International Journal of Mathematical Archive-7(11), 2016, 112-116
Available online through www.ijma.info ISSN 2229 – 5046

FREE AND COFREE GAMMA ACTS

MEHDI S. ABBAS[1], ABDULQADER FARIS*[2]

Al-Mustansiriyah University,
College of Science, Department of Mathematics, Baghdad, Iraq.

(Received On: 31-10-16; Revised & Accepted On: 28-11-16)

ABSTRACT.

Let M be a Gamma Act over a Gamma semigroup S. In this paper, we investigate the notion of Free gamma acts and Cofree gamma acts and study some of their properties. Among other results we show that every gamma act is an epimorphic image of free gamma act. And any gamma act can be embedded in a cofree gamma act. As an applications of free and cofree gamma acts, we show that concept of monomorphism (epimorphosm)is coincides with injective (surjective) homomorphism in the category of gamma acts.

1. INTRODUCTION

The concept of Γ-semigroups has been introduced by M.K. Sen and N. K. Senin [2] and [3] as follows: if S and Γ are nonempty sets, S is called a Γ-semigroup if there is a mapping from S×Γ×S into S and the following condition is satisfied:

$$(s_1\alpha s_2)\,\beta s_3 = s_1\alpha\,(s_2\beta s_3) \text{ for all } s_1, s_2, s_3 \in S \text{ and } \alpha, \beta \in \Gamma.$$

And S is called Γ-semigroup with identity if there is an element $1 \in S$ such that $1\alpha s = s\alpha 1 = s$ for all $s \in S$ and $\alpha \in \Gamma$.

As a generalization of Γ-semigroup the authors in [1] the notion of gamma act: let S be a Γ-semigroup. A nonempty set M is called left S_Γ-act of S (denoted by ${}_{S_\Gamma}M$), if there is a mapping from S×Γ×M into M written (s, α, m) by $s\alpha m$ such that the following condition is satisfied

$$(s_1\alpha s_2)\beta m = s_1\alpha(s_2\beta m) \text{ for all } s_1, s_2 \in S,\ \alpha, \beta \in \Gamma \text{ and } m \in M.$$

Similarity we can define a right S_Γ-acts.

A left S_Γ-act M is called unitary if S is a Γ-semigroup with identity 1 and there exist $\alpha_\circ \in \Gamma$ such that $1\alpha_\circ m = m$ for all $m \in M$. We denote the element $1\alpha_\circ$ by $1_{\alpha_\circ}$ i.e $1_{\alpha_\circ}m = m$ for all $m \in M$. A nonempty subset N of M is called S_Γ-subact if $S\Gamma N \subseteq N$, Where $S\Gamma N = \{s\alpha n \mid s \in S, \alpha \in \Gamma, n \in N\}$. In this case we write $N \le M$.

The notion of S_Γ-homomorphism was introduced in [1] as follows: Let M and N be two S_Γ-acts. A mapping $f: M \to N$ is called S_Γ-homomorphism if $f(s\alpha m) = s\alpha f(m)$ for all $s \in S$, $\alpha \in \Gamma$ and $m \in M$. and $f: M \to N$ is called

1. injective if f is one to one mapping .
2. surjective if f is onto mapping
3. bijective if f is one to one and onto mapping .

If $f: M \to N$ is an bijective S_Γ-homomorphism, then we called M isomorphic to N and denoted by $M \cong N$.

Let M be an S_Γ-act, X a nonempty subset from M. Then the set define by $[X]_M \coloneqq \bigcap\{B \mid X \subseteq B, B \le M$, is the smallest S_Γ-subact of M contains X. and we showed $[X]_M = \bigcup_{u\in X} S\Gamma u$ where $S\Gamma u = \{s\alpha u \mid s \in S \text{ and } \alpha \in \Gamma\}$. If $M = [X]_M$ for some nonempty subset X of M, then X is called the generating set of M. That is for any element $m \in M$ implies that $m = s_1\alpha_1 u$ for some $s \in S$, $\alpha \in \Gamma$, $u \in X$, and if $X = [m]_M$ for some $m \in M$, then we call M is a cyclic gamma act generating by m. If $f: M \to N$ is an S_Γ-homomorphism, then $f([A]_M) = [f(A)]_N$ see [1].

Corresponding Author: Abdulqader Faris[2]*
Al-Mustansiriyah University, College of Science, Department of Mathematics, Baghdad, Iraq.

2. FREE GAMMA ACTS

In this section we introduce the concept of free gamma acts and show the structure of free gamma act. and study their properties.

