

EQUIVALENCE CHROMATIC PARTITION OF A GRAPH

N. SARADHA*¹, V. SWAMINATHAN²

¹Assistant Professor, Department of Mathematics,
SCSVMV University, Enathur, Kanchipuram, Tamil Nadu, India.

²Coordinator, Ramanujan Research Center in Mathematics,
Saraswathi Narayanan College, Madurai, Tamil Nadu, India.

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ABSTRACT

Let $G = (V, E)$ be a simple finite undirected graph. A subset S of V is called an equivalence set if every component of the induced sub graph $\langle S \rangle$ is complete. An equivalence number $\beta_e(G)$ is the maximum cardinality of an equivalence set of G [3]. A proper coloring of a graph is a partition of $V(G)$ into independent sets and the minimum cardinality of such a partition is called the chromatic number of G ($\chi(G)$). A partition of V into equivalence sets is called an equivalence partition of V . The minimum cardinality of an equivalence partition is called the equivalence chromatic number of G and is denoted by $\chi_{eq}(G)$. In this paper, a partition of $V(G)$ into equivalence sets is defined and results are derived.

Keywords: Equivalence set, Equivalence graph, β_e -excellent, rigid β_e -excellent.

1. INTRODUCTION

Let $G = (V, E)$ be a simple finite undirected graph. A subset S of V is called an equivalence set if every component of the induced sub graph $\langle S \rangle$ is complete. An equivalence number $\beta_e(G)$ is the maximum cardinality of an equivalence set of G [3]. A proper coloring of a graph is a partition of $V(G)$ into independent sets and the minimum cardinality of such a partition is called the chromatic number of G ($\chi(G)$). Since an equivalence set is a generalization of an independent set, every proper color partition is an equivalence partition. Thus, equivalence partition may be considered as a generalization of proper color partition. A vertex u in $V(G)$ is said to be β_e -good if u belongs to a β_e set of G . G is said to be β_e -excellent [12] if every vertex of G is β_e -good. An equivalence graph is a vertex disjoint union of complete graphs. Several variations of partitions of the vertex set have been studied. Since an equivalence set is a generalization of an independent set, every proper color partition is an equivalence partition. Thus, equivalence partition may be considered as a generalization of proper color partition. The concept of equivalence set, sub chromatic number, generalized coloring and equivalence covering number were studied in [1], [2], [4], [5], [6], [8], [10], [12]. In this paper, a partition of $V(G)$ into equivalence sets is defined and results are derived.

2. EQUIVALENCE CHROMATIC PARTITION

Definition 2.1 [3]: Recall that a subset S of V is an equivalent set if every component of $\langle S \rangle$ is complete.

Definition 2.2: A partition of V into equivalence sets is called an equivalence partition of V . The minimum cardinality of an equivalence partition is called the equivalence chromatic number of G and is denoted by $\chi_{eq}(G)$.

Remark 2.3: Any proper color cross partition is an equivalence partition. Therefore, $\chi_{eq}(G) \leq \chi(G)$.

*Corresponding Author: N. Saradha*¹*

*¹Assistant Professor, Department of Mathematics,
SCSVMV University, Enathur, Kanchipuram, Tamil Nadu, India.*

To find $\chi_{eq}(G)$ for standard graphs

1. $\chi_{eq}(K_n) = 1$
2. $\chi_{eq}(\overline{K}_n) = 1$
3. $\chi_{eq}(K_{1,n}) = 2$
4. $\chi_{eq}(K_{m,n}) = 2$
5. $\chi_{eq}(C_n) = 2$
6. $\chi_{eq}(P_n) = 2$
7. $\chi_{eq}(W_n) = \begin{cases} 2 & \text{if } n = 5 \\ 3 & \text{if } n \geq 6 \end{cases}$
8. $\chi_{eq}(P) = 3$, where P is the Petersen graph.

Proposition 2.4: $\chi_{eq}(G) = 1$ if and only if G is component wise complete.

Proof: Obvious.

Proposition 2.5: $\chi_{eq}(G) = 2$ iff $V(G)$ is the disjoint union of two maximal equivalence sets.

