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CREEPING FLOW FROM A CIRCULAR ORIFICE IN A PLANE WALL

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ABSTRACT

Creeping motion of a viscous fluid through a circular hole in a plane when the boundary conditions on the plane are prescribed is studied and integral solution for the problem presented. Infinite series representation and asymptotic expansion are developed to understand the far field behavior of the fluid flow and to facilitate numerical work.

INTRODUCTION

Forste [1], parmet and saibel [2] and viviand and Berger [3] have considered the two-dimensional and axisymmetric creeping motion of a viscous incompressible fluid in the half plane z>0 for an arbitrarily prescribed velocity field in the plane z = 0. Integral solutions for asymmetric jet type of flows have been given by vijaya lakshmi [4, 5]. In this paper, integral solution is presented for an axisymmetric flow from a circular hole in the plane wall z = 0 when the velocity of flow through the hole is constant. The solution is presented alternatively in the form of an infinite series and asymptotic expansion is also developed to understand the far field behavior of the fluid flow and to facilitate numerical work.

MATHEMATICAL FORMULATION AND SOLUTION

The equations of motion for steady creeping flow are given by Batchelor [6]

Equation of momentum: $\nabla^2\bar{\mu} = \frac{1}{u}\nabla p$ (1)

Equation of mass-conservation: $\nabla.\bar{u} = 0$ (2)

Where ∇^2 is the two-dimensional laplacian, $\bar{\mu}$ is the velocity vector, p the pressure and u the coefficient of viscosity. Let (μ, v, w) be the components of velocity along the directions of the cylindrical polar coordinates (z, p, Ø) respectively, the plane wall being defined by z = 0. Consider the steady flow of a viscous incompressible fluid with a constant velocity μ_0 through a circular orifice p=a in the wall. For an axisymmetric flow we have $\frac{\partial()}{\partial \emptyset} = 0$ and $w = 0$

$$\mu = z\frac{\partial \emptyset}{\partial z} - \emptyset$$

$$v = z\frac{\partial \emptyset}{\partial p} \quad (3)$$

$$w = 0$$

then the eq's (1) and (2) are satisfied, provided that

$$p = p_0 + 2\mu\frac{\partial \emptyset}{\partial z} \quad (4)$$

and

$$\nabla^2 \emptyset = 0, \nabla^2 = \frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p} + \frac{\partial^2}{\partial z^2} \quad (5)$$

Where p_0 is a constant

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The boundary conditions are

$$\left.\begin{array}{lll}\mu = \mu_0 \text{ (constant)} & p < a & \\ \quad = 0 & p > a & \\ \quad = 0 & & \end{array}\right\} \quad \text{on } z = 0 \tag{6}$$

$\mu, v \to 0$ as $z \to \infty$

In terms of Ø the boundary conditions at z = 0 Are

$$\left.\begin{array}{ll}\text{Ø} = -\mu_0 & p < a \\ \quad = 0 & p > a \\ \quad = 0 & \end{array}\right\} \begin{array}{l} \\ \text{inside the orifice} \\ \text{outside the orifice}\end{array} \tag{7}$$

Now, we have to solve (5) subject to the boundary conditions (7). That is, we have to seek solution for the dirichlet problem for the domain z > 0 and the appropriate solution is Lebedev [7]

$$\text{Ø}(z, p) = -\mu_0 a \int_0^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda \tag{8}$$

We therefore have

$$\mu = \mu_0\, az\int_0^\infty \lambda e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda + \mu_0\, a\int_0^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda \tag{9}$$

$$v = \mu_0\, a\int_0^\infty \lambda e^{-\lambda z} J_1(\lambda p)\, J_1(\lambda a)\, d\lambda \tag{10}$$

$$p = p_0 + 2\,\mu\, \mu_0 a \int_0^\infty \lambda e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda \tag{11}$$

Here we have used the result Lebedev [7]

$$J_0^1(z) = -J_1(z)$$

We discuss the following cases here:

Case-(1): Putting $\mu = 2q$, $p = x$, $v = 1$, $t = \lambda$ in the formula sneddon [9]

$$\int_0^\infty t^u\ e^{-pt}\ J_v(at)dt = \frac{2v\mu_a}{(a^2+p^2)^{\frac{1}{2}\mu+\frac{1}{2}v+\frac{1}{2}}}\ \frac{\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}v+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}v+1\right)}{\Gamma(v+1)\,\Gamma\left(\frac{1}{2}\right)}\ 2^{F_1}\ \left(\frac{1}{2}\mu+\frac{1}{2}v+\frac{1}{2}, \left(\frac{1}{2}v-\frac{1}{2}\mu, 1+v, \frac{a^2}{a^2+p^2}\right)\right. \tag{I.1}$$