Definition 2.1: Let M be an S_Γ-act. A generating set U of M is said to be basis of M, if every element m ∈ M can be uniquely presented in the form m = $s\alpha m_o$ for some s ∈ S, α∈Γ and m_o ∈ U, that is m = $s_1\alpha_1 m_1 = s_2\alpha_2 m_2$ if and only if $s_1 = s_2$, $\alpha_1 = \alpha_2$ and $m_1 = m_2$.

A Γ-semigroup S with identity 1 is free S_Γ-act with a basis {1} where Γ = {α_0}, otherwise in case Γ is non singleton, then the S_Γ-act S may not be free see example. (2.2) (2) below. If an S_Γ-act M has a basis U, then it is called a free S_Γ-act, more precisely |U|-free S_Γ-act. If U is one element set, then M is called 1-free. And every 1-free is a cyclic, but the converse is not true in general. The following examples show that the converse is not true.

Examples 2.2:
1. Let S = ℕ and Γ = {1}⊆ℕ. Then S is free S_Γ-act under multiplication of natural numbers with a basis {1}.
2. ℤ is not free $\mathbb{Z}_{\mathbb{N}}$-act since {1} is the only generating set of ℤ but 9 ∈ ℤ and 9 = 3.3.1 = 9.1.1, are two distance presentation of 9, observe that ℤ cyclic generating by {1}.
3. ℤ is $\mathbb{Z}_{\mathbb{Q}}$-act by the mapping ℤαℚαℤ →ℤ which is define by (n, $\frac{p}{q}$, m) ⟼np .for n, p, q, m ∈ℤ with q ≠ 0. Then ℤ is generating set of ℤ but it is not a base of ℤ. Since n = n$\frac{1}{q}$q for all n ≠ 0 and 0 = q$\frac{0}{q}$q for any q ≠ 0. However n = n$\frac{1}{n}$n = n$\frac{1}{q}$q for q ≠0 are two distance presentation of an integer n.

Proposition 2.3: Let M, N be S_Γ-acts, M ≅ N, and M free. Then so is N.

Proof: Clearly f(U) is a generating set of N. Now for n ∈ N and n = $s_1\alpha_1 f(u_1) = s_2\alpha_2 f(u_2)$, i.e n = $f(s_1\alpha_1 u_1) = f(s_2\alpha_2 u_2)$. This implies that (since f is injective) $s_1\alpha_1 u_1 = s_2\alpha_2 u_2$, and since M be |U|-free. we get $s_1 = s_2$, $\alpha_1 = \alpha_2$, $u_1 = u_2$ implies that $f(u_1) = f(u_2)$. We have N is $|f(U)|$-free S_Γ-act.

Corollary 2.4: If f: M → N is an injective S_Γ-homomorphism and M is |U|- free, then f(M) is |f(U)|-free.

Corollary 2.5: The union of free S_Γ-subacts is also free.

Proof: Let {N_i | i ∈ I} be collection of free S_Γ-subacts of M, and U_i be a basis of N_i for all i ∈ I. Then $[\bigcup_{i\in I} U_i] = \bigcup_{i\in I}[U_i] = \bigcup_{i\in I} N_i$, so it is easy to see that $\bigcup_{i\in I} U_i$ is a basis of $\bigcup_{i\in I} N_i\alpha$

The following gives the structure of free gamma acts.

It is known that in acts, F is free S-act where S is a monoid if and only if it is isomorphic to disjoint union of S-acts each is isomorphic to the S-act S see [2]. Unlike, for S_Γ-acts we have the following.

Theorem 2.6: Let M be an S_Γ-act. If M is |U|- free, then it is isomorphic to disjoint union of S_Γ-acts each is isomorphic to the S_Γ-act S, i.e M = $\bigcup_{i\in I} S_i$ where S_i ≅S for all i ∈ I.