Remark 2.6: If G is a complete k-partite graph then $\chi_{eq}(G) = k$.

Remark 2.7: Given a positive integer k, there exists a graph G such that $|\chi_{eq}(G) - \chi(G)| = k$.

For example, take $G = K_{k+1}$.

$$\chi(G) = k + 1, \chi_{eq}(G) = 1.$$

Definition 2.8 [3]: A dominating set S of G is called an equivalence dominating set if $\langle S \rangle$ is complete. The equivalence dominating number $\gamma_e(G)$ of G is the cardinality of its smallest equivalence dominating set.

Remark 2.9: There is no relationship between $\chi_{eq}(G)$ and $\gamma_e(G)$.

For: $\gamma_e(K_n) = 1$ and $\chi_{eq}(K_n) = 1$

$$\gamma_e(P_3) = 1, \chi_{eq}(P_3) = 2$$

$$\gamma_e(\overline{K}_n) = n, \chi_{eq}(\overline{K}_n) = 1$$

Remark 2.10: Given a positive integer k, there exists a graph G such that $|\gamma_e(G) - \chi_{eq}(G)| = k$.

Proof: Let G be a $(k+2)$ -partite graph, where each partite sets contains at least two elements. Then $\gamma_e(G) = 2$ (Since the set containing two vertices from different partite sets forms a K_2 and this set is a dominating set) and $\chi_{eq}(G) = k + 2$. Therefore $\chi_{eq}(G) - \gamma_e(G) = k$.

Remark 2.11: Let G be a bipartite graph of order ≥ 3 which contains an edge. Then $\gamma_{eq}(G) = 2$. Therefore, for any tree T with at least 3 vertices, $\chi_{eq}(T) = 2$.

Proposition 2.12: Let G be an unicyclic graph. Then $\chi_{eq}(G) = 2$.

Proof: Let G be an unicyclic graph. Let C be the unique cycle in G.

$$\text{Let } V(C) = \{u_1, u_2, u_3, \dots, u_k\}$$

Case I: k is even.

Then $\chi_{eq}(C)$ contains the partitions $S_1 = \{u_1, u_3, \dots, u_{k-1}\}; S_2 = \{u_2, u_4, \dots, u_k\}$.

Let P_i be a path attached at u_i , $1 \leq i \leq k$.

Let $V(P_i) = \{u_{i1}, u_{i2}, \dots, u_{iri}\}$ where r_i may be zero. Let $S_{i1} = \{u_{i1}, u_{i3}, \dots, u_{iri}\}$ if r_i is odd and $S_{i1} = \{u_{i1}, \dots, u_{iri-1}\}$ if r_i is even. Let $S_{i2} = \{u_{i2}, u_{i4}, \dots, u_{iri}\}$ or $\{u_{i2}, u_{i4}, \dots, u_{iri-1}\}$ if r_i is even or odd. Join S_{i2} with S_1 if $u_i \in S_1$. Otherwise join S_{i2} with S_2 . Also join S_{i1} with S_2 if $u_i \in S_1$. Otherwise, join S_{i1} with S_1 . The resulting partition Π of $V(G)$ contains only two elements and hence $\chi_{eq}(G) = 2$.

Case II: k is odd.

Then $\chi_{eq}(C)$ contains the partitions $S_1 = \{u_1, u_3, \dots, u_{k-2}, u_k\}; S_2 = \{u_2, u_4, \dots, u_{k-1}\}$.

Let P_i be a path attached at u_i , $1 \leq i \leq k$.

Let $V(P_i) = \{u_{i1}, u_{i2}, \dots, u_{iri}\}$ where r_i may be zero. Let $S_{i1} = \{u_{i1}, u_{i3}, \dots, u_{iri}\}$ if r_i is odd and $S_{i1} = \{u_{i1}, \dots, u_{iri-1}\}$ if r_i is even. Let $S_{i2} = \{u_{i2}, u_{i4}, \dots, u_{iri}\}$ or $\{u_{i2}, u_{i4}, \dots, u_{iri-1}\}$ if r_i is even or odd.