Where $$2^{F_1}\ (a,b,c,x) = \sum_{r=0}^{\infty} \frac{(a)_{r}(b)_r}{(r!)(c)_r}\ x^r \tag{I.2}$$

With

$(a)_r = a(a+1)(a+2)\ldots\ \ldots(a+r-1)$, is the generalized hypergeometric function, and also using the result Lebedev [7]

$$J_m(z) = \sum_{q=0}^{\infty} \frac{\frac{z}{2}^{m+2v}(-1)^q}{q!(m+q)!} \tag{I.3}$$

In $$I_1 = \int_0^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda$$

We obtain

$$I_1 = \int_0^\infty e^{-\lambda z} \sum_{q=0}^{\infty} \frac{(-1)^q \lambda^{2q} p^{2q}}{2^{2q}(q!)^2}\ J_1(\lambda a)\, d\lambda$$

$$= \sum_{q=0}^{\infty} \frac{(-1)^q p^{2q}}{2^{2q}(q!)^2} \int_0^\infty \lambda^{2q} e^{-\lambda z}\ J_1(\lambda a)\, d\lambda$$

$$\sum_{q=0}^{\infty} \frac{(-1)^q (2q+1)!p^{2q}}{(q!)^2}\ \frac{a}{(z^2+a^2)^{q+1}}\, F\left(q+1, \frac{1}{2}-q, 2, \frac{a^2}{a^2+z^2}\right) \tag{I.4}$$

We can derive similarly

In $$I_2 = \int_0^\infty \lambda e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda$$

$$= \sum_{q=0}^{\infty} \frac{(-1)^q p^{2q}}{2^{2q}(q!)^2} \int_0^\infty \lambda^{2q+1} e^{-\lambda z}\ J_1(\lambda a)\, d\lambda$$

$$= \sum_{q=0}^{\infty} \frac{(-1)^q (2q+1)!(q+1)p^{2q}}{(q!)^2}\ \frac{a}{(z^2+a^2)^{q+\frac{3}{2}}}\, F\left(q+\frac{3}{2}, -q, 2, \frac{a^2}{a^2+z^2}\right)$$

And

$$I_3 = \int_0^\infty \lambda e^{-\lambda z} J_1(\lambda p)\, J_1(\lambda a)\, d\lambda$$

$$= \sum_{q=0}^{\infty} \frac{(-1)^q p^{2q+1}}{2^{2q+1}(q!)^{q+1}(q+1)!} \int_0^\infty \lambda^{2q+2} e^{-\lambda z} J_1(\lambda a)\, d\lambda$$

$$\sum_{q=0}^{\infty} \frac{(-1)^q (2q+3)!p^{2q+1}}{(q+1)!q!2^{2q+2}}\ \frac{a}{(z^2+a^2)^{2q+2}}\, F\left(q+2, -q\frac{1}{2}, 2, \frac{a^2}{a^2+z^2}\right) \tag{I.6}$$

Case-(2): Laplace transform of Bessel function $J_0(\lambda p)$ is given by Sneddon [9]

$$L\{J_0(\lambda p)\} = \int_0^\infty e^{-\lambda z} J_0(\lambda p)\, d\lambda = (z^2+p^2)^{\frac{-1}{2}} \tag{II.1}$$

We therefore have on using (I.3)

$$I_1 = L\{J_0(\lambda p)J_1(\lambda a)\} = \int_0^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda$$
$$= \int_0^\infty e^{-\lambda z} J_0(\lambda p)\left(\sum_{q=0}^\infty \frac{(-1)^q p^{2q+1}\lambda^{2q+1}}{2^{2q+1}q!(q+1)!}\right) d\lambda$$
$$= \sum_{q=0}^\infty \frac{(-1)^q a^{2q+1}}{2^{2q+1}q!(q+1)!}\int_0^\infty \lambda^{2q+1} e^{-\lambda z} J_0(\lambda p)\, d\lambda,$$

Interchanging the order of summation and integration,

$$I_1 = \sum_{q=0}^\infty \frac{(-1)^{q+1} a^{2q+1}}{2^{2q+1}q!(q+1)!}\frac{\partial^{2q+1}}{\partial z^{2q+1}}((z^2+p^2)^{-\frac{1}{2}}$$
$$= \frac{a}{2}\frac{z}{(z^2+a^2)^{\frac{3}{2}}} + \frac{a^3}{2!\,2^3}\left\{\frac{9z}{(z^2+a^2)^{\frac{5}{2}}} - \frac{15z^3}{(z^2+a^2)^{\frac{7}{2}}}\right\} + \ldots\ldots\ldots \tag{II.2}$$

