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PROPERTIES OF BIPOLAR-VALUED Q-FUZZY SUBGROUPS OF A GROUP

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ABSTRACT

In this paper, we study some of the properties of bipolar-valued Q-fuzzy subgroup of a group and prove some results on these.

Key words: *Bipolar-valued Q-fuzzy set, bipolar-valued Q-fuzzy subgroup, product, bipolar-valued Q-fuzzy relation.*

INTRODUCTION

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued Q-fuzzy subgroup and established some results.

1. PRELIMINARIES

1.1 Definition: A bipolar-valued Q-fuzzy set (BVQFS) A in X is defined as an object of the form $A = \{< (x, q), A^+(x, q), A^-(x, q) > / x \text{ in } X \text{ and } q \text{ in } Q\}$, where $A^+: X\times Q\to [0, 1]$ and $A^- : X\times Q\to [-1, 0]$. The positive membership degree $A^+(x, q)$ denotes the satisfaction degree of an element (x, q) to the property corresponding to a bipolar-valued Q-fuzzy set A and the negative membership degree $A^-(x, q)$ denotes the satisfaction degree of an element (x, q) to some implicit counter-property corresponding to a bipolar-valued Q-fuzzy set A. If $A^+(x, q) \neq 0$ and $A^-(x, q) = 0$, it is the situation that (x, q) is regarded as having only positive satisfaction for A and if $A^+(x, q) = 0$ and $A^-(x, q) \neq 0$, it is the situation that (x, q) does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element (x, q) to be such that $A^+(x, q) \neq 0$ and $A^-(x, q) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

1.1 Example: $A = \{< (a, q), 0.7, -0.4 >, < (b, q), 0.6, -0.7 >, < (c, q), 0.5, -0.8 >\}$ is a bipolar-valued Q-fuzzy subset of X= {a, b, c}, where Q = {q}.

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1.2 Definition: Let G be a group and Q be a non-empty set. A bipolar-valued Q-fuzzy subset A of G is said to be a bipolar-valued Q-fuzzy subgroup of G (BVQFSG) if the following conditions are satisfied,

(i) $A^+(xy, q) \geq \min\{A^+(x, q), A^+(y, q),$
(ii) $A^+(x^{-1}, q) \geq A^+(x, q),$
(iii) $A^-(xy, q) \leq \max\{A^-(x, q), A^-(y, q)\},$
(iv) $A^-(x^{-1}, q) \leq A^-(x, q)$, for all x and y in G and q in Q.

1.2 Example: Let G = {1, –1, i, –i} be a group with respect to the ordinary multiplication and Q = {q}. Then A = {< (1, q), 0.5, – 0.6 >, < (–1, q), 0.4, – 0.5 >, < (i, q), 0.2, –0.4 >, < (–i, q), 0.2, –0.4 >} is a bipolar-valued Q-fuzzy subgroup of G.

1.3 Definition: Let $A = \langle A^+, A^-\rangle$ and $B = \langle B^+, B^-\rangle$ be any two bipolar-valued Q-fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by A×B, is defined as $A\times B = \{\langle ((x, y), q), (A\times B)^+((x, y), q), (A\times B)^-((x, y), q)\rangle$ / for all x in G and y in H and q in Q}, where $(A\times B)^+((x, y), q) = \min\{A^+(x, q), B^+(y, q)\}$ and $(A\times B)^-((x, y), q) = \max\{A^-(x, q), B^-(y, q)\}$, for all x in G and y in H and q in Q.

1.4 Definition: Let $A = \langle A^+, A^-\rangle$ be a bipolar-valued Q-fuzzy subset in a set S, the strongest bipolar-valued Q-fuzzy relation on S, that is a bipolar-valued Q-fuzzy relation on A is $V = \{\langle((x, y), q), V^+((x, y), q), V^-((x, y), q)\rangle$ / x and y in S and q in Q} given by $V^+((x, y), q) = \min\{A^+(x, q), A^+(y, q)\}$ and $V^-((x, y), q) = \max\{A^-(x, q), A^-(y, q)\}$, for all x and y in S and q in Q.

