DIRECT PRODUCT OF (Q, L)-FUZZY SUBGROUPS AND THEIR PROPERTIES

A. SOLAIRAJU*1, S. THIRUVENI2

1Associate Professor of Mathematics, Jamal Mohamed College, Trichy, India.
2Assistant Professor in Mathematics, K. S. R. College of Engineering, Tiruchengode, Namakkal Dist. Tamil Nadu, India.

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ABSTRACT

In this paper, some properties of (Q, L)-fuzzy subgroups of a group are discussed, and obtained some algebraic properties on the direct product of (Q, L)-fuzzy subgroups by means of Q-level sets.

Keywords: (Q, L)-fuzzy subset, (Q, L)-fuzzy subgroups, (Q, L) – fuzzy normal subgroup, Q-level subsets.

SECTION 1 – INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [7]. Rosenfield [6] gave the idea of subgroups. Solairaju and Nagarajan [4, 5] introduced and defined a new algebraic structure of Q-fuzzy groups. Asokkumer Ray [1] defined a product of fuzzy groups. Goguen [2] studied the fuzzy set theory by studying L-fuzzy sets. In this paper, we discuss some equivalent characterizations of direct product of (Q, L)-fuzzy groups by means of Q-level subsets.

SECTION 2 – BASIC DEFINITIONS

Definition 2.1: Let X and Q be any two non-empty sets. A mapping $\mu : X \times Q \rightarrow [0, 1]$ is called a Q-fuzzy set in X.

Definition 2.2: Let X be a non-empty set and L = (L, $\leq$) be a lattice with least element 0 and greatest element 1 and Q be a non-empty set . A (Q, L)-fuzzy subset $A$ of X is a function $A : X \times Q \rightarrow L$.

Definition 2.3: A (Q, L) - fuzzy subset $\lambda$ of G is said to be a (Q, L)-fuzzy subgroup of G if for all $x, y \in G$ and $q \in Q$

(i) $(xy, q) \geq \lambda(x, q) \land \lambda(y, q)$

(ii) $\lambda(x^{-1}, q) = \lambda(x, q)$

SECTION 3 – PROPERTIES ON (Q, L) – FUZZY SUBGROUP

Theorem 3.1: A (Q, L)-fuzzy subset $\lambda$ of G is a (Q, L)-fuzzy subgroup of G if and only if

$(xy^{-1}, q) \geq \lambda(x, q) \land \lambda(y, q), \forall x, y \in G$ and $q \in Q$.

Proof: $\lambda$ is a (Q, L)-fuzzy subgroup of G.

$\Leftrightarrow \lambda(xy, q) \geq \lambda(x, q) \land \lambda(y, q)$ and $\lambda(x^{-1}, q) = \lambda(x, q)$

$\Leftrightarrow \lambda(xy^{-1}, q) \geq \lambda(x, q) \land \lambda(y, q), \forall x, y \in G$ and $q \in Q$

Definition 3.2: Let A be a (Q, L)-fuzzy subgroup of G. For $\alpha \in L$, a Q-level subset of A corresponding to $\alpha$ is the set $A_\alpha = \{x \in G, qeQ : A(x, q) \geq \alpha \}$

Corresponding Author: A. Solairaju*1

*1Associate Professor of Mathematics, Jamal Mohamed College, Trichy, India.
Theorem 3.3: If A is a (Q, L)-fuzzy subset of a group G. Then A is a (Q, L)-fuzzy subgroup of G if and only if \( A_{\alpha} \) is a subgroup of a group G for all \( \alpha \in L \).

**Proof:** Let \( x, y \in G, q \in Q \):

\[
A(xy^{-1}, q) \geq A(x, q) \wedge A(y, q)
\]

\[
A(xy^{-1}, q) \geq \alpha
\]

\[
\Rightarrow xy^{-1} \in A_{\alpha} \Rightarrow A_{\alpha} \text{ is a subgroup of } G \text{ for all } \alpha \in L.
\]

**Definition 3.4:** A (Q, L)-fuzzy subgroup A of group G is a (Q, L)-fuzzy normal subgroup of G if for some \( \alpha \in L \), \( A_{\alpha} = A \).

