VERY $\beta_e$-EXCELLENCE OF A GRAPH

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(Received On: 04-10-16; Revised & Accepted On: 22-10-16)

ABSTRACT

Let $G = (V, E)$ be a simple finite undirected graph. A subset $S$ of $V$ is called an equivalence set if every component of the induced sub graph $\langle S \rangle$ is complete. The equivalence number $\beta_e(G)$ is the maximum cardinality of an equivalence set of $G$ [3]. A vertex $u$ in $V(G)$ is said to be $\beta_e$-good if $u$ belongs to a $\beta_e$ set of $G$. $G$ is said to be $\beta_e$-excellent if every vertex of $G$ is $\beta_e$-good. A graph $G = (V,E)$ is said to be very $\beta_e$-excellent if there exists a $\beta_e$-set $S$ of $G$ such that for every $u$ in $V-S$, there exists a vertex $v$ in $S$ such that $(S - \{v\}) \cup \{u\}$ is a $\beta_e$-set of $G$. $S$ is called a very $\beta_e$-excellent set of $G$ and $G$ is called a very $\beta_e$-excellent graph. An equivalence graph is a vertex disjoint union of complete graphs. The concept of equivalence set, sub chromatic number, generalized coloring and equivalence covering number were studied in [1], [2], [4], [5], [6], [8], [10]. In this paper the concept of very $\beta_e$-excellence is studied.

Keywords: Equivalence set, Equivalence graph, $\beta_e$-excellence, Very $\beta_e$-excellence.

1. INTRODUCTION

Gred.H. Fricke et al [7] called a vertex $u$ of a graph $G = (V, E)$ to be $\mu$-good if $u$ is contained in a $\mu(G)$-set of $G$ (where $\mu$ is a parameter). $G$ is said to be $\mu$-excellent if every vertex in $V$ is $\mu$-good. A number of results has been proved by taking $\mu$ as the domination parameter. Sridharan and Yamuna [12], [13] introduced several types of excellence, one of them being rigid excellence. A graph $G$ is said to be rigid $\mu$-excellent if every vertex of $G$ belongs to a unique $\mu$-set of $G$. Rigid $\gamma$-excellence was studied in [13]. A similar study was made with respect to the parameter $\beta_0$ in [11]. A sub set $S$ of $V(G)$ is said to be an equivalence set if every component of $\langle S \rangle$ is complete. A graph $G$ is said to be an equivalence graph if $V(G)$ is an equivalence set. The maximum cardinality of an equivalence set is denoted by $\beta_e(G)$ [3]. In this paper, very $\beta_e$-excellence is defined and several results are derived.

2. Very $\beta_e$-Excellence of a Graph

Definition 2.1: A graph $G = (V,E)$ is said to be very $\beta_e$-excellent if there exists a $\beta_e$-set $S$ of $G$ such that for every $u$ in $V-S$, there exists a vertex $v$ in $S$ such that $(S - \{v\}) \cup \{u\}$ is a $\beta_e$-set of $G$. $S$ is called a very $\beta_e$-excellent set of $G$ and $G$ is called a very $\beta_e$-excellent graph.
Example 2.2: Consider $P_4$ with $V(P_4) = \{u_1, u_2, u_3, u_4\}$.

A graph which is very $\beta_e$-excellent

Figure-2.1

$S = \{u_1, u_2, u_4\}$ is a $\beta_e$-set of $P_4$. Also $P_4$ is $\beta_e$-excellent. $V - S = \{u_3\}$ and $(S - \{u_2\}) \cup \{u_3\}$ is a $\beta_e$ set of $P_4$.

Therefore, $S$ is a very $\beta_e$-excellent set of $P_4$ and $P_4$ is a very $\beta_e$-excellent graph.

Remark 2.3: Any very $\beta_e$-excellent graph is a $\beta_e$-excellent graph.

Proof: Let $G$ be a very $\beta_e$-excellent graph and let $S$ be a very $\beta_e$-excellent set of $G$. Let $u \in V - S$. Then there exist $v \in S$ such that $(S - \{v\}) \cup \{u\}$ is a $\beta_e$-set of $G$. Therefore, every vertex of $V - S$ is an element of a $\beta_e$-set of $G$.

Since $S$ is a $\beta_e$-set of $G$, every element of $V(G)$ is in a $\beta_e$-set of $G$. Therefore, $G$ is $\beta_e$-excellent.