2. PROPERTIES

2.1 Theorem: Let $A = \langle A^+, A^-\rangle$ be a bipolar-valued Q-fuzzy subgroup of G. Then $A^+(x^{-1}, q) = A^+(x, q)$ and $A^-(x^{-1}, q) = A^-(x, q)$, $A^+(x, q) \leq A^+(e, q)$ and $A^-(x, q) \geq A^-(e, q)$, for all x in G and the identity element e in G and q in Q.

Proof: Let x be in G and q in Q. Now, $A^+(x, q) = A^+((x^{-1})^{-1}, q) \geq A^+(x^{-1}, q) \geq A^+(x, q)$. Therefore $A^+(x, q) = A^+(x^{-1}, q)$, for all x in G and q in Q. And $A^-(x, q) = A^-((x^{-1})^{-1}, q) \leq A^-(x^{-1}, q) \leq A^-(x, q)$. Therefore $A^-(x^{-1}, q) = A^-(x, q)$, for all x in G and q in Q. Now, $A^+(e, q) \geq \min\{A^+(x, q), A^+(x^{-1}, q)\} = A^+(x, q)$. Therefore $A^+(e, q) \geq A^+(x, q)$, for all x in G and q in Q. And $A^-(e, q) \leq \max\{A^-(x, q), A^-(x^{-1}, q)\} = A^-(x, q)$. Therefore $A^-(e, q) \leq A^-(x, q)$, for all x in G and q in Q.

2.2 Theorem: Let $A = \langle A^+, A^-\rangle$ be a bipolar-valued Q-fuzzy subgroup of G. Then

(i) $A^+(xy^{-1}, q) = A^+(e, q)$ implies that $A^+(x, q) = A^+(y, q)$, for x and y in G and q in Q.
(ii) $A^-(xy^{-1}, q) = A^-(e, q)$ implies that $A^-(x, q) = A^-(y, q)$, for x and y in G and q in Q.

Proof: Now, $A^+(x, q) \geq \min\{A^+(xy^{-1}, q), A^+(y, q)\} = \min\{A^+(e, q), A^+(y, q)\} = A^+(y, q) \geq \min\{A^+(yx^{-1}, q), A^+(x, q)\} = \min\{A^+(e, q), A^+(x, q)\} = A^+(x, q)$. Therefore $A^+(x, q) = A^+(y, q)$, for x and y in G and q in Q. And $A^-(x, q) \leq \max\{A^-(xy^{-1}, q), A^-(y, q)\} = \max\{A^-(e, q), A^-(y, q)\} = A^-(y, q) \leq \max\{A^-(yx^{-1}, q), A^-(x, q)\} = \max\{A^-(e, q), A^-(x, q)\} = A^-(x, q)$. Therefore $A^-(x, q) = A^-(y, q)$, for x and y in G and q in Q.

2.3 Theorem: Let $A = \langle A^+, A^-\rangle$ be a bipolar-valued Q-fuzzy subgroup of a group G.

(i) If $A^+(xy^{-1}, q) = 1$, then $A^+(x, q) = A^+(y, q)$, for x and y in G and q in Q.
(ii) If $A^-(xy^{-1}, q) = -1$, then $A^-(x, q) = A^-(y, q)$, for x and y in G and q in Q.

Proof: Now, $A^+(x, q) \geq \min\{A^+(xy^{-1}, q), A^+(y, q)\} = \min\{1, A^+(y, q)\} = A^+(y, q) = A^+(y^{-1}, q) \geq \min\{A^+(x^{-1}, q), A^+(xy^{-1}, q)\} = \min\{A^+(x^{-1}, q), 1\} = A^+(x^{-1}, q) = A^+(x, q)$. Therefore $A^+(x, q) = A^+(y, q)$, for x and y in G and q in Q. Hence (i) is proved. Also $A^-(x, q) \leq \max\{A^-(xy^{-1}, q), A^-(y, q)\} = \max\{-1, A^-(y, q)\} = A^-(y, q) = A^-(y^{-1}, q) \leq \max\{A^-(x^{-1}, q), A^-(xy^{-1}, q)\} = \max\{A^-(x^{-1}, q), -1\} = A^-(x^{-1}, q) = A^-(x, q)$. Therefore $A^-(x, q) = A^-(y, q)$, for x and y in G and q in Q. Hence (ii) is proved.