**Theorem 4.1:** If A and B are two (Q, L)-fuzzy normal subgroups of group G. Then A and B are said to be (Q, L)-fuzzy conjugate subgroup of G if for some \( \alpha \in L \), \( A_{\alpha} = B_{\alpha} \).

**Definition 3.6:** Let A and B be two (Q, L)-fuzzy subgroups of G. Then A and B are said to be (Q, L)-fuzzy conjugate subgroup of G if for some \( \alpha \in L \), \( A_{\alpha} = B_{\alpha} \).

**Theorem 4.2:** Let A be a (Q, L)-fuzzy normal subgroup of a group G. Then A is a (Q, L)-fuzzy normal subgroup of G if and only if \( A_{\alpha} \) is a normal subgroup of a group G for all \( \alpha \in L \).

**Proof:** Let A be a (Q, L)-fuzzy normal subgroup of a group G. Then A is a (Q, L)-fuzzy normal subgroup of G if and only if \( A_{\alpha} \) is a normal subgroup of a group G for all \( \alpha \in L \).

**Definition 3.7:** Let A be a (Q, L)-fuzzy subgroup in a set S, the strongest (Q, L) fuzzy relation on S, that is (Q, L)-fuzzy relation on A is V given by \( \mu_V((x, y), q) = \mu_A(x, q) \wedge \mu_B(y, q) \forall x, y \in S \).

**SECTION 4: DIRECT PRODUCT OF (Q, L)-FUZZY SUBGROUPS**

**Definition 4.1:** Let A and B be two (Q, L)-fuzzy subgroups of X and Y respectively. Then the Cartesian product of A and B is denoted by \( A \times B \) and is defined as

\[
A \times B = \{ (x, y), (x, y) \mid \mu_A((x, y), q) \wedge \mu_B((x, y), q) > 0 \} \forall x \in X, y \in Y, q \in Q
\]

where \( \mu_{A \times B}((x, y), q) = \mu_A(x, q) \wedge \mu_B(y, q) \).

**Theorem 4.2:** If A and B be two (Q, L)-fuzzy subgroups of X and Y respectively, then \((A \times B)_\alpha = A_\alpha \times B_\alpha\) for \( \alpha \in L \).

**Proof:** Let \((x, y) \in (A \times B)_\alpha \) and \( q \in Q \).

Then \( \mu_{A \times B}((x, y), q) \geq \alpha \)

\[
\Leftrightarrow \mu_A(x, q) \wedge \mu_B(y, q) \geq \alpha
\]

\[
\Leftrightarrow \mu_A(x, q) \geq \alpha, \mu_B(y, q) \geq \alpha
\]

\[
\Leftrightarrow x \in A_{\alpha}, y \in B_{\alpha}
\]

\[
\Leftrightarrow (x, y) \in A_{\alpha} \times B_{\alpha} \text{ for } \alpha \in L
\]

Hence, \((A \times B)_\alpha = A_\alpha \times B_\alpha\) for \( \alpha \in L \).

**Theorem 4.3:** Let A and B be two (Q, L)-fuzzy subgroups of group \( G_1 \) and \( G_2 \) respectively. Then \( A \times B \) is a (Q, L)-fuzzy subgroup of group \( G_1 \times G_2 \).

**Proof:** Since A and B are \((Q, L)-fuzzy \) subgroups of group \( G_1 \) and \( G_2 \) respectively. Then \( A_\alpha \) and \( B_\alpha \) are subgroups of group \( G_1 \) and \( G_2 \) respectively.

\[
\Rightarrow A_\alpha \times B_\alpha \text{ is a subgroup of } G_1 \times G_2, \text{ for } \alpha \in L.
\]

\[
\Rightarrow (A \times B)_\alpha \text{ is a subgroup of } G_1 \times G_2, \text{ for } \alpha \in L. \text{ (By thm 2.6)}
\]

\[
\Rightarrow A \times B \text{ is a } (Q, L)-fuzzy \text{ subgroup of group } G_1 \times G_2.
\]

**Theorem 4.4:** Let A and B be two \((Q, L)-fuzzy \) normal subgroups of group \( G_1 \) and \( G_2 \) respectively. Then \( A \times B \) is a \((Q, L)-fuzzy \) normal subgroup of group \( G_1 \times G_2 \).