We also have

$$I_2 = \int_0^\infty \lambda e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda$$
$$= \sum_{q=0}^\infty \frac{(-1)^q a^{2q+1}}{2^{2q+1}q!(q+1)!}\frac{\partial^{2q+2}}{\partial z^{2q+2}}(z^2+p^2)^{-\frac{1}{2}}$$
$$= \frac{a}{2}\left\{-\frac{1}{(z^2+p^2)^{\frac{3}{2}}} + \frac{3z^2}{(z^2+p^2)^{\frac{5}{2}}}\right\} + \frac{(-1)a^3}{2^3 1!2!}\left\{\frac{9}{(z^2+p^2)^{\frac{5}{2}}} - \frac{90z^2}{(z^2+p^2)^{\frac{7}{2}}} + \frac{105z^4}{(z^2+p^2)^{\frac{9}{2}}}\right\} + \ldots\ldots$$

And

$$I_3 = \int_0^\infty \lambda e^{-\lambda z} J_1(\lambda p)\, J_1(\lambda a)\, d\lambda$$
$$= \sum_{q=0}^\infty \frac{(-1)^q a^{2q+1}}{2^{2q+1}q!(q+1)!}\frac{\partial^{2q+2}}{\partial z^{2q+2}}(z^2+p^2)^{-\frac{1}{2}}$$
$$= -\frac{3a}{2}\frac{pz}{(z^2+p^2)^{\frac{5}{2}}} - \frac{a^3}{2^3 2!}\left\{\frac{45pz}{(z^2+p^2)^{\frac{7}{2}}} + \frac{105pz^3}{(z^2+p^2)^{\frac{9}{2}}}\right\} + \ldots\ldots\ldots\ldots \tag{II.4}$$

Case-(3): We break up the integral I_1 as follows

$$I_1 = \int_0^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda$$
$$= \int_0^k e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda + \int_k^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda \ldots\ldots\ldots\ldots \tag{III.1}$$

And then estimate the second integral asymptotically for large k. For large z we have Lebedev [7]

$$J_v(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{v}{2}\pi - \frac{\pi}{4}\right) \ldots\ldots\ldots \tag{III.2}$$

So that

$$\int_k^\infty e^{-\lambda z} J_0(\lambda p)\, j_1(\lambda a)\, d\lambda = -\frac{1}{\pi(ar)^{\frac{1}{2}}}\left[\,\mathrm{Re}\int_k^\infty \frac{e^{-\lambda[z+i(p+a)]}}{\lambda} d\lambda + \mathrm{I_m}\int_k^\infty \frac{e^{-\lambda[z+i(p-a)]}}{\lambda} d\lambda\right] \ldots.. \tag{III.3}$$

Where Re and I_m denote the real and imaginary parts respectively. Noticing that each of the above integrals is expressible as the exponential integral Lebedev [7] defined by

$$\mathrm{Ei}\,(z) = \int_{-\infty}^z \frac{e^t}{t} dt \qquad |\arg(-z)| < \pi \ \ldots\ldots\ldots \tag{III.4}$$

And using the asymptotic representation Lebedev [7]

$$\mathrm{Ei}(z) = \frac{e^z}{z}\sum_{m=0}^n \frac{m!}{z^m} + 0\,(|z|^{-n-1}),\ |\arg(-z)| < \pi - \delta \ \ldots\ldots\ldots \tag{III.5}$$

Where δ is an arbitrarily small positive number, we finally obtain, for sufficiently large k

$$\int_0^\infty e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda \approx \int_0^k e^{-\lambda z} J_0(\lambda p)\, J_1(\lambda a)\, d\lambda - \frac{1}{\pi(ar)^{\frac{1}{2}}} e^{-kz}$$
$$\sum_{m=0}^n \frac{m!}{k^{m+1}}\left\{\frac{\cos[k(p+a)+(m+1)Tan^{-1}(p+a)/z]}{[z^2+(p+a)^2]^{m+1/2}} + \frac{\sin[k(p-a)+(m+1)Tan^{-1}(p-a)/z]}{[z^2+(p-a)^2]^{m+1/2}}\right\}\ldots\ldots\ldots\ldots \tag{III.6}$$

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