2.4 Theorem: Let $A = \langle A^+, A^-\rangle$ be a bipolar-valued Q-fuzzy subgroup of a group G.

(i) If $A^+(xy^{-1}, q) = 0$, then either $A^+(x, q) = 0$ or $A^+(y, q) = 0$, for x, y in G and q in Q.
(ii) If $A^-(xy^{-1}, q) = 0$, then either $A^-(x, q) = 0$ or $A^-(y, q) = 0$, for x, y in G and q in Q.

Proof: Let x and y in G and q in Q.

(i) By the definition $A^+(xy^{-1}, q) \geq \min\{A^+(x, q), A^+(y, q)\}$, which implies that $0 \geq \min\{A^+(x, q), A^+(y, q)\}$. Therefore, either $A^+(x, q) = 0$ or $A^+(y, q) = 0$.
(ii) By the definition $A^-(xy^{-1}, q) \leq \max\{A^-(x, q), A^-(y, q)\}$, which implies that $0 \leq \max\{A^-(x, q), A^-(y, q)\}$. Therefore, either $A^-(x, q) = 0$ or $A^-(y, q) = 0$, for x, y in G and q in Q.

2.5 Theorem: If $A = \langle A^+, A^- \rangle$ be a bipolar-valued Q-fuzzy subgroup of G, then
(i) $A^+(xy, q) = A^+(yx, q)$ if and only if $A^+(x, q) = A^+(y^{-1}xy, q)$, for x and y in G and q in Q.
(ii) $A^-(xy, q) = A^-(yx, q)$ if and only if $A^-(x, q) = A^-(y^{-1}xy, q)$, for x and y in G and q in Q.

Proof: Let x and y be in G and q in Q. Assume that $A^+(xy, q) = A^+(yx, q)$, so, $A^+(y^{-1}xy, q) = A^+(y^{-1}yx, q) = A^+(x, q)$. Therefore $A^+(x, q) = A^+(y^{-1}xy, q)$, for x and y in G and q in Q. Conversely, assume that $A^+(x, q) = A^+(y^{-1}xy, q)$, we get, $A^+(xy, q) = A^+(xyxx^{-1}, q) = A^+(yx, q)$. Therefore $A^+(xy, q) = A^+(xy, q)$, for x and y in G and q in Q. Hence $A^+(xy, q) = A^+(yx, q)$ if and only if $A^+(x, q) = A^+(y^{-1}xy, q)$, for x and y in G and q in Q. Also assume that $A^-(xy, q) = A^-(yx, q)$, we get, $A^-(y^{-1}xy, q)= A^-(y^{-1}yx, q) = A^-(x, q)$. Therefore $A^-(x, q) = A^-(y^{-1}xy, q)$, for x and y in G and q in Q. Conversely, assume that $A^-(x, q) = A^-(y^{-1}xy, q)$, so, $A^-(xy, q) = A^-(xyxx^{-1}, q) = A^-(yx, q)$. Therefore $A^-(xy, q) = A^-(xy, q)$, for x and y in G and q in Q. Hence $A^-(xy, q) = A^-(yx, q)$ if and only if $A^-(x, q) = A^-(y^{-1}xy, q)$, for x and y in G and q in Q.

2.6 Theorem: If $A = \langle A^+, A^- \rangle$ is a bipolar-valued Q-fuzzy subgroup of a group G, then $H = \{x\in G \mid A^+(x, q) = 1, A^-(x, q) = -1\}$ is either empty or is a subgroup of G.

