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THE LOCALIZED RADIAL BASIS FUNCTION MESHLESS METHOD FOR SOLVING TWO–DIMENSIONAL SHALLOW WATER EQUATIONS

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ABSTRACT

Proposed to demonstrate the localized radial basis function (LRBF) is a stable and accurate tool for simulating two–dimensional shallow water equations (SWEs) (de Saint Venant equation). There are three unknowns in SWEs. Which the water depth h and the water fluxes uh,vh are firstly estimated directly by their values and spatial derivatives in the previous time step. Comparing with global–type meshless methods, the present methods are more appropriate to large–scale problem with complex shapes. For real hydraulic engineering applications, this numerical model is applied to simulate a dam–break problem.

***Keywords**: Shallow water equations; Meshless methods; localized radial basis function; Dam breaking.*

1. INTRODUCTION

The Shallow water equations are known for its application in the field of hydraulic and ocean engineering problems such as dam break analyze the open channel flow, tidal currents and tsunami propagations, floods and inundation waves[5]. The system of equations, which can be obtained asymptotic analysis and depth-averaging of the Navier–Stokes equations however that the fluid itself must be incompressible [1]. The phenomena of shallow water flow very complex and in general nonlinear partial differential equations in nature as well [4]. Most of the studies in this field carried out by finite difference method (FDM), finite element method (FEM) as well as finite volume method (FVM) have been used to simulate shallow water flows [1][4][5].

During the past two decades, meshless methods have been demonstrated as attractive tools for solving partial differential and boundary value problem. They do not require the mesh or grid generation employed. By the other methods they are a cloud of points, and less computationally expansive then mesh-generating methods. The idea of meshless methods could mainly be traced back to the interpolation of scattered data [3]. The time dependent data distribution is fitted either globally or locally by sought functions and the gradient of the spatially distributed data is then determined. The concept of Meshless methods is mainly divided into two categories: one is boundary type and other one is domain type, boundary type meshless methods have advantages from requiring relatively fewer computing nodes only around the boundary of the domain, domain type (mesh free) methods have wider applications to the partial differential equations without fundamental solutions but need more nodes distributed in the whole domain [6]. At each and every node the partial derivatives of the sought solution are approximated by one of this meshless interpolation (interpolating) approaches. A global matrix equation and the solution is obtained by solving this linear algebraic system [7].

Shallow water equation describes the motion of a free surface incompressible fluid under gravity, over a variable bottom topography. The shallow water equations are widely accepted as a mathematical model for the fluid dynamics. Two–dimensional shallow water equations specifically focused on values of water depth h at discretized nodes can be treated as scattered data in the interpolation approach. The water fluxes uh and vh could be treated in the same way so their gradients are also obtained. The three unknowns in shallow water equations, h, uh and vh is estimated directly by their values and their spatial derivatives in the previous time step [1] [2].

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2. GENERAL SHALLOW WATER EQUATIONS

Today real world problem such as interesting system of nonlinear differential equations appearing in fluid dynamics is given by three dimensional incompressible Navier–Stokes equations and can be simplified to the two-dimensional shallow water equations (SWEs, which are also called the de Saint–Venant equations) used to describe flow behaviors in bodies of water [1][2][5]. Where the horizontal length scales are much greater than the flow depth as depicted in (fig.1).