Remark 2.4: A very $\beta_e$-excellent graph need not be a rigid $\beta_e$-excellent graph. For example, $P_4$ is a very $\beta_e$-excellent graph. But is not a rigid $\beta_e$-excellent graph.

Very $\beta_e$-excellence for standard graphs

1. $K_n$ is very $\beta_e$-excellent for all $n$.
2. $K_{1,n}$ is not a very $\beta_e$-excellent graph for any $n \geq 2$.
3. $\overline{K}_n$ is a very $\beta_e$-excellent for all $n$.
4. $W_n$ is not very $\beta_e$-excellent for $n \geq 5$.
5. $K_{m,n}$ is not very $\beta_e$-excellent.
6. Petersen graph is not very $\beta_e$-excellent.
7. Any equivalence graph is very $\beta_e$-excellent.

Proposition 2.5: $P_n$ is very $\beta_e$-excellent iff $n = 2, 3, 4, 6, 7, 9, 12$.

Proof: When $n \equiv 2(\text{mod } 3)$, $P_n$ is not $\beta_e$-excellent and hence not very $\beta_e$-excellent.

Therefore, the possible values of $n$ are $n = 15r , n = 15r + 1, n = 15r + 3, n = 15r + 4, n = 15r + 6, n = 15r + 7, n = 15r + 9, n = 15r + 10, n = 15r + 12, n = 15r + 13, n = 15r + 15 (r \geq 1)$.

Case I: $n = 15r$. Let $n = 3k$. Then $k = 5r$; if $n = 3k$ then $\beta_e = 2k = 10r$.

Since the number of vertices is $15r$, there are $3r$ consecutive five vertices set. For very $\beta_e$-excellence, from each set at most 3 vertices can be taken. Therefore, at most $3(3r) = 9r$ vertices can be taken for constructing a very $\beta_e$-excellent set. But $\beta_e(P_n) = 10r$ where $n = 15r$. Therefore, $P_n$ where $n = 15r$ is not very $\beta_e$-excellent.

Case II: $n = 15r + 1$.

$n = 3k + 1$ implies $k = 5r$; $\beta_e = 2k + 1 = 10r + 1$. 

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Since there are $15r+1$ vertices, we have $3r$ five consecutive element sets. From these sets as per the definition of very $\beta_e$-excellent set, at most 3 vertices can be taken from each set. The number of possible vertices chosen is $3(3r)+1=9r+1$. But $\beta_e = 10r + 1$. Therefore, $P_n$ where $n = 15r+1$ is not very $\beta_e$-excellent.

Case III: $n = 15r + 3$

$n = 3k$ where $k = 5r + 1$, $\beta_e = 2k = 2(5r + 1)$.

The number of possible vertices in a very $\beta_e$-excellent set chosen is $3(3r) + 2 = 9r + 2$.

Hence, $P_n$ where $n = 15r+3$ is not very $\beta_e$-excellent.

Case IV: $n = 15r + 4$. $n = 3k + 1$ where $k = 5r + 1$, $\beta_e = 2k + 1 = 2(5r + 1) + 1 = 10r + 3$.

The number of possible vertices chosen with respect to the definition of very $\beta_e$-excellent set is $3(3r) + 3 = 9r + 3$.

But $\beta_e = 10r + 3$.

Therefore, $P_n$ where $n = 15r+4$ is not very $\beta_e$-excellent.

Case V: $n = 15r + 6$. $n = 3k$ where $k = 5r + 2$, $\beta_e = 2k = 2(5r + 2) = 10r + 4$.

There are $3r+1$ five consecutive elements sets and from each set at most 3 vertices can be chosen is at most $3(3r+1) + 1 = 9r + 4$. But $\beta_e = 10r + 4$.

Therefore, $P_n$ where $n = 15r+6$ is not very $\beta_e$-excellent.

Case VI: $n = 15r + 7$. $n = 3k + 1$ where $k = 5r + 2$, $\beta_e = 2k + 1 = 2(5r + 2) + 1 = 10r + 5$.

The number of maximum possible vertices chosen for a very $\beta_e$-excellent set is $3(3r+1) + 2 = 9r + 5$.

But $\beta_e = 10r + 5$.

Therefore, $P_n$ where $n = 15r+7$ is not very $\beta_e$-excellent.