**Proof:** Since A and B are \((Q, L)-fuzzy \) normal subgroups of group \( G_1 \) and \( G_2 \) respectively. Then \( A_\alpha \) and \( B_\alpha \) are normal subgroups of group \( G_1 \) and \( G_2 \) respectively.
\[ \Rightarrow A_x \times B_y \text{ is a normal subgroup of } G_1 \times G_2, \text{ for } \alpha \in L. \]

\[ \Rightarrow (A \times B)_x \text{ is a normal subgroup of } G_1 \times G_2, \text{ for } \alpha \in L. \text{ (By thm 2.6)} \]

\[ \Rightarrow A \times B \text{ is a (Q, L)–fuzzy normal subgroup of group } G_1 \times G_2. \]

**Remark 4.5:** Let A and B be (Q, L)–fuzzy subgroups of group G_1 and G_2 respectively. If \( A \times B \) is a (Q, L)–fuzzy subgroup of group \( G_1 \times G_2 \), then it is not necessary that both A and B should be (Q, L)–fuzzy subgroups of group \( G_1 \times G_2 \).

**Example 4.6:** Let \( G_1 = \{e_1, x\} \) where \( x^2 = e_1, G_2 = \{e_2, a, b, ab\} \) where \( a^2 = b^2 = e_2 \) and \( ab = ba \).

Then \( G_1 \times G_2 = \{(e_1, e_2), (e_1, a), (e_1, b), (x, e_2), (x, a), (x, b), (x, ab)\} \).

Let \( A = \{(e_1, q), (0.5, q) >, (x, q), (0.8, q) >\} \) and \( B = \{(e_2, q), (0.7, q) >, (a, q), (1, q) >, (b, q), (0.8, q) >\} \) be (Q, L)-fuzzy subsets of \( G_1 \) and \( G_2 \) respectively.

Then \( A \times B = \{(e_1, e_2, q), (0.5, q) >, (e_1, a, q), (0.5, q) >, (e_1, b, q), (0.5, q) >, (e_1, ab, q), (0.7, q) >\} \).

Here \( A \times B \) is a (Q, L)-fuzzy subgroup of \( G_1 \times G_2 \) where A is a (Q, L)-fuzzy subgroup of \( G_1 \) but B is not a (Q, L)-fuzzy subgroup of \( G_2 \).

**Theorem 4.7:** Let A and B be (Q, L)–fuzzy subgroups of \( G_1 \) and \( G_2 \) respectively. Suppose that \( e_1 \) and \( e_2 \) are the identity element of \( G_1 \) and \( G_2 \) respectively. If \( A \times B \) is a (Q,L)-Fuzzy subgroup of \( G_1 \times G_2 \), then at least one of the two statements must hold.

(i) \( \mu_A(e_2, q) \geq \mu_A(x, q) \) for all \( x \in G_1 \) (ii) \( \mu_A(e_1, q) \geq \mu_B(y, q) \) for all \( y \in G_2 \).

**Proof:** Let \( A \times B \) be a (Q, L)-Fuzzy subgroup of \( G_1 \times G_2 \).

Suppose that (i) and (ii) does not hold.

Then we can find some \( x \in G_1 \) and \( y \in G_2 \) such that \( \mu_A(x, q) > \mu_A(e_2, q) \) and \( \mu_A(e_1, q) < \mu_B(y, q) \).

Now \( \mu_{A \times B}(x, y, q) = \mu_A(x, q) \wedge \mu_B(y, q) > \mu_A(e_2, q) \wedge \mu_A(e_1, q) = \mu_{A \times B}(e_1, e_2, q) \).

which implies that \( A \times B \) is not a (Q, L)-Fuzzy subgroup of \( G_1 \times G_2 \), which is a contradiction.

Hence either \( \mu_B(e_2, q) \geq \mu_A(x, q) \) for all \( x \in G_1, q \in Q \) or \( \mu_A(e_1, q) \geq \mu_B(y, q) \) for all \( y \in G_2, q \in Q \).

**Theorem 4.8:** Let A and B be (Q, L)-fuzzy subsets of \( G_1 \) and \( G_2 \) respectively such that \( \mu_A(x, q) \leq \mu_B(e_2, q), x \in G_1, e_2 \) be the identity element of \( G_2 \), \( q \in Q \). If \( A \times B \) is a (Q, L)-fuzzy subgroup of \( G_1 \times G_2 \), then A is a (Q, L)-fuzzy subgroup of \( G_1 \).