Proof: If no element satisfies this condition, then H is empty. If x and y in H and q in Q, then $A^+(xy^{-1}, q) \geq \min\{A^+(x, q), A^+(y, q)\}= 1$. Therefore $A^+(xy^{-1}, q) = 1$. And $A^-(xy^{-1}, q) \leq \max\{ A^-(x, q), A^-(y, q)\}= -1$. Therefore $A^-(xy^{-1}, q) = -1$. That is $xy^{-1}\in H$. Hence H is a subgroup of G. Hence H is either empty or is a subgroup of G.

2.7 Theorem: If $A = \langle A^+, A^- \rangle$ is a bipolar-valued Q-fuzzy subgroup of G, then $H = \{x\in G \mid A^+(x, q) = A^+(e, q)$ and $A^-(x, q) = A^-(e, q) \}$ is a subgroup of G.

Proof: Here $H = \{x\in G \mid A^+(x, q) = A^+(e, q)$ and $A^-(x, q) = A^-(e, q)\}$, by Theorem 2.1, $A^+(x^{-1}, q) = A^+(x, q) = A^+(e, q)$ and $A^-(x^{-1}, q) = A^-(x, q) = A^-(e, q)$. Therefore $x^{-1}\in H$. Now, $A^+(xy^{-1}, q) \geq \min\{A^+(x, q), A^+(y, q)\}= \min\{ A^+(e, q), A^+(e, q)\} = A^+(e, q)$, and $A^+(e, q) = A^+((xy^{-1})(xy^{-1})^{-1}, q) \geq \min\{A^+(xy^{-1}, q), A^+(xy^{-1}, q)\}= A^+(xy^{-1}, q)$. Hence $A^+(e, q)= A^+(xy^{-1}, q)$. Also, $A^-(xy^{-1}, q)\leq \max\{A^-(x, q), A^-(y, q)\}= \max\{A^-(e, q), A^-(e, q)\} = A^-(e, q)$, and $A^-(e, q) = A^-((xy^{-1})(xy^{-1})^{-1}, q) \leq \max\{A^-(xy^{-1}, q), A^-(xy^{-1}, q)\}= A^-(xy^{-1}, q)$. Therefore $A^-(e, q) = A^-(xy^{-1}, q)$. Hence $A^+(e, q) = A^+(xy^{-1}, q)$ and $A^-(e, q) = A^-(xy^{-1}, q)$. Therefore $xy^{-1}\in H$. Hence H is a subgroup of G.

2.8 Theorem: Let G be a group and Q be a non-empty set. If $A = \langle A^+, A^- \rangle$ is a bipolar-valued Q-fuzzy subgroup of G, then $A^+(xy, q) = \min\{A^+(x, q), A^+(y, q)\}$ and $A^+(xy, q) = \max\{A^-(x, q), A^-(y, q) \}$ for each x and y in G with $A^+(x, q) \neq A^+(y, q)$ and $A^-(x, q) \neq A^-(y, q)$, where q in Q.

Proof: Assume that $A^+(x, q)> A^+(y, q)$ and $A^-(x, q)< A^-(y, q)$. Then $A^+(y, q)\geq \min\{A^+(x^{-1}, q), A^+(xy, q)\}= \min\{A^+(x, q), A^+(xy, q)\} = A^+(xy, q) \geq \min\{A^+(x, q), A^+(y, q)\}= A^+(y, q)$. Therefore $A^+(xy, q) = A^+(y, q) = \min\{A^+(x, q), A^+(y, q)\}$. And $A^-(y, q) \leq \max\{A^-(x^{-1}, q), A^-(xy, q)\}= \max\{A^-(x, q), A^-(xy, q)\}= A^-(xy, q)\leq \max\{A^-(x, q), A^-(y, q)\}= A^-(y, q)$. Therefore $A^-(xy, q)=A^-(y, q) = \max\{A^-(x, q), A^-(y, q)\}$.