Case VII: $n = 15r + 9$. $n = 3k + 3$ where $k = 5r + 3$, $\beta_e = 2k = 2(5r + 3) = 10r + 6$.

The number of possible vertices chosen for constructing a very $\beta_e$-excellent set is $3(3r+1) + 3 = 9r + 6$.

But $\beta_e = 10r + 6$.

Therefore, $P_n$ where $n = 15r+9$ is not very $\beta_e$-excellent.

Case VIII: $n = 15r + 10$. $n = 3k + 1$ where $k = 5r + 3$, $\beta_e = 2k + 1 = 2(5r + 3) + 1 = 10r + 7$.

The number of possible vertices chosen for constructing a very $\beta_e$-excellent set is $3(3r+2) = 9r + 6$.

But $\beta_e = 10r + 7$.

Therefore, $P_n$ where $n = 15r+10$ is not very $\beta_e$-excellent.

Case IX: $n = 15r + 12$. $n = 3k$ where $k = 5r + 4$, $\beta_e = 2k = 2(5r + 4) = 10r + 8$.

The number of possible vertices chosen for constructing a very $\beta_e$-excellent set is $3(3r+2)+2 = 9r+8$.

But $\beta_e = 10r + 8$.

Therefore, $P_n$ where $n = 15r+12$ is not very $\beta_e$-excellent set.

Case X: $n = 15r + 13$. $n = 3k + 1$ where $k = 5r + 4$, $\beta_e = 2k + 1 = 2(5r + 4) + 1 = 10r + 9$.

The number of possible vertices chosen for constructing a very $\beta_e$-excellent set is $3(3r+2)+2$. But $\beta_e = 10r + 9$.

Therefore $P_n$ where $n = 15r+13$ is not very $\beta_e$-excellent set.
Case XI: \( n = 15r + 15 \cdot n = 3k \) where \( k = 5r + 5 \); \( \beta_e = 2k = 2(5r + 5) = 10r + 10 \).

The number of possible vertices chosen for constructing a very \( \beta_e \)-excellent set is \( 3(3r+5) = 9r+15 \). But \( \beta_e = 10r + 10 \). Therefore \( P_n \) where \( n = 15r + 15 \) is not very \( \beta_e \)-excellent set.

When \( n = 1,2,3,4 \) \( P_n \) is clearly very \( \beta_e \)-excellent.

When \( n = 6 \), \( \{u_1,u_2,u_5,u_6\} \) is a very \( \beta_e \)-excellent set where \( V(P_6) = \{u_1,u_2,u_3,u_4,u_5,u_6\} \).

When \( n = 7 \), \( \{u_1,u_2,u_4,u_6,u_7\} \) is a very \( \beta_e \)-excellent set.

When \( n = 9 \), \( \{u_1,u_3,u_4,u_6,u_7\} \) is a very \( \beta_e \)-excellent set.

When \( n = 10 \); \( n = 3k + 1 \) where \( k = 3 \). There are two five consecutive elements set in \( P_{10} \) and at most 6 element are possible for a very \( \beta_e \)-excellent. Hence \( P_n \) is not very \( \beta_e \)-excellent.

When \( n = 12 \), \( n = 3k \) where \( k = 4 \). \( \beta_e(P_n) = 8 \).

The set \( \{u_1,u_2,u_4,u_6,u_7,u_9,u_{11},u_{12}\} \) is a very \( \beta_e \)-excellent and hence \( P_{12} \) is a very \( \beta_e \)-excellent graph.

When \( n = 13 \), \( n = 3k+1 \) where \( k=4 \). \( \beta_e(P_{13}) = 9 \). There are two five consecutive element sets with 3 elements remaining in the last. Hence at most 6 elements can be taken from the two consecutive elements sets and all the three remaining elements are to be taken for having 9 elements. This might will not give a \( \beta_e \)-set, since 3 consecutive elements cannot be taken in a \( \beta_e \)-set. Hence \( P_{13} \) is not very \( \beta_e \)-excellent.

**Proposition 2.6:** \( C_n \) is very \( \beta_e \)-excellent only if \( n = 3,4,5,7,10,13 \).

**Proof:** Arguing as in the previous proposition 2.5 the above result is obtained.

**Remark 2.7:** If a graph \( G \) has a unique \( \beta_e \)-set which is not \( V(G) \) then \( G \) is not very \( \beta_e \)-excellent.

