called the Strong Roman domination number of G: Then we study the graph theoretic properties of this variant of the domination number of a graph. In this paper we are presenting the value of $\gamma_{SR}(G)$ for several classes of graphs.

2. EXACT VALUES OF STRONG ROMAN DOMINATION NUMBERS

In this section, we illustrate the Strong Roman domination number by presenting the value of $\gamma_{SR}(G)$ for several classes of graphs.

The following table shows that the Strong Roman dominating function and Strong Roman domination number of some graphs of smaller sizes.

S. No.	Graph	Function	SRDF	Minimal SRDF	SRDN ($\gamma_{SR}(G)$)
1	K_1	0	No	----	------
2	K_1	1	No	----	------
3	K_1	2	Yes	Yes	2
4	K_1	3	Yes	No	
5	K_2	0_______3	Yes	Yes	3
6	K_2	1_______2	Yes	Yes	3
7.	K_2	0_______2	No	----	----
8	K_2	0_______1	No	----	----
9	K_3	2 1 1	Yes	No	----
10	K_3	3 0 0	Yes	Yes	3
11	K_3	0 1 2	No	---	---

S. No.	Graph	Function	SRDF	Minimal SRDF	SRDN (γ_{SR}(G))
12	K_3	 0 3 2	No		
13		 1 2 2 1	Yes	No	
14		 0 3 3 0	Yes	No	
15		 0 3 2 0	Yes	Yes	5
16		 0 3 2 1	Yes	No	
17	P_2	 0 3 0	Yes	Yes	3
18	P_2	 0 3 2	Yes	No	
19	P_4	 1 0 3 2 0	Yes	Yes	6
20	C_5	 0 0 3 3 0 0	Yes	Yes	6

Theorem 2.1: If G = P_3; then γ_{SR}(G) = 3:

Proof: G can be drawn as

Define $f(v_1) = 0$, $f(v_2) = 3$, $f(v_3) = 0$:

Then f is a Strong Roman dominating function with f(V) = 3: We have to prove that f is minimal. Suppose there is a minimal SRDF g such that g < f:

Case-(i): Let $g(v_1) = 0$, then $g(v_2)$ must be equal to 3. If $g(v_3)$ 6= 0, then g is not minimal, a contradiction.

Case-(ii): Let $g(v_1) = 1$, then $g(v_2)$ must be equal to 2.

If $g(v_3)$ 0, then $g(V) = 3$ which implies g, f, a contradiction.

Case-(iii): Let $g(v_1) = 2$:

If $g(v_2) = 0$, then $g(v_3)$ must be equal to 3. Here $g(V) = 5 > f(V)$, a contradiction.

If $g(v_2)$ 6= 0, then $g(V)=3$ which implies g, f, a contradiction.

Case-(iv): Let $g(v_1) = 3$, then $g(V)$ 3 which implies g f, a contradiction. Hence f is minimal SRDF. Thus $_{SR}(P_3) = 3$:

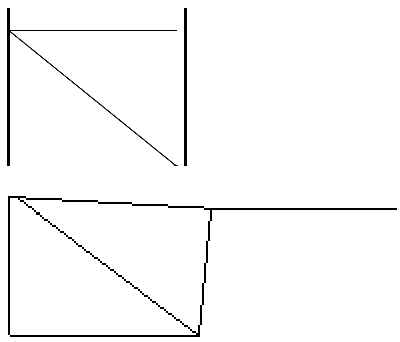
Definition 2.2: **L**et G be a graph and P_3 be a path of order 3. Let v be any vertex in V (G): Then we de ne a graph $G_v + P_3$ such that rst draw G and then draw P_3 starting from the vertex v in V (G):

Example 2.3:

v

G :

$G+P_3 =$



Theorem 2.4: Let $v \in V(G)$ and let f be a minimal SRDF of G and $f(v) = 0$: Now de ne g in $G_v + P_3$ such that $g(u) = f(u)$ if $u \in V(G)$ and $g(v_1) = 0$, $g(v_2) = 3$, $g(v_3) = 0$ if $v_i \in P_3$: Then g is minimal in $G_v + P_3$:

Proof: Suppose g is not minimal in $G_v + P_3$: Let h be a SRDF such that $h < g$:

Case-(i): Let $h(v_3) = 0$, then $h(v_2) = 3$:

If $h(v_1)$ 6= 0, then h is not minimal, a contradiction.

Case-(ii): Let $h(v_3) = 1$, then $h(v_2) = 2$ which implies any value of $h(v_1)$, h g, a contradiction.