2.9 Theorem: If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are two bipolar-valued Q-fuzzy subgroups of a group G, then their intersection $A\cap B$ is a bipolar-valued Q-fuzzy subgroup of G.

Proof: Let $A = \{< (x, q), A^+(x, q), A^-(x, q) > / x\in G$ and q in Q$\}$, $B = \{< (x, q), B^+(x, q), B^-(x, q) > / x\in G$ and q in Q$\}$. Let $C = A\cap B$ and $C = \{< (x, q), C^+(x, q), C^-(x, q) > / x\in G$ and q in Q$\}$. Now, $C^+(xy^{-1}, q) = \min\{A^+(xy^{-1}, q), B^+(xy^{-1}, q)\}\geq \min\{\min\{A^+(x, q), A^+(y, q)\}, \min\{B^+(x, q), B^+(y, q)\}\}\geq \min\{\min\{A^+(x, q), B^+(x, q)\}, \min\{A^+(y, q), B^+(y, q)\}\}= \min\{C^+(x, q), C^+(y, q)\}$. Therefore $C^+(xy^{-1}, q) \geq \min\{C^+(x, q), C^+(y, q)\}$. Also, $C^-(xy^{-1}, q) = \max\{A^-(xy^{-1}, q), B^-(xy^{-1}, q) \}\leq \max\{\max\{A^-(x, q), A^-(y, q)\}, \max\{B^-(x, q), B^-(y, q)\}\}\leq \max\{\max\{A^-(x, q), B^-(x, q)\}, \max\{A^-(y, q), B^-(y, q)\}\} = \max\{C^-(x, q), C^-(y, q)\}$. Therefore $C^-(xy^{-1}, q) \leq \max\{ C^-(x, q), C^-(y, q)\}$. Hence $A\cap B$ is a bipolar-valued Q-fuzzy subgroup of G.

2.10 Theorem: The intersection of a family of bipolar-valued Q-fuzzy subgroups of a group G is a bipolar-valued Q-fuzzy subgroup of G.

Proof: It is trivial.

2.11 Theorem: If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are any two bipolar-valued Q-fuzzy subgroups of the groups G_1 and G_2 respectively, then $A\times B = \langle (A\times B)^+, (A\times B)^- \rangle$ is a bipolar-valued Q-fuzzy subgroup of $G_1\times G_2$.

Proof: Let A and B be two bipolar-valued Q-fuzzy subgroups of the groups G_1 and G_2 respectively. Let x_1 and x_2 be in G_1, y_1 and y_2 be in G_2 and q in Q. Then (x_1, y_1) and (x_2, y_2) are in $G_1\times G_2$. Now, $(A\times B)^+[(x_1, y_1)(x_2, y_2)^{-1}, q] = (A\times B)^+((x_1x_2^{-1}, y_1y_2^{-1}), q) = \min\{A^+(x_1x_2^{-1}, q), B^+(y_1y_2^{-1}, q)\} \geq \min\{\min\{A^+(x_1, q), A^+(x_2, q)\}, \min\{B^+(y_1, q), B^+(y_2, q)\}\} = \min\{\min\{A^+(x_1, q), B^+(y_1, q)\}, \min\{A^+(x_2, q), B^+(y_2, q)\} = \min\{(A\times B)^+((x_1, y_1), q), (A\times B)^+((x_2, y_2), q)\}$. Therefore, $(A\times B)^+[(x_1, y_1)(x_2, y_2)^{-1}, q] \geq \min\{(A\times B)^+((x_1, y_1), q), (A\times B)^+((x_2, y_2), q)\}$. Also, $(A\times B)^-[(x_1, y_1)(x_2, y_2)^{-1}, q] = (A\times B)^-((x_1x_2^{-1}, y_1y_2^{-1}), q) = \max\{A^-(x_1x_2^{-1}, q), B^-(y_1y_2^{-1}, q)\} \leq \max\{\max\{A^-(x_1, q), A^-(x_2, q)\}, \max\{B^-(y_1, q), B^-(y_2, q)\}\} = \max\{\max\{A^-(x_1, q), B^-(y_1, q)\}, \max\{A^-(x_2, q), B^-(y_2, q)\}\} = \max\{(A\times B)^-((x_1, y_1), q), (A\times B)^-((x_2, y_2), q)\}$. Therefore, $(A\times B)^-[(x_1, y_1)(x_2, y_2)^{-1}, q] \leq \max\{(A\times B)^-((x_1, y_1), q), (A\times B)^-((x_2, y_2), q)\}$. Hence A×B is a bipolar-valued Q-fuzzy subgroup of $G_1\times G_2$.