Join S_{i2} with S_1 if $u_i \in S_1$. Otherwise join S_{i2} with S_2 . Also join S_{i1} with S_2 if $u_i \in S_1$. Otherwise join S_{i1} with S_1 . The resulting partition Π of $V(G)$ contains two elements and hence $\chi_{eq}(G) = 2$.

Theorem 2.13: Given any positive integer $k \geq 3$, the problem of deciding $\chi_{eq}(G) \geq k$ is NP-complete for any graph G with $\chi(G) \geq k$.

Proof: Let $\Pi = \{V_1, V_2, \dots, V_r\}$ be a chromatic partition of G where $r = \chi(G)$. Let $V(G) = \{u_1, u_2, \dots, u_n\}$. Add vertices $u_{11}, u_{12}, u_{21}, u_{22}, \dots, u_{n1}, u_{n2}$ to V . Join u_{i1} and u_{i2} with u_i , $1 \leq i \leq n$. Also join u_{i1} and u_{i2} , $1 \leq i \leq n$. Let H be the resulting graph. Then $\chi_{eq}(H) = r = \chi(G)$. The problem of deciding $\chi(G) \geq k$ is NP-complete if $k \geq 3$. Therefore, the problem of deciding $\chi_{eq}(H) \geq k$ is NP-complete.

Definition 2.14: Let $u \in V(G)$. Let S be a maximum equivalence set of G containing u . $|S|-1$ is called the equivalence degree of u and is denoted by $\deg_{eq}(u)$.

Example 2.15: Let $G = K_{1,n}$.

Let u be the central vertex. Then $\deg_{eq}(u) = 1; \deg_{eq}(v) = n-1$ where v is the pendant vertex.

Example 2.16: Let $G = K_n$. $\deg_{eq}(u) = n-1$ for any vertex u of G .

Example 2.17: Let $G = P_5$. The vertices 1,2,4,5 have \deg_{eq} as 3 and $\deg_{eq}(3) = 2$.

Example 2.18: Let

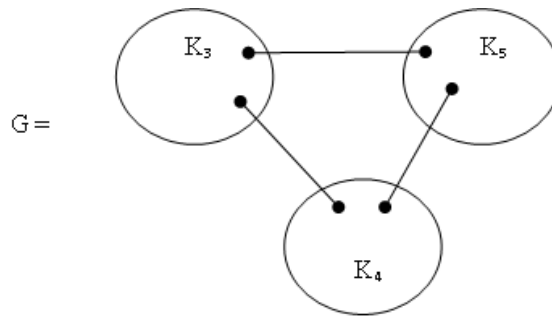


Figure-2.1: Illustration of \deg_{eq}

The \deg_{eq} of any vertex of K_3 is 2, \deg_{eq} of any vertex of K_4 is 3 and \deg_{eq} of any vertex of K_5 is 4.

Definition 2.19 [3]: A graph G is said to be an equivalence graph if $V(G)$ is an equivalence set. The maximum cardinality of an equivalence set is denoted by $\beta_e(G)$.

Definition 2.20 [12]: A graph G is said to be a β_e -excellent graph if every vertex of G is contained in a β_e -set of G .

Remark 2.21: Let G be a β_e -excellent graph. Then $\deg_{eq}(u) = \beta_e - 1$ for every vertex u of G . Therefore, G is \deg_{eq} -regular.

Definition 2.22:

$$\Delta_{eq}(G) = \max\{\deg_{eq}(u) : u \in V(G)\}$$

$$\delta_{eq}(G) = \min\{\deg_{eq}(u) : u \in V(G)\}$$

Remark 2.23: $\Delta_{eq}(G) = \beta_e(G) - 1$.

Theorem 2.24: Let G be a graph of order n . Then $\frac{n}{\beta_e(G)} \leq \chi_{eq}(G) \leq n - \beta_e(G) + 1$.

Proof: Let $\chi_{eq}(G) = k$.