**Proposition 2.8:** \( C_n \circ K_1 \) is not very \( \beta_e \)-excellent.

**Proof:**

**Case I:** Let \( n \) be even.

Let \( V(C_n \circ K_1) = \{u_1,u_2,...,u_n,v_1,v_2,...,v_n\} \). Any \( \beta_e \)-set \( S \) of \( C_n \circ K_1 \) consists of all \( v_i \)'s and alternate \( u_i \)'s. Any vertex outside \( S \) cannot come inside by replacing a vertex of \( S \) without affecting the equivalence nature of \( S \). Therefore, \( C_n \circ K_1 \) is not very \( \beta_e \)-excellent.

**Case II:** Let \( n \) be odd.

A similar argument as before shows that there exist no \( \beta_e \) excellent set which is very \( \beta_e \)-excellent.

**Observation 2.9:** A very \( \beta_e \)-excellent graph may have isolates. Also, there are non-equivalence graphs which have isolates and which are very \( \beta_e \)-excellent.
For example, $K_m \cup \overline{K_n}$ is a very $\beta_e$-excellent graph which have isolates, but this is an equivalence graph. $C_4 \cup K_1$ is a non equivalence graph which is very $\beta_e$-excellent and which has an isolate.

**Remark 2.10:** If $G$ is a very $\beta_e$-excellent graph then $G \cup K_m$ is also very $\beta_e$-excellent.

**Proposition 2.11:** Let $G$ be a very $\beta_e$-excellent graph without isolates. Let $S$ be a very $\beta_e$-excellent set of $G$. Then for any $u \in S$, $|pn[u,S]| \geq 1$.

**Proof:** Let $G$ be a very $\beta_e$-excellent graph and let $S$ be a very $\beta_e$-excellent set of $G$. Let $u \in S$. Suppose $u$ is an isolate of $S$ and any neighbor of $u$ in $G$ is adjacent with some vertex of $S$ other than $u$. Then $pn[u,S] = 1$. Also, if all the neighbors of $u$ form a complete sub graph with $u$, then $pn[u,S] = 1$.

**Corollary 2.12:** $P_6$ is very $\beta_e$-excellent. $S = \{u_1,u_2,u_5,u_6\}$ is a very $\beta_e$-excellent set of $G$ and $pn[u_5,S] = 2 > 1$.

**Remark 2.13:** Let $G$ be a graph without isolates. Let $S$ be a very $\beta_e$-excellent set of $G$. Let $x \in V - S$. Then there exist $u \in S$ such that $(S - \{u\}) \cup \{x\}$ is a $\beta_e$-set of $G$. For example, $P_7$ is very $\beta_e$-excellent. Let $V(P_7) = \{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$. Let $S = \{u_1,u_2,u_4,u_5,u_6,u_7\}$.

**Theorem 2.14:** Let $G$ be a graph without isolates. Suppose there exist a $\beta_e$-set $S$ of $G$ such that for every $x \in V - S$, there exist $u \in S$ such that $x \in pn(u,S)$. Then $G$ is very $\beta_e$-excellent.

**Proof:** By hypothesis, there exist a $\beta_e$-set $S$ of $G$ such that for every $x \in V - S$, there exist $u \in S$ such that $x \in pn(u,S)$. Then $(S - \{u\}) \cup \{x\}$ is a $\beta_e$-set of $G$. Therefore $S$ is a very $\beta_e$-excellent set of $G$. Hence $G$ is a very $\beta_e$-excellent graph.

**Illustration 2.15:** Let $V(C_4) = \{u_1,u_2,u_3,u_4\}$. Let $S = \{u_1,u_2\}$. Then $u_3$ and $u_4$ are private neighbours of $S$. $(S - \{u_1\}) \cup \{u_3\}$ is a $\beta_e$-set. $(S - \{u_2\}) \cup \{u_4\}$ is a $\beta_e$-set.

**Theorem 2.16:** Let $G$ be a graph such that $G$ is an equivalence graph. Let $V_1,V_2,...,V_k$ be the components of $G$ which are complete. Add vertices $u_1, u_2,...,u_k$. Join $u_i$ only with every vertex of $V_i$, $1 \leq i \leq k$. Let $H$ be the resulting graph. Then $H$ is very $\beta_e$-excellent.

**Proof:** Clearly $H$ is an equivalence graph and $H$ is very $\beta_e$-excellent.

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Source of support: Nil, Conflict of interest: None Declared

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