2.12 Theorem: Let $A = \langle A^+, A^-\rangle$ and $B = \langle B^+, B^-\rangle$ be any two bipolar-valued Q-fuzzy subsets of the groups G and H respectively. Suppose that e and e^l are the identity elements of G and H respectively. If A×B is a bipolar-valued Q-fuzzy subgroup of G×H, then at least one of the following two statements must hold.

(i) $B^+(e^l, q) \geq A^+(x, q)$ and $B^-(e^l, q) \leq A^-(x, q)$, for all x in G and q in Q,
(ii) $A^+(e, q) \geq B^+(y, q)$ and $A^-(e, q) \leq B^-(y, q)$, for all y in H and q in Q.

Proof: Let A×B is a bipolar-valued Q-fuzzy subgroup of G×H. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in G and b in H and q in Q such that $A^+(a, q) > B^+(e^l, q)$, $A^-(a, q) < B^-(e^l, q)$ and $B^+(b, q) > A^+(e, q)$, $B^-(b, q) < A^-(e, q)$. We have, $(A\times B)^+((a, b), q) = \min\{A^+(a, q), B^+(b, q)\} > \min\{A^+(e, q), B^+(e^l, q)\} = (A\times B)^+((e, e^l), q)$. Also, $(A\times B)^-((a, b), q) = \max\{A^-(a, q), B^-(b, q)\} < \max\{A^-(e, q), B^-(e^l, q)\} = (A\times B)^-((e, e^l), q)$. Thus A×B is not a bipolar-valued Q-fuzzy subgroup of G×H. Hence either $B^+(e^l, q) \geq A^+(x, q)$ and $B^-(e^l, q) \leq A^-(x, q)$, for all x in G and q in Q or $A^+(e, q) \geq B^+(y, q)$ and $A^-(e, q) \leq B^-(y, q)$, for all y in H and q in Q.

2.13 Theorem: Let $A = \langle A^+, A^-\rangle$ and $B = \langle B^+, B^-\rangle$ be any two bipolar-valued Q-fuzzy subsets of the groups G and H, respectively and A×B is a bipolar-valued Q-fuzzy subgroup of G×H. Then the following are true:

(i) if $A^+(x, q) \leq B^+(e^l, q)$ and $A^-(x, q) \geq B^-(e^l, q)$, for all x in G and q in Q, then A is a bipolar-valued Q-fuzzy subgroup of G, where e^l is identity element of H.
(ii) if $B^+(x, q) \leq A^+(e, q)$ and $B^-(x, q) \geq A^-(e, q)$, for all x in H and q in Q, then B is a bipolar-valued Q-fuzzy subgroup of H, where e is identity element of G.
(iii) either A is a bipolar-valued Q-fuzzy subgroup of G or B is a bipolar-valued Q-fuzzy subgroup of H, where e and e^l are the identity elements of G and H respectively.

Proof: Let A×B be a bipolar-valued Q-fuzzy subgroup of G×H and x, y be in G. Then (x, e^l) and (y, e^l) are in G×H and q in Q. Now, using the property if $A^+(x, q) \leq B^+(e^l, q)$ and $A^-(x, q) \geq B^-(e^l, q)$, for all x in G and q in Q, where e^l is identity element of H, we get, $A^+(xy^{-1}, q) = \min\{A^+(xy^{-1}, q), B^+(e^le^l, q)\} = (A\times B)^+((xy^{-1}, e^le^l), q) = (A\times B)^+[(x, e^l)(y^{-1}, e^l), q] \geq \min\{(A\times B)^+((x, e^l), q), (A\times B)^+((y^{-1}, e^l), q)\} = \min\{\min\{A^+(x, q), B^+(e^l, q)\}, \min\{A^+(y^{-1}, q), B^+(e^l, q)\}\} = \min\{A^+(x, q), A^+(y^{-1}, q)\} \geq \min\{A^+(x, q), A^+(y, q)\}$. Therefore, $A^+(xy^{-1}, q) \geq \min\{A^+(x, q), A^+(y, q)\}$, for all x, y in G and q in Q. Also, $A^-(xy^{-1}, q) = \max\{A^-(xy^{-1}, q), B^-(e^le^l, q)\} = (A\times B)^-((xy^{-1}, e^le^l), q) = (A\times B)^-[(x, e^l)(y^{-1}, e^l), q] \leq \max\{(A\times B)^-((x, e^l), q), (A\times B)^-((y^{-1}, e^l), q)\} = \max\{A^-(x, q), B^-(e^l, q)\}, \max\{A^-(y^{-1}, q), B^-(e^l, q)\}\} = \max\{A^-(x, q), A^-(y^{-1}, q)\} \leq \max\{A^-(x, q), A^-(y, q)\}$. Therefore, $A^-(xy^{-1}, q) \leq \max\{A^-(x, q), A^-(y, q)\}$, for all x, y in G and q in Q. Hence A is a bipolar-valued Q-fuzzy subgroup of G. Thus (i) is proved. Now, using the property $B^+(x, q) \leq A^+(e, q)$ and $B^-(x, q) \geq A^-(e, q)$, for all x in H and q in Q, we get, $B^+(xy^{-1}, q) = \min\{B^+(xy^{-1}, q), A^+(ee, q)\} = (A\times B)^+((ee, xy^{-1}), q) = (A\times B)^+[(e, x)(e, y^{-1}), q] \geq \min\{(A\times B)^+((e, x), q), (A\times B)^+((e, y^{-1}), q)\} = \min\{\min\{A^+(e, q), B^+(x, q)\}, \min\{A^+(e, q), B^+(y^{-1}, q)\}\} = \min\{B^+(x, q), B^+(y^{-1}, q)\} \geq \min\{B^+(x, q), B^+(y, q)\}$. Therefore, $B^+(xy^{-1}, q) \geq \min\{B^+(x, q), B^+(y, q)\}$, for all x, y in H and q in Q. Also, $B^-(xy^{-1}, q) = \max\{B^-(xy^{-1}, q), A^-(ee, q)\} = (A\times B)^-((ee, xy^{-1}), q) = (A\times B)^-[(e, x)(e, y^{-1}), q] \leq \max\{(A\times B)^-((e, x), q), (A\times B)^-((e, y^{-1}), q)\} = \max\{\max\{A^-(e, q), B^-(x, q)\}, \max\{A^-(e, q), B^-(y^{-1}, q)\}\} = \max\{B^-(x, q), B^-(y^{-1}, q)\} \leq \max\{B^-(x, q), B^-(y, q)\}$. Therefore, $B^-(xy^{-1}, q) \leq \max\{B^-(x, q), B^-(y, q)\}$, for all x, y in H and q in Q. Hence B is a bipolar-valued Q-fuzzy subgroup of H. Thus (ii) is proved. Hence (iii) is clear.

2.14 Theorem: Let $A = \langle A^+, A^-\rangle$ be a bipolar-valued Q-fuzzy subset of a group (G, .) and $V = \langle V^+, V^-\rangle$ be the strongest bipolar-valued Q-fuzzy relation of G. Then A is a bipolar-valued Q-fuzzy subgroup of G if and only if V is a bipolar-valued Q-fuzzy subgroup of G×G.

Proof: Suppose that A is a bipolar-valued Q-fuzzy subgroup of G. Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in G×G and q in Q. We have, $V^+(xy^{-1}, q) = V^+[(x_1, x_2)(y_1, y_2)^{-1}, q] = V^+((x_1y_1^{-1}, x_2y_2^{-1}), q) = \min\{A^+(x_1y_1^{-1}, q), A^+(x_2y_2^{-1}, q)\} \geq \min\{\min\{A^+(x_1, q), A^+(y_1, q)\}, \min\{A^+(x_2, q), A^+(y_2, q)\}\} = \min\{\min\{A^+(x_1, q), A^+(x_2, q)\}, \min\{A^+(y_1, q), A^+(y_2, q)\}\} = \min\{V^+((x_1, x_2), q), V^+((y_1, y_2), q)\} = \min\{V^+(x, q), V^+(y, q)\}$. Therefore, $V^+(xy^{-1}, q) \geq \min\{V^+(x, q), V^+(y, q)\}$, for all x, y in G×G and q in Q. Also we have, $V^-(xy^{-1}, q) = V^-[(x_1, x_2)(y_1, y_2)^{-1}, q] = V^-((x_1y_1^{-1}, x_2y_2^{-1}), q) = \max\{A^-(x_1y_1^{-1}, q), A^-(x_2y_2^{-1}, q)\} \leq \max\{\max\{A^-(x_1, q), A^-(y_1, q)\}, \max\{A^-(x_2, q), A^-(y_2, q)\}\} = \max\{\max\{A^-(x_1, q), A^-(x_2, q)\}, \max\{A^-(y_1, q), A^-(y_2, q)\}\} = \max\{V^-((x_1, x_2), q), V^-((y_1, y_2), q)\} = \max\{V^-(x, q), V^-(y, q)\}$. Therefore,

$V^-(xy^{-1}, q) \leq \max\{V^-(x, q), V^-(y, q)\}$, for all x, y in G×G and q in Q. This proves that V is a bipolar-valued Q-fuzzy subgroup of G×G. Conversely, assume that V is a bipolar-valued Q-fuzzy subgroup of G×G, then for any $x=(x_1, x_2)$ and $y=(y_1, y_2)$ are in G×G, we have $\min\{A^+(x_1y_1^{-1}, q), A^+(x_2y_2^{-1}, q)\} = V^+((x_1y_1^{-1}, x_2y_2^{-1}), q) = V^+[(x_1, x_2)(y_1, y_2)^{-1}, q] = V^+(xy^{-1}, q) \geq \min\{V^+(x, q), V^+(y, q)\} = \min\{V^+((x_1, x_2), q), V^+((y_1, y_2), q)\} = \min\{\min\{A^+(x_1, q), A^+(x_2, q)\}, \min\{A^+(y_1, q), A^+(y_2, q)\}\}$. If we put $x_2 = y_2 = e$, we get, $A^+(x_1y_1^{-1}, q) \geq \min\{A^+(x_1, q), A^+(y_1, q)\}$, for all x_1, y_1 in G and q in Q. Also we have, $\max\{A^-(x_1y_1^{-1}, q), A^-(x_2y_2^{-1}, q)\} = V^-((x_1y_1^{-1}, x_2y_2^{-1}), q) = V^-[(x_1, x_2)(y_1, y_2)^{-1}, q] = V^-(xy^{-1}, q) \leq \max\{V^-(x, q), V^-(y, q)\} = \max\{V^-((x_1, x_2), q), V^-((y_1, y_2), q)\} = \max\{\max\{A^-(x_1, q), A^-(x_2, q)\}, \max\{A^-(y_1, q), A^-(y_2, q)\}\}$. If we put $x_2 = y_2 = e$, we get, $A^-(x_1y_1^{-1}, q) \leq \max\{A^-(x_1, q), A^-(y_1, q)\}$, for all x_1, y_1 in G and q in Q. Hence A is a bipolar-valued Q-fuzzy subgroup of G.

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