Let $\Pi = \{V_1, V_2, \dots, V_k\}$ be a χ_{eq} -partition of $V(G)$. Each V_i is an equivalence set and hence $|V_i| \leq \beta_e(G)$.

Therefore, $n = |V_1| + |V_2| + \dots + |V_k| \leq \beta_e(G) + \beta_e(G) + \dots + \beta_e(G)_{(k \text{ times})}$.

Therefore, $n \leq k\beta_e(G)$.

Therefore, $\frac{n}{\beta_e(G)} \leq k = \chi_{eq}(G)$.

Let S be a β_e -set of G . Consider the partition $\Pi = \{S, \{v_1\}, \{v_2\}, \dots, \{v_{n-\beta_e(G)}\}\}$ where $v_i \in V - S$, $1 \leq i \leq n - \beta_e(G)$.

Therefore, $|\Pi| \geq \chi_{eq}(G)$.

That is, $n - \beta_e(G) + 1 \geq \chi_{eq}(G)$.

Therefore, $\chi_{eq}(G) \leq n - \beta_e(G) + 1$.

Remark 2.25: In the case of K_n , $\beta_e(K_n) = n$, $\chi_{eq}(K_n) = 1$ and $\frac{n}{\beta_e(K_n)} = \frac{n}{n} = 1 = \chi_{eq}(K_n)$.
 $\chi_{eq}(K_n) = 1 = n - \beta_e(K_n) + 1$.

Problem 2.26: $\chi_{eq}(G) \leq 1 + \Delta_{eq}(G)$

It is known that, $\chi_{eq}(G) \leq \chi(G) \leq 1 + \Delta(G)$.

Problem 2.27: What is the relationship between $\Delta_{eq}(G)$ and $\Delta(G)$?

In the case of star $K_{1,n}$, $\Delta_{eq}(G) = n - 1$, $\Delta(G) = n$.

In the case of P_5 , $\Delta_{eq} = 3$ and $\Delta = 2$.

In the case of K_n , $\Delta_{eq} = \Delta = n - 1$.

Remark 2.28: If G has a full degree vertex, $\Delta(G) = n - 1$ and $\Delta_{eq}(G) \leq n - 1$.

Remark 2.29: $\Delta_{eq}(G) = n - 1$ iff $G = K_n$ or $\overline{K_n}$ or G is an equivalence graph.

$\Delta_{eq}(G) \geq \Delta(G)$ if G is an equivalence graph and $\Delta_{eq}(G) = \Delta(G)$ in the case of equivalence graph G iff $G = K_n$

In the case of equivalence graph, $\chi_{eq}(G) = 1$ and $\Delta_{eq}(G) = n - 1$.

Remark 2.30: Let Π be a χ_{eq} -partition of G . Let the number of β_e -sets in the partition be denoted by $\chi_{eq}^{\beta_e}$.

1. In the case of K_n ,
 $\chi_{eq} = 1$ and $\chi_{eq}^{\beta_e} = 1$.
2. In the case of P_6 ,
 $\chi_{eq} = 2$ and $\chi_{eq}^{\beta_e} = 1$.
3. In the case of complete tri-partite graph, where each partite set contains 3 elements,
 $\chi_{eq} = 3$ and $\chi_{eq}^{\beta_e} = 3$.

In general, in the case of complete k -partite graph, where each partite set contains k elements, $\chi_{eq} = k$ and $\chi_{eq}^{\beta_e} = k$.

Remark 2.31: In a χ_{eq} -partition of G , it may happen that none of the sets in the partition is a β_e -set. For example,

In W_7 , $\beta_e = 4$, $\Pi = \{\{1,6,7\}, \{2,4,5\}, \{3\}\}$ is a χ_{eq} -partition and it does not contain any β_e -set.

Remark 2.32: Let S be an equivalence set in G with k components, each component being complete. Then in \overline{G} , the vertices of S constitute a complete k -partite graph where each component of S becomes a partite set in \overline{G} . Therefore, S becomes an equivalence set in \overline{G} if and only if $\langle S \rangle$ is a complete subgraph of G or a totally disconnected subgraph of G .

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