Proof: Let M be a |U|- free S_Γ-act where U ={x_i | i ∈ I}is a basis of M. Then $S\Gamma x_i$ is S_Γ-subact of M for all i ∈ I, and $S\Gamma x_i \cap S\Gamma x_j = \emptyset$ for all distinct i, j ∈ I. Now for each i ∈ I define f_i: $S\Gamma x_i$ → S by $f_i(s\alpha x_i) = s\alpha 1$ for all s ∈ S, α∈ Γ and x_i ∈ U. Then $s_1\alpha_1 x_i = s_2\alpha_2 x_i$ implies that $s_1 = s_2$ and $\alpha_1 = \alpha_2$ and hence $s_1\alpha_1 1 = s_2\alpha_2 1$. For each $s\alpha x_i$ ∈ $S\Gamma x_i$, s' ∈ S and α' ∈ Γ implies that $f_i(s'\alpha'(s\alpha x_i)) = f_i((s'\alpha's)\alpha x_i) = (s'\alpha's)\alpha 1 = s'\alpha'(s\alpha 1) = s'\alpha'\ f_i(s\alpha x_i)$. and injective (if $f_i(s_1\alpha_1 x_i) = f_i(s_2\alpha_2 x_i)$ implies $s_1\alpha_1 1 = s_2\alpha_2 1$ we get $(s_1\alpha_1 1)_0 x_i = (s_2\alpha_2 1)_0 x_i$, where $1_0 = 1_0$ is the identity of Γ-semigroup S, then $s_1\alpha_1(1_0 x_i) = s_2\alpha_2(1_0 x_i)$ we have $s_1\alpha_1 x_i = s_2\alpha_2 x_i$ implies $s_1 = s_2$, $\alpha_1 = \alpha_2$) and surjective S_Γ-homomorphism. Then M ≅ $\bigcup_{i\in I} S_i$ where $S\Gamma x_i = S_i$ ≅S for all i ∈ I

The converse of the above theorem is true, however if Γ is a singleton set, then converse is true.

In the following we will show that any nonempty set gives a free gamma acts under arbitrary Γ-semigroup. Let S be a Γ-semigroup, X a nonempty set and F(X) be the set of all expression of the form sαx where s ∈ S, α ∈ Γ and x ∈ X such that $s_1\alpha_1 x_1 = s_2\alpha_2 x_2$ if and only if $x_1 = x_2$, $\alpha_1 = \alpha_2$ and $x_1 = x_2$. Define a mapping SαΓαF(X) → F(X) by (s', α', (sαx)) ⟼ (s'α's)αx, Under this multiplication F(X) becomes |X|-free S_Γ-act. The above construction of |X|-free S_Γ-act has the following universal property.

Theorem 2.7: Let M be a free S_Γ-act with a basis B = {x_i | i ∈ I]. If θ is any mapping from B into any S_Γ-act N, then there exists a unique S_Γ-homomorphism ψ: M → N such that ψ | B = θ.

Proof: Define a mapping ψ : M → N by $\psi(s\alpha x_i) = s\alpha\theta(x_i)$ where s ∈ S, α∈ Γ and x_i ∈ B. Since B is a basis of M we give ψ be well define and clearly that ψ be a S_Γ-homomorphism, ψ | B =θ. To show ψ is unique let ψ': M → N be another S_Γ-homomorphism with ψ' | B = θ. Then $\psi'(s\alpha x_i) = s\alpha\psi'(x_i) = s\alpha\theta(x_i) = s\alpha\psi(x_i) = \psi(s\alpha x_i)$ for all s ∈ S, α ∈ Γ and x_i∈ B.

The next proposition gives the importance of free S_Γ-act.

Proposition 3.8: For any S_Γ-act M, there exist a free S_Γ-act F, and an surjective SΓ-homomorphism from F to M.

Proof: For S_Γ-act M consider the free S_Γ-act F(M) as in construction of free gamma acts. Consider θ = I_M, Theorem **(2.7).** there exists an S_Γ-homomorphism ψ : F(M)→ M such that ψ | M = I_M. Since I_M is surjectiveImplies ψ is an surjective.

As an application of the universal property of free gamma acts we have the following results.

We recalled in [4] the definition of S_Γ-monomorphism in category as : an S_Γ-homomorphism f : M →N is called S_Γ-monomorphism if for all S_Γ-act H and all S_Γ-homomorphism ∂_1, ∂_2 : H → M such that $f\partial_1 = f\partial_2$, implies $\partial_1 = \partial_2$.

Proposition 2.9: Let M and N be S_Γ-acts and : M → N S_Γ-homo- morphism. Then the following are equivalent

1. f is a S_Γ-monomorphism.
2. f is injective mapping.