Case-(iii): Let $h(v_3) = 2$:

If $h(v_2) = 0$, then $h(v_1) = 3$, which implies h g, a contradiction. If $h(v_2) = 1$, then $h(v_1) = 2$, here also h g, a contradiction.

Case-(iv): Let $h(v_3) = 3$: For any value of $h(v_1)$ & $h(v_2)$, h g, a contradiction. Hence g is minimal in $G_v + P_3$:

Theorem 2.5: If $n = 3m$, then $\gamma_{SR}(P_n) = n$:

Proof: We will prove this result by using mathematical induction on m: Let m = 1, then n = 3:

By Theorem 2.1, $\gamma_{SR}(P_3) = 3$:

Assume the result is true for m: We have to prove that the results is true for m+ 1:

Now, $P_n = P_{3(m+1)} = P_{3m+3} = P_{3m} + P_3$:

By our induction hypothesis, $\gamma_{SR}(P_{3m}) = 3m$:

Then by Theorem 2.4, we have

$$_{SR}(P_{3m} + P_3) = 3m + 3:$$

$$\text{(ie:)} \; _{SR}(P_{3(m+1)}) = 3(m + 1):$$

Hence, the result is true for all the values of m:

Theorem 2.6: If G = P_4, then $\gamma_{SR}(G) = 5$:

Proof: G can be drawn as v_1 v_2 v_3 v_4

Define $f(v_1) = 0$, $f(v_2) = 3$, $f(v_3) = 0$, $f(v_4) = 2$: Then f is a SRDF with f(v) = 5: We have to prove that f is minimal SRDF. Suppose these is a minimal SRDF g such that g < f:

Case-(i): Let $g(v_1) = 0$, then $g(v_2) = 3$:

If $g(v_3) = 0$, then $g(v_4)$ must be 2 or 3, which implies g f, a contradiction. If $g(v_3) = 1$, then $g(v_4) = 2$: Here g > f, a contradiction.

If $g(v_3) = 2$, then $g(v_4)$ 6= 0, now g is not minimal, a contradiction. If $g(v_3) = 3$, then g > f, a contradiction.

Case-(ii): Let $g(v_1) = 1$, then $g(v_2) = 2$:

If $g(v_3) = 0$, then $g(v_4) = 3$, which implies g > f, a contradiction. If $g(v_3) = 1$ or 2, then $g(v_4)$ 6= 0, here g f, a contradiction.

If $g(v_3) = 3$, then obviously g > f, a contradiction.

Case-(iii): Let $g(v_1) = 2$:

If $g(v_2) = 0$, then $g(v_3) = 3$, which implies g > f, a contradiction.

If $g(v_2) = 1$, then for any value of $g(v_3)$ and $g(v_4)$, g f, a contradiction. If $g(v_2) = 2$, then for any value of $g(v_3)$ and $g(v_4)$, g f, a contradiction. If $g(v_2) = 3$, then clearly g > f, a contradiction.

Case-(iv): Let $g(v_1) = 3$:

If $g(v_2) = 0$ and $g(v_3) = 0$, then $g(v_4) = 3$, here g > f, a contradiction.

If $g(v_2) = 0$ and $g(v_3) = 1$, then $g(v_4) = 2$, which implies g > f, a contradiction.

If $g(v_2) = 0$ and $g(v_3) = 2$, then $f(v_4)$ 6= 0, which implies g > f, a contradiction.

If $g(v_2) = 0$ and $g(v_3) = 3$, then any value of $g(v_4)$, g > f, a contradiction. If $g(v_2) = 1$, then $g(v_3) = 2$, here g > f, a contradiction.

If $g(v_2) = 2$, then all the values of $g(v_3)$ & $g(v_4)$, g > f, a contradiction. If $g(v_3) = 3$, clearly g > f, a contradiction.

Thus all the above cases, we get a contradiction. Hence f is minimal SRDF.

Theorem 2.7: If G = P_5, then $\gamma_{SR}(G) = 6$:

Proof: G can be split into two graphs as $P_5 = P_2 + P_3$.
P_2 can be drawn as v_1 —— v_2

Define $f(v_1)=3$ and $f(v_2) = 0$
Then f is a SRDF with f(V) = 3: We have to prove that f is minimal SRDF. Suppose there is a minimal SRDF g such that g < f:

Case-(i): Let $g(v_1) = 0$, then $g(v_2) = 3$, which implies g = f:

Case-(ii): Let $g(v_1) = 1$, then $g(v_2) = 2$, which implies g = f:

Case-(iii): Let $g(v_1) = 2$, then $g(v_2) = 1$, which implies g = f:

Case-(iv): Let $g(v_1) = 3$, then $g(v_2) = 0$, which implies g = f:

Thus all the above cases, we get a contradiction. Hence f is minimal SRDF in P_2: Then by Theorem 2.4, the minimal SRDF of $P_2 + P_3$ is defined as $f(v_1) = 3$, $f(v_2) = 0$, $f(v_3) = 0$, $f(v_4) = 3$ and $f(v_5) = 0$, which is the minimal SRDF of P_5: Hence $\gamma_{SR}(P_5) = 6$:

Theorem 2.8: If n = 3m + 1, then $\gamma_{SR}(P_n) = n + 1$:

Proof: We will prove this result by using induction on m: Let m = 1, then n = 4: By Theorem 2.6, we have $_{SR}(P_4) = 5$:

Assume the result is true for m: We have to prove that the result is true for m + 1:

Now, $P_n = P_{3(m+1)+1} = P_{3m+4} = P_{3m+1} + P_3$:

Then f is a SRDF with f(V) = 3: We have to prove that f is minimal SRDF. Suppose there is a minimal SRDF g such that g < f:

Case-(i): Let $g(v_1) = 0$, then $g(v_2) = 3$, which implies g = f:

Case-(ii): Let $g(v_1) = 1$, then $g(v_2) = 2$, which implies g = f:

Case-(iii): Let $g(v_1) = 2$, then $g(v_2) = 1$, which implies g = f:

Case-(iv): Let $g(v_1) = 3$, then $g(v_2) = 0$, which implies g = f:

Thus all the above cases, we get a contradiction. Hence f is minimal SRDF in P_2: Then by Theorem 2.4, the minimal SRDF of $P_2 + P_3$ is de ned as $f(v_1) = 3$, $f(v_2) = 0$, $f(v_3) = 0$, $f(v_4) = 3$ and $f(v_5) = 0$, which is the minimal SRDF of P_5: Hence $\gamma_{SR}(P_5) = 6$:

Theorem 2.8: If n = 3m + 1, then $\gamma_{SR}(P_n) = n + 1$:

Proof: We will prove this result by using induction on m: Let m = 1, then n = 4: By Theorem 2.6, we have $_{SR}(P_4) = 5$:

Assume the result is true for m: We have to prove that the result is true for m + 1:

Now, $P_n = P_{3(m+1)+1} = P_{3m+4} = P_{3m+1} + P_3$:

By our induction hypothesis, $\gamma_{SR}(P_{3m+1}) = 3m + 2$: Then by Theorem 2.4, we have

$$\gamma_{SR}(P_{3m+1} + P_3) = 3m + 2 + 3:$$

(ie:) $\gamma_{SR}(P_{3(m+1)+1}) = 3m + 5$:

Hence the result is true for all the values of m:

Theorem 2.9: If n = 3m + 2, then $\gamma_{SR}(P_n) = n + 1$:

Proof: We will prove this result by using induction on m: Let m = 1, then n = 5: By Theorem 2.7, we have $\gamma_{SR}(P_5) = 6$:

Assume the result is true for m: We have to prove that the result is true for m + 1:

Now, $P_n = P_{3(m+1)+2} = P_{3m+5} = P_{3m+2} + P_3$:

By our induction hypothesis, $\gamma_{SR}(P_{3m+2}) = 3m + 3$: Then by Theorem 2.4, we have

$$\gamma_{SR}(P_{3m+2} + P_3) = 3m + 3 + 3:$$

(ie:) $\gamma_{SR}(P_{3(m+1)+2}) = 3m + 6$:

(ie:) $\gamma_{SR}(P_{3m+5}) = 3m + 6$:

Hence the result is true for all the values of m:

The proof the following theorems are straight forward from the above results.

Theorem 2.10: If n = 3m, $\gamma_{SR}(C_n) = n$:

Theorem 2.11: If n = 3m + 1, $\gamma_{SR}(C_n) = n + 1$:

Theorem 2.12: If n = 3m + 2, $\gamma_{SR}(C_n) = n + 1$:

Theorem 2.13: $\gamma_{SR}(S_n) = n$ for all n:

3. CONCLUSION AND SCOPE

In this paper, we presented the values of Strong Roman domination number $\gamma_{SR}(G)$ of some classes of graphs such as paths, cycles and stars.

Among the many questions raised by this research and the particular interest of the authors, we propose the following open problems.

Problem 3.1: Can you find other classes of Strong Roman graphs?

Problem 3.2: Can you construct on algorithm for computing the value of $\gamma_{SR}(G)$?

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