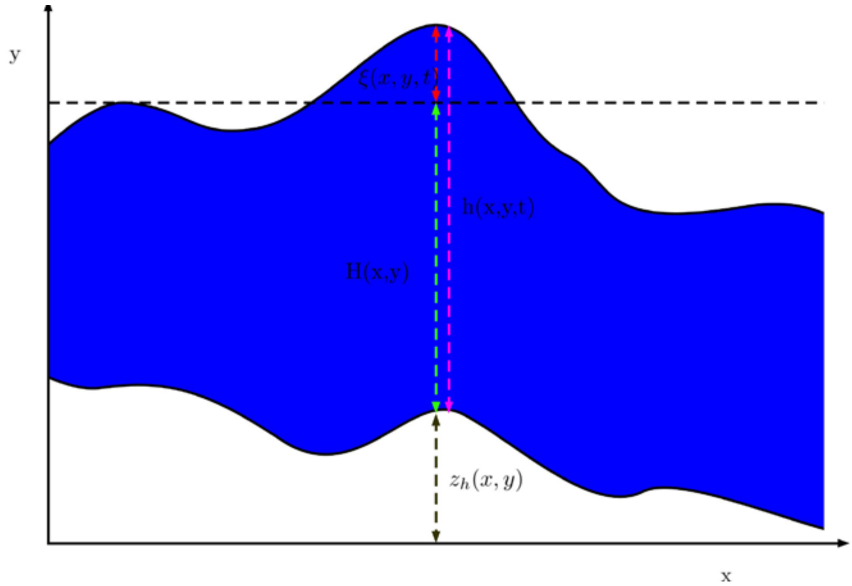

Figure-1: Sketch for the water body in 2D shallow water flow.

By the assumptions of the hydrostatic pressure and uniform velocity profile along the vertical direction, a continuity equation and a momentum equation in each of the x and y directions respectively. The forcing terms $F_i F_j$, variation of bottom elevation, wind shear stress, bed friction and rotation of the plant earth as follows

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0 \tag{1}$$

$$\frac{\partial (uh)}{\partial t} + \frac{\partial (u^2 h)}{\partial x} + \frac{\partial (uvh)}{\partial y} + gh\frac{\partial h}{\partial x} = v_e\left(\frac{\partial}{\partial x}\left(h\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right)\right) + F_i \tag{2}$$

$$\frac{\partial (vh)}{\partial t} + \frac{\partial (uvh)}{\partial x} + \frac{\partial (v^2 h)}{\partial y} + gh\frac{\partial h}{\partial y} = v_e\left(\frac{\partial}{\partial x}\left(h\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial y}\right)\right) + F_j \tag{3}$$

Many marine hydrodynamic problems viscous effect of kinematic eddy viscosity can be neglected, momentum equations can be simplified as follows

$$h\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x}\right) = F_i \tag{4}$$

$$h\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y}\right) = F_j \tag{5}$$

$$F_i = F_{Pi} + F_{Wi} + F_{Bi} + F_{Ri} \tag{6}$$

$$F_j = F_{Pj} + F_{Wj} + F_{Bj} + F_{Rj} \tag{7}$$

$$F_{pi} = gh\frac{\partial z_h}{\partial x}$$

$$F_{pj} = gh\frac{\partial z_h}{\partial y} \tag{8}$$

$$F_{wi} = \frac{\rho_a}{\rho_w} C_w u_w \sqrt{u_w^2 + v_w^2}$$

$$F_{wj} = \frac{\rho_a}{\rho_w} C_w v_w \sqrt{u_w^2 + v_w^2} \tag{9}$$

$$F_{bi} = -\frac{gu\sqrt{u_w^2 + v_w^2}}{C_z^2}$$

$$F_{bj} = -\frac{gv\sqrt{u_w^2 + v_w^2}}{C_z^2} \tag{10}$$

$$F_{Ri} = f_c vh$$

$$F_{Rj} = -f_c uh \tag{11}$$

Where t is time, $g = 9.81m/s^2$ is the gravity acceleration,$h = \xi - Z_h$ is the total water depth,ξ is the free surface elevation,z_h is the height of the bed surface which is usually negative in coastal and ocean engineering as $z = 0$ is set at the mean sea level, ν_e is the kinematic eddy viscosity which is smaller than the bed shear stress in water,ρa is the density of air,ρw is the density of water,$C_w \approx 2.6 \times 10^-3$ is a wind coefficient. $C_z = rach^{1/6}n$ is the Chezh friction coefficient and n is the manning coefficient, u and v denote depth averaged velocity components in the horizontal x and y directions respectively. u_x, u_y and v_x, v_y are the derivatives of depth averaged velocity components to x and y direction respectively. A viscous effect of kinematic eddy viscosity can be neglected in many marine hydrodynamics problems.

3. LOCALIZED RADIAL BASIS FUNCTION METHOD

In order to apply the LRBFM, the following procedure may be adopted to solve physical variable q(x,t) with any given operator. The computation points in the global domain can be defined as$x_i \in \Psi, i = 1,2,3,\dots,N$ where N is the total number of global points. q is approximated by a series of integrated basis functions

$$q(x) \cong \sum_{j=1}^{N} \alpha_j \, \Phi(\| x - x_j \|) \quad x \in \Psi \tag{12}$$

where Φ is the integrated basis function and α_j are unknown weighting coefficients to determined. α_j are solved in order to approximation the solution in computation domain, $\Phi(\|x - x_j\|)$ is the given and $\|x - x_j\|$ is Euclidian distance between x and x_j each global point x_i has local influence domain χ for all points within χ can be defined as $x_{i,k} \in \chi$, $k = 1,2,\dots,N_l$, where N_l is the number of local points within χ_i we consider that each local influence domainχ as

$$q(x) \cong \sum_{k=1}^{N_l} \alpha_j \, \Phi(\| x - x_{i,m} \|) \quad x \in \chi_i \tag{13}$$

The approximate derivative of q(x) can be obtained by a linear combination of the derivatives of LRBF

$$\frac{\partial q(x)}{\partial x_i} \cong \sum_{j=1}^{N_l} \alpha_j \frac{\partial}{\partial x_i} \Phi(\|x - x_j\|) \tag{14}$$

If $N_l = N.$ Then the method become the global. In the local region we only have N_l collocation nodes. It means we have N_l equations in the matrix form (13) considered larger of N_l is the more influences. it does not mean theaccuracy can be improved. We have to more supporting nodes are considered the higher conditional number of matrix is induced by (Figure 2).

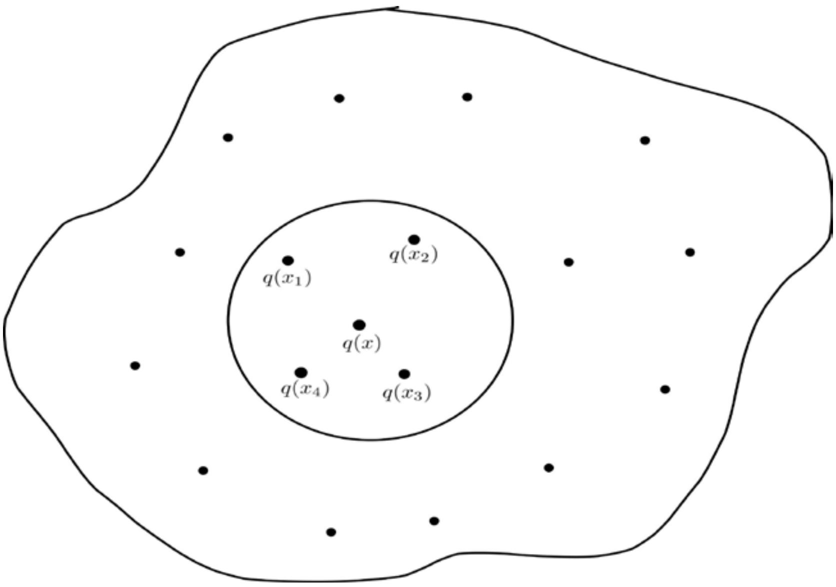

Figure-2: Local RBF Interpolation value q(x).

$$\boldsymbol{q} = \Phi\alpha \tag{15}$$

Where the approximate value vector

$$q(x) = [q(x_{i,1}, t), q(x_{i,2}, t), \dots, q(x_{i,Nl}, t)]^T,$$

$\Phi = [\Phi(\| x_{i,k} - x_{i,m} \|)]N_l \times N_l$ symmetric square LRBF matrix as

$$\Phi = \begin{bmatrix} \Phi(\| x_1 - x_1 \|) & \Phi(\| x_1 - x_2 \|) & \dots & \Phi(\| x_1 - x_{N_l} \|) \\ \Phi(\| x_2 - x_1 \|) & \Phi(\| x_2 - x_2 \|) & \dots & \Phi(\| x_2 - x_{N_l} \|) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(\| x_{N_l} - x_1 \|) & \Phi(\| x_{N_l} - x_2 \|) & \dots & \Phi(\| x_{N_l} - x_{N_l} \|) \end{bmatrix}$$

and unknown coefficient vector $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2} \dots, \alpha_{i,Nl}\}$ can be obtained by

$$\alpha = \Phi^{-1}\boldsymbol{q} \tag{16}$$

In index from we have

$$\alpha_j = \sum_{k=1}^{N_l} (\Phi^{-1})_{jk}\, q(x_j) \tag{17}$$

Therefore, equation (13)

$$q(x) = \varpi^T q = \sum_{j=1}^{N_l} \varpi_j\, q(x_j) \tag{18}$$

where $\varpi = \{\varpi_1, \varpi_2, \dots \varpi_{N_L}\}^T$ and

$$\varpi_j = \sum_{k=1}^{N_l} (\Phi^{-1})_{jk}\, \Phi(\| x - x_j \|) \tag{19}$$

The derivative of $q(x)$ are approximated as

$$\frac{\partial q(x)}{\partial x} = \varpi^{xT} q = \sum_{j=1}^{N_l} \varpi_j^x\, q(x_j) \tag{20}$$

where $\varpi^x = \{\varpi_1^x, \varpi_2^x, \dots \varpi_{N_l}^x\}^T$ and

$$\varpi_j^x = \sum_{k=1}^{N_l} (\Phi^{-1})_{jk} \frac{\partial \Phi(\| x - x_j \|)}{\partial x} \tag{21}$$

3.1. Integrated Multiquadrics radial basis functions

In this section, we have ϕ is integrated multiquadrics radial basis function (MQ-RBFs) given applied. The integrated MQ-RBFs are $\Phi(\|x - x_j\|) = \sqrt{\|x - x_j\|^2 + c^2}$. Where c is a shape parameter, we suggest to use c = 1. If the more local nodes are chosen, the parameter c should be reduced.

4. NUMERICAL PROCEDURES

In the section, the numerical procedures for the approximation of the shallow water equations via the LRBF method will be explained in spatial domain and fully– implicit scheme in temporal direction, the discrete form at i^{th} node can be written as follows:

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} + u_i^{n+1} \sum_{j=1}^{N_l} \varpi_j^x\, h_j^{n+1} + v_i^{n+1} \sum_{j=1}^{N_l} \varpi_j^y\, h_j^{n+1} + h_i^{n+1}\Big(\sum_{j=1}^{N_l} \varpi_j^x\, u_j^{n+1} + \sum_{j=1}^{N_l} \varpi_j^y\, v_j^{n+1}\Big) = 0 \tag{22}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^{n+1} \sum_{j=1}^{N_l} \varpi_j^x\, u_j^{n+1} + v_i^{n+1} \sum_{j=1}^{N_l} \varpi_j^y\, u_j^{n+1} + g \sum_{j=1}^{N_l} \varpi_j^x\, h_j^{n+1} + g \sum_{j=1}^{N_l} \varpi_j^x\, z_{bj} - \frac{\rho_a}{\rho_w (h_i^{n+1})^2} C_w u_w \sqrt{u_w^2 + v_w^2 h_i^{n+1}} + g \frac{\sqrt{(u_i^{n+1})^2 + (v_i^{n+1})^2}}{h_i^{n+1} C_z^2} u_i^{n+1} - f_c v_i^{n+1} = 0 \tag{23}$$

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + u_i^{n+1} \sum_{j=1}^{N_l} \varpi_j^x\, v_j^{n+1} + v_i^{n+1} \sum_{j=1}^{N_l} \varpi_j^y\, v_j^{n+1} + g \sum_{j=1}^{N_l} \varpi_j^y\, h_j^{n+1} + g \sum_{j=1}^{N_l} \varpi^y\, z_{bj} - \frac{\rho_a}{\rho_w (h_i^{n+1})^2} C_w u_w \sqrt{u_w^2 + v_w^2 h_i^{n+1}} + g \frac{\sqrt{(u_i^{n+1})^2 + (v_i^{n+1})^2}}{h_i^{n+1} C_z^2} v_i^{n+1} + f_c u_i^{n+1} = 0 \tag{24}$$

Where ϖ_j^x and ϖ_j^y are weighting coefficient matrix of the first derivative in x and y directions respectively. In our numerical procedures, the Coriolis effect and wind shear stress are not considered. Therefore, bed slopes are zero.

$$\frac{h_i^{n+1}-h_i^n}{\Delta t}+u_i^{n+1}\sum_{j=1}^{N_l}\varpi_j^x\, h_j^{n+1}+v_i^{n+1}\sum_{j=1}^{N_l}\varpi_j^y\, h_j^{n+1}+h_i^{n+1}\Big(\sum_{j=1}^{N_l}\varpi_j^x\, u_j^{n+1} + \sum_{j=1}^{N_l}\varpi_j^y\, v_j^{n+1}\Big) = 0 \tag{25}$$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+u_i^{n+1}\sum_{j=1}^{N_l}\varpi_j^x\, u_j^{n+1}+v_i^{n+1}\sum_{j=1}^{N_l}\varpi_j^y\, u_j^{n+1}+ g\sum_{j=1}^{N_l}\varpi_j^x\, h_j^{n+1}+g\sum_{j=1}^{N_l}\frac{\sqrt{(u_i^{n+1})^2+(v_i^{n+1})^2}}{h_i^{n+1}C_z^2}u_i^{n+1}=0 \tag{26}$$

$$\frac{v_i^{n+1}-v_i^n}{\Delta t}+u_i^{n+1}\sum_{j=1}^{N_l}\varpi_j^x\, v_j^{n+1}+v_i^{n+1}\sum_{j=1}^{N_l}\varpi_j^y\, v_j^{n+1}+ g\sum_{j=1}^{N_l}\varpi_j^y\, h_j^{n+1}+g\sum_{j=1}^{N_l}\frac{\sqrt{(u_i^{n+1})^2+(v_i^{n+1})^2}}{h_i^{n+1}C_z^2}v_i^{n+1}=0 \tag{27}$$

by solving equation (25) the water depth h_i^{n+1} and the velocity $u_i^{n+1}$and v_i^{n+1}can beobtained

5. A TWO-DIMENSIONAL DAM-BREAK PROBLEM

We study now the ability of the scheme in dealing with a two-dimensional dam-break the simulation of the dam break problem is a classic example in free-surface simulations, the geometry of the computational domain is consider as 200×200 a square region with frictionless, the bottom is flat and free slip boundary at wall. A non-symmetric breach with gate of 75 m wide is located in the middle of domain with 10 m thickness. The initial water level of the downstream water is 5 m high and the upstream of the dam water 10 m high of the dividing wall. At the initial time $t = 0$ with dam failure, and water is released through the 75 m wide the upstream water is goto the downstream side through the breach, as shown in Fig 3. The boundary conditions at $x = 0$ and $x = 200$ m are assumed to be trans missive and all other boundaries are considered as reflective. The time step is given by the Courant–Friedrichs–Lewy (CFL) condition as follows: initial water evaluation of dam break caused by the gravity we take the value of u_{max} by the velocity caused from the difference of the initial water elevation. The main difference causing the dam break is on the x-direction, we also focus on the rule of x-direction

$$C = \frac{u_{max}\,\Delta t}{\Delta d_{min}} \le 1 \tag{28}$$

Where C denotes the Courant number. u_{max} is the maximum physical velocityΔd_{min} distance between any two points in a domain and the base size per time step Δt which should numerical stability ensured can be estimated. For SWEs, the velocity can be denoted by the maximum characteristic velocity along two coordinates direction can be rewritten as

$$\Delta t \le C\frac{d_{min}}{max(|u \pm \sqrt{gh}|, |v \pm \sqrt{gh}|)} \tag{29}$$

The value of Courant Number C is set to be less than unity ensuring the stability of the scheme, this definition is still not specified enough, and the required value is usually mush smaller then unity. However, this meshless methods which usually have randomly nodal distribution and alternative number of supporting nodes.

For MQ– based meshless method, the sensitivity of the shape parameter c is a crucial and common issue then greatly affects the performance of the modeling. The main idea uses c q_{max}^i. Instead of the original c in the local RBF which q_{max}^i is the maximum distance from central node x to the support node x_i in the sub domain χ_i give by

$$q_{max}^i = max(\| x - x_{i,m} \|) = max\,(q^i) \qquad x_i \in \chi_i \tag{30}$$

The numerical predictions are in good agreement with other results reported in the literature, notably those provided in [2]. To check convergence of our results, the contour solutions at 13,000 nodes are shown for two different time with t=6.5 sand t=8 s are demonstrated in the Fig(4) and Fig(5); therefore highlight that the wet or dry interface is handled in a very robust way.

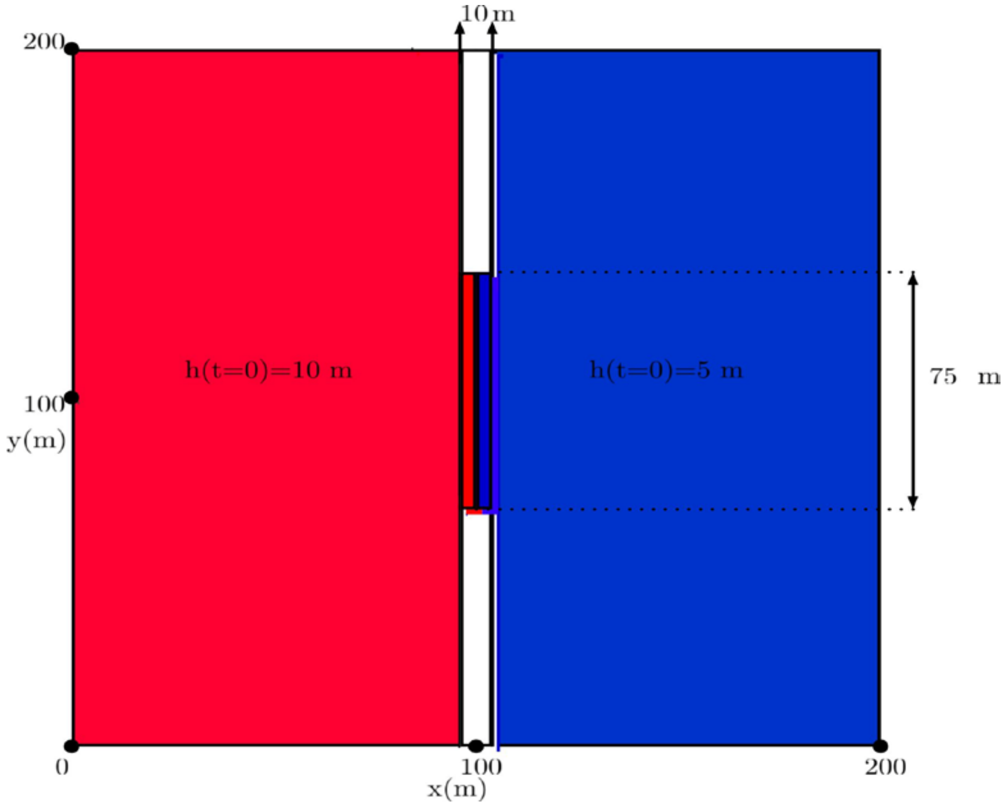

Figure-3: 2-D dam break problem.

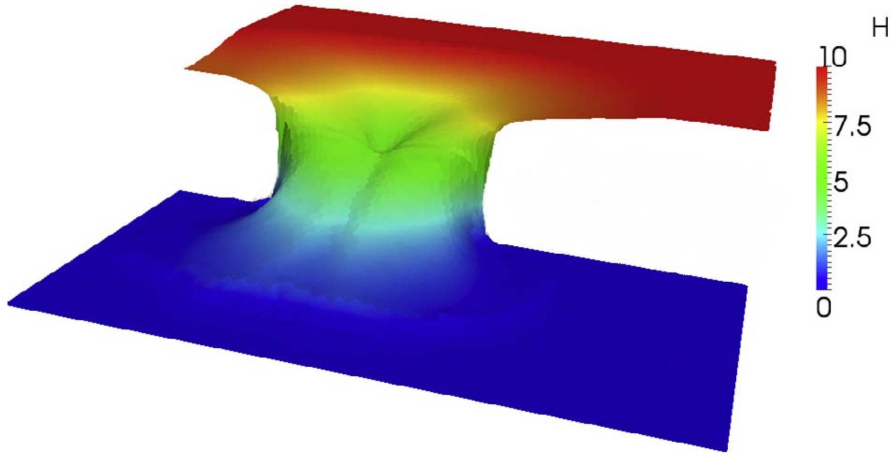

Figure-4: Two–dimensional dam break– free surface profile of t=6.5 s.

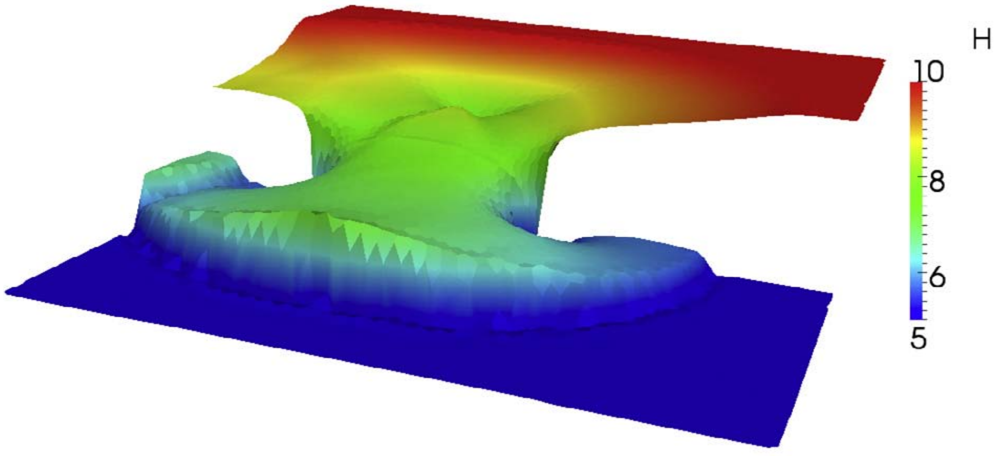

Figure-5: Two–dimensional dam break– free surface profile of t=8 s.

6. CONCLUSION

The LRBF has been proposed to solve two–dimensional shallow water equations (de Saint–Venant equations) based on MQ for used. A numerical model for the simulation of two–dimensional shallow water flows is developed. At each time step the three unknowns in SWEs, which the water depth h and the water fluxes uh, vh spatial derivatives of h, uh, vh for the further time marching processes. The data of dam break experiment in a two–dimensional numerical result show the accuracy of LRBF scheme and demonstrate. The local RBF method is work to develop for more complex shallow water flow for engineering applications.

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