**Proof:** Let \( x, y \in G_1 \). Then \( (x, e_2), (y, e_2) \in G_1 \times G_2 \).

Since \( \mu_A(x, q) \leq \mu_B(e_2, q) \), for all \( x \in G_1, e_2 \in G_2, q \in Q \).

\[
\begin{align*}
\mu_A(xy^{-1}, q) &= \mu_A(xy^{-1}, q) \wedge \mu_B(e_2, q) \\
&= \mu_{A \times B}(xy^{-1}, e_2, q) = \mu_{A \times B}((x, e_2)(y^{-1}, e_2), q) \\
&\geq \mu_{A \times B}(e_2, q) \wedge \mu_{A \times B}((y^{-1}, e_2), q) \quad (\text{since } A \times B \text{ is a (Q, L)-Fuzzy subgroup of } G_1 \times G_2) \\
&= (\mu_A(x, q) \wedge \mu_B(e_2, q)) \wedge ((\mu_A(y^{-1}, q) \wedge \mu_B(e_2, q))) \\
&= \mu_A(x, q) \wedge \mu_A(y^{-1}, q) \\
&\geq \mu_A(x, q) \wedge \mu_A(y, q)
\end{align*}
\]

Hence A is an (Q, L)-fuzzy subgroup of \( G_1 \).

**Corollary 4.9:** Let A and B be (Q, L)-fuzzy subsets of \( G_1 \) and \( G_2 \) respectively such that \( \mu_B(y, q) \leq \mu_A(e_1, q) \) holds for all \( y \in G_2, q \in Q, e_1 \) being the identity element of \( G_1 \). If \( A \times B \) is a (Q, L)-fuzzy subgroup of \( G_1 \times G_2 \), then B is a (Q, L)-fuzzy subgroup of \( G_2 \).
SECTION 5: OTHER PROPERTIES ON (Q, L) – FUZZY SUBGROUPS

Theorem 5.1: Let A, C be (Q, L)-fuzzy subgroups of G1 and B, D be (Q, L)-fuzzy subgroups of G2 respectively such that A, C be (Q, L)-fuzzy conjugate subgroups of G1 and B, D be (Q, L)-fuzzy conjugate subgroups of G2. Then A × B of G1 × G2 is conjugate to the (Q, L)-fuzzy conjugate subgroup C × D of G1 × G2.

Proof: Since A and C are (Q, L)-fuzzy conjugate subgroups of G1, ∃g1 ∈ G1 such that μA(x, q) = μC(g1−1xg1, q), ∀x ∈ G1.

Since B and D are (Q, L)-fuzzy conjugate subgroups of G2, ∃g2 ∈ G2 such that μB(y, q) = μD(g2−1yg2, q), ∀y ∈ G2.

Now μA×B((x,y), q) = μA(x,q)∧μD(y,q) = μC(g1−1xg1, q)∧μD(g2−1yg2, q)

= μC×D((g1−1xg1, g2−1yg2), q) = μC×D((g1−1, g2−1)(x,y)(g1g2), q)

Hence the (Q, L)–fuzzy subgroup A X B is conjugate to the (Q, L)–fuzzy subgroup C × D.

Theorem 5.2: Let A be a(Q, L)-fuzzy subset of a group G and V be the strongest fuzzy (Q, L)-fuzzy relation on G. Then A is a(Q, L)-fuzzy subgroup of G iff V is (Q, L)-fuzzy subgroup of G × G.

Proof: Let A be a (Q, L)-fuzzy subgroup of G.

Let x = (x1, x2), y = (y1, y2) ∈ G × G. We have

μV(x,y,q) = μV((x1,y1)x2y2,q) = μV((x1y1,x2y2),q) = μA(x1y1,q)∧μA(x2y2,q)

≥ (μA(x1,q)∧μA(y1,q))∧(μA(x2,q)∧μA(y2,q))

= (μA(x1,q)∧μA(x2,q))∧(μA(y1,q)∧μA(y2,q))

= μV((x1,x2),q)∧μV((y1,y2),q).

Hence V is a (Q, L)-fuzzy subgroup of G × G.

Lemma 5.3: For a, b ∈ L, m is positive integer (i) If a < b, then am < bm (ii) (a ∧ b)m = am ∧ bm

Proof: It is obvious.

Theorem 5.4: Let A be a (Q, L)-fuzzy subgroup of G. Then A = {< (x,q), (μA(x,q))m >: x ∈ G, q ∈ Q} is a (Q, L)-fuzzy subgroup of Gm, where m is a positive integer.

Proof: Let G be a group. Then (G, .) is a group. Hence (Gm, .) is also a group.

Let A be a (Q, L)-fuzzy subgroup of G. Let x, y ∈ G and q ∈ Q

μA+m(xy,q) = (μA(xy,q)m) ≥ (μA(x,q)∧μA(y,q))m

= (μA(x,q)m)∧(μA(y,q)m)

= μA+m(x,q)∧μA+m(y,q)

μA+m(x−1,q) = (μA(x−1,q))m

= (μA(x,q))m

= μA+m(x,q)

Hence Am is a (Q, L)-fuzzy subgroup of Gm.

Theorem 5.5: If A and A′ are (Q, L)-fuzzy subgroups of G, A is a constant (Q, L)-fuzzy subset of G.

Proof: Since A and A′ are (Q, L)-fuzzy subgroups of G, it follows that

μA(xx−1,q) ≥ μA(x,q)∧μA(x−1,q), ∀x, x−1 ∈ G, q ∈ Q

μA′(xx−1,q) ≥ μA′(x,q)∧μA′(x−1,q), ∀x, x−1 ∈ G, q ∈ Q

(1)
Proof: Since $n < m$, then it follows that $A^n \subset A^m$ and $\mu_{A^n}(x, q) \leq \mu_{A^m}(x, q)$.

Now

$$\mu_{A^n \lor A^m}(xy, q) = (\mu_{A^n}(xy, q))^{\lor} \mu_{A^m}(xy, q) = (\mu_{A^n}(xy, q))^n \lor (\mu_{A^m}(xy, q))^m$$

$$= (\mu_{A^n}(xy, q))^n \lor (\mu_{A^m}(xy, q))^m$$

$$= (\mu_{A^n}(x, q))^n \lor (\mu_{A^m}(x, q))^m$$

$$= (\mu_{A^n}(x, q))^n \lor (\mu_{A^m}(x, q))^m$$

Therefore $A^n \lor A^m$ is also a $(Q, L)$-fuzzy subgroup of $G^m$.

Theorem 5.6: If $A^n$ and $A^m$ are $(Q, L)$-fuzzy subgroups of $G^m$, then $A^n \lor A^m$ is also an $(Q, L)$-fuzzy subgroup of $G^m$ if $n < m$.

Proof: Since $n < m$, then it follows that $A^n \subset A^m$ and $\mu_{A^n}(x, q) \leq \mu_{A^m}(x, q)$.

Now

$$\mu_{A^n \lor A^m}(x, q) = \mu_{A^n}(x, q) \lor \mu_{A^m}(x, q) = (\mu_{A^n}(x, q))^n \lor (\mu_{A^m}(x, q))^m$$

$\therefore A^n \lor A^m$ is also a $(Q, L)$-fuzzy subgroup of $G^m$.

Theorem 5.7: If $A^n(n = 1, 2, \ldots ), A^i \subset A^j$ for $i \leq j$ is a $(Q, L)$-fuzzy subgroup, then $A = A \lor A^2 \lor A^3 \lor \ldots$ is also a $(Q, L)$-fuzzy subgroup.

Proof: Since $A^i \lor A^j$ is also a $(Q, L)$-fuzzy subgroup for $i \leq j$, also $A^i \subset A^j$ for $i \leq j$.

Hence, $A = A \lor A^2 \lor A^3 \lor \ldots$ is a $(Q, L)$-fuzzy subgroup.

CONCLUSION

In this paper, we have discussed the direct product of $(Q, L)$-fuzzy groups, $(Q, L)$-fuzzy conjugate groups, and direct product of $(Q, L)$-fuzzy conjugate groups. Also we have conclude that positive integral powers of a $(Q, L)$-fuzzy group is a $(Q, L)$-fuzzy group. This concept can be extended for new results.

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