Proof:
(1) → (2): Let a, a' ∈ M with f(a) = f(a') and {x} is any one element set. Define g_1, g_2 : {x} → M by $g_1(x) = a$, $g_2(x) = a'$. Since F({x}) is a free with a base{x}, then by Theorem (2.7), implies that there are S_Γ-homomorphisms ∂_1, ∂_2 : F({x}) → M such that $\partial_1|\{x\} = g_1$ and $\partial_2|\{x\} = g_2$ by similar argument, there are ∂'_1, ∂'_2 : F({x}) → N such that $fg_1 = \partial'_1|\{x\}$ and $fg_2 = \partial'_2|\{x\}$. Since $fg_1 = fg_2$ by the uniqueness in Theorem (2.7) of S_Γ-homomorphism implies that $\partial'_1 = \partial'_2$ and hence $f\partial_1 = \partial'_1 = \partial'_2 = f\,\partial_2$. by the hypothesis implies that $\partial_1 = \partial_2$. Consequently a = $\partial_1(x) = \partial_2(x)$ = a'.

(2) → (1): follows [1].

Proposition 2.10: Let F be a free gamma act. Then for any S_Γ-acts X and Y, any S_Γ-epimorphism g: X → Y and homomorphism f: P → Y. there exist a homomorphism $\bar{f}$: P → X such that $f = g\bar{f}$.

Proof: Let I be a basis of F, for any S_Γ-homomorphism f: F→ Y, any S_Γ-epimorphism g: X→ Y. define the mapping $\bar{f}$: I → X by $\bar{f}$ (i) $\in g^{-1}(f(i))$ for all i ∈ I, which is possible since g is surjective. Then by Theorem (2.7). there exist $\check{f}$: F → X such that $f = g\check{f}$.

Proposition 2.11: Let F be a free S_Γ-act with a basis I. Then for any two S_Γ-acts X, Y and any S_Γ-homomorphisms f, g: G → Y, f ≠ g implies that fα ≠ gα for some α: G → X. where G any S_Γ-act.

Proof: Take f, g: X → Y be S_Γ-homomorphism with f≠ g. Then there is x ∈ X such that f(x)≠ g(x), let ∂: I → X be a mapping define by∂(i) = x for all i ∈ I. by Theorem (2.7). there exist S_Γ-homomorphism ∂ ': F → X such that ∂ = ∂' | I therefore, (f∂') (i) = *f(x) ≠ g(x)* = (*g*∂') (i) consequently *f*∂' ≠ *g*∂'.

3. COFREE GAMMA ACTS

In the following we introduce the dully concept of free gamma cats and study their properties.

Definition 3.1: An S_Γ-act K is called Cofree. If the exist a non- empty set I and mapping $\varkappa$: K → I such that for every S_Γ-act X and every mapping ϵ : X → I there exists exactly one S_Γ-homomorphism ϵ^* : X → K such that $\varkappa\epsilon^* = \epsilon$.

We sometimes write Cof(I) or (Cof(I),) for K and say that K is I-Cofree or |I|-Cofree. The set I is called Cobasis for K.

The following proposition is clear by definition of Cofree gamma acts.

Proposition 3.2: If |I| = |J| for sets I and J, Cof(I) and Cof(J) are |I|- and |J|-cofreeS_Γ-acts respectively, then Cof(I) ≅Cof(J).

In the following we constract of |I|-Cofree gamma acts. Let S be Γ-semigroup, and I a nonempty set. Consider the set I^S which is $I^S = \{f: S \to I \mid f \text{ is a mapping}\}$, and define the mapping S ⍰ Γ ⍰ $I^S \to I^S$ by $(s,\alpha, f) \longmapsto s\alpha f$ where $(s\alpha f)(s^*) = f(s\alpha s^*)$ for all $s^* \in S$.Under this multiplication I^S is an S_Γ-act.

The following proposition gives the existence of |I|-Cofree gamma act for a nonempty set I.

Proposition 3.3: Let S be Γ-semigroup and I a nonempty set. Then the S_Γ-act I^S with the mapping $\varkappa: I^S \to I$ define by $\varkappa(\alpha) = \alpha(1)$ for all $\alpha \in I^S$ is an |I|-Cofree.

Proof: Let X be S_Γ-act and let $\epsilon: X \to I$. Define a mapping
$$\epsilon^*: X \to I^S \text{ by } \epsilon^*(x)(s) = \epsilon(s_0 x) \text{ for all } s \in S, x \in X.$$
Then it is straightforward that $\varkappa^*$ is the unique mapping for which $\varkappa\epsilon^* = \epsilon$ and it is an S_Γ -homomorphism.

Corollary 3.4: Let S be Γ-semigroup. Then any |X|-Cofree gamma act is isomorphic to X^S.

Proof: From Proposition (3.3). For any nonempty set I there is an S_Γ-act K such that Cof(I) = K. and by Proposition (3.2) any Cofree S_Γ-act of a nonempty set X is isomorphic to X^S.

Corollary 3.5: Every S_Γ-act M can be embedding in a Cofree gamma act.

Proof: Exactly in [1] p150 proposition (4.3)

As an application of the notion of cofree gamma acts we have the following results.

We recalled the in [4] definition of SΓ-epimorphism in category as : an S_Γ-homomorphism $f: M \to N$ is called S_Γ-epimorphism if for all S_Γ-act H and all S_Γ-homomorphism $\partial_1, \partial_2: N \to H$ such that $\partial_1 f = \partial_2 f$, then $\partial_1 = \partial_2$.

Proposition 3.6: Let M and N be S_Γ-acts and $f: M \to N$ be S_Γ-homomorphism. Then the following are equivalent.

1. f is S_Γ-epimorphism .
2. f is surjective.

Proof:

(1) → (2): Let I be nonempty set with |I| > 1, $\{a, b\} \subseteq I$, $a \neq b$ and $\partial_1, \partial_2: X \to \{a, b\}$ such that $\partial_1 \mid Im(f) = \partial_2 \mid Im(f)$, $\partial_1(X \setminus Im(f)) = \{a\}$ and $\partial_2(X \setminus Im(f)) = \{b\}$. Consequently $\partial_1 f = \partial_2 f$. Then by definition of Cofree there exist S_Γ-homomorphisms $\varkappa_1, \varkappa_2: X \to \mathrm{Cof}(I)$ and $\varkappa'_1, \varkappa'_2: Y \to \mathrm{Cof}(I)$ as required of definition of Cofree by the uniqueness of S_Γ-homomorphism implies that $\varkappa_1 f = \varkappa'_1 = \varkappa'_2 = \varkappa_2 f$ and by hypothesis of (1), $\varkappa_1 = \varkappa_2$ thus $Im(f) = X$ i.e f is surjectiv.

(2) → (1): follows [1].

Proposition 3.7: Let Q be S_Γ-act. Then. For any S_Γ-act B, S_Γ-subact A of B and any S_Γ-homomorphism $f: A \to Q$ there exist homomorphism $\bar{f}: B \to Q$ such that $\bar{f} \mid A = f$.

Proof: Let B be S_Γ-act, A be S_Γ-subact of B, X^S a Cofree S_Γ-act for nonempty set X and $\varphi: A \to X^S$ An S_Γ-homomorphism. Let $y \in X$ be a fixed element, define a mapping $\overline{\varphi}: B \to X^S$ by
$$\overline{\varphi}(b)(t) = \begin{cases} \varphi(t_0 b)(1) & \text{if } t_0 b \in A \\ y & \text{other wise} \end{cases}$$

for all $t \in S$. It follows easily from definition of X^S that $\overline{\varphi}$ is a S_Γ-homomorphism. Let $a \in A$ then for any $t \in S$ we have $t_0 a \in A$. Hence $\overline{\varphi}(b)(t) = \varphi(t_0 b)(1) = t_0\varphi(b)(1) = \varphi(b)(1_0 t) = \varphi(b)(t)$. This mean $\overline{\varphi} \mid A = \varphi$.

Proposition 3.8: Let K = (Cof(I), δ) S_Γ-act for |I| > 1. Then for any S_Γ-acts X, Y and any S_Γ-homomorphisms f, g: X → Y with $f \neq g$ implies that $f\partial \neq g\partial$ for some S_Γ-homomorphism $\partial: Y \to C$. where C is any S_Γ-act.

Proof: Let f, g: X → Y be S_Γ-homomorphisms with $f \neq g$. Then there is $x \in X$ such that $f(x) \neq g(x)$. Take $\varkappa: Y \to I$ be a mapping such that $\varkappa f(x) \neq \varkappa g(x) \in I$. By definition of Cofree, there exist S_Γ-homomorphism $\varkappa': Y \to K$ such that $\varkappa = \delta\varkappa'$. This implies $\varkappa' f \neq \varkappa' g$.

REFERENCES

1. M. S. Abbas and Abdulqader, Gamma acts, International of advanced research,, Volume 4, Issue 6, (2016), 1592-1601
2. M. K sen. On Γ-semigroups. Proceeding of International Conference on Algebra and its Applications, Decker Publication, New York, (1981) 301.

3. M. K. SenandN. K. Saha, on Γ-semigroup I, Bulletin of the Calcutta Mathematical Society 78(1986), 181-186.
4. Matikilp, Ulrich knauerand Alexander V. Mikhalev. On monoids acts and categories. Walter de gruyter. Berlin. New York 2000.

Source of support: Nil, Conflict of